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# On relation between uniform asymptotic stability and exponential stability of linear differential equations

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**Abstract.** We present such a restriction on parameters of linear functional differential equations of retarded type that is sufficient for the uniform asymptotic stability of an equation to be equivalent to its exponential stability.

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## Introduction

Different notions of stability for ordinary differential equations complement each other in the following sense. The definition of asymptotic stability requires Lyapunov stability, and there are examples showing that all solutions to an equation may tend to zero while its trivial solution is not Lyapunov stable. Further, one should distinguish between global and local stability, and the stability of a solution is unrelated to its boundedness.

However, definitions of stability for linear equations, as well as relations between different kinds of stability, are simplified. The purpose of the paper is to investigate some of these relations. In the first section we consider linear ordinary differential equations. The second section, which is the main one, is devoted to functional differential equations.

Let $\mathbb{C}^r$ be an $r$-dimensional linear complex space with some norm, $\mathbb{C}^{r\times r}$ be the algebra of $r \times r$ complex matrices with unit $E$ and zero $\Theta$, the norm in $\mathbb{C}^{r\times r}$ being consistent with the norm in $\mathbb{C}^r$. The norms will be denoted by $|\cdot|$. Denote $\mathbb{R}_+ = [0, \infty)$, $\Delta = \{(t, s) \in \mathbb{R}^2_+ \mid t \geq s\}$.

## 1 Interconnection between the types of stability for linear ordinary differential equations

Consider a homogeneous differential equation of the form

$$\dot{x}(t) + A(t)x(t) = 0, \qquad t \in \mathbb{R}_+, \tag{1.1}$$

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where $A\colon \mathbb{R}_+ \to \mathbb{C}^{r\times r}$ is a matrix function with locally integrable components.

We shall say that a *solution* of equation (1.1) is an absolutely continuous function satisfying the equality (1.1) almost everywhere.

It is easy to see that there is no distinction between local and global stability for equation (1.1). This makes the use of the term "the stability of an equation" correct.

The definitions of uniform and uniform asymptotic stability suggest that one should consider the family of equations (1.1) with an arbitrary starting point, instead of a single equation. However, this complexity of the object of research can be avoided by introducing the notion of the *Cauchy function*.

Let $X = X(t)$ be the fundamental solution of equation (1.1), and $X(0) = E$. As it is known, $X(t)$ is invertible for any $t$. Therefore for all $(t,s) \in \Delta$ the matrix $C(t,s) = X(t)X^{-1}(s)$ is defined, which is called the Cauchy function of equation (1.1).

Let us remark the useful property of the Cauchy function, which follows directly from the definition: for any $t, s, \tau$ there holds the equality

$$C(t,s) = C(t,\tau)C(\tau,s). \tag{1.2}$$

Equality (1.2) is often called the *semigroup property* of the Cauchy function.

It follows from the definition of the Cauchy function that each solution of equation (1.1) defined for $t \geq s$ may be represented in the form $x(t) = C(t,s)x(s)$. It is obvious that all definitions of stability for equation (1.1) can be reformulated in terms of the Cauchy function.

**Definition 1.1.** Equation (1.1) is:

- *Lyapunov stable*, if for every $s \in \mathbb{R}_+$ there exists $K_s > 0$ such that $|C(t,s)| \leq K_s$ for every $t \geq s$;
- *asymptotically stable*, if for every fixed $s \in \mathbb{R}_+$ we have $\lim_{t\to\infty} |C(t,s)| = 0$;
- *uniformly stable*, if there is $K > 0$ such that for all $(t,s) \in \Delta$ we have $|C(t,s)| \leq K$;
- *uniformly asymptotically stable*, if it is uniformly stable, and $|C(t,s)| \to 0$ as $t - s \to \infty$ uniformly with respect to $s$;
- *exponentially stable*, if there are $K, \gamma > 0$ such that for all $(t,s) \in \Delta$ we have $|C(t,s)| \leq K\exp(-\gamma(t-s))$.

Note that the definition of asymptotic stability does not include Lyapunov stability, and the definition of exponential stability does not include uniform stability. In this connection the following question arises. Suppose that the Cauchy function tends to zero as $t - s \to \infty$ uniformly with respect to $s$. Does this imply that equation (1.1) is uniformly stable? The answer is given by the following example.

**Example 1.2.** Consider a scalar equation of the form (1.1)

$$\dot{x}(t) + a(t)x(t) = 0, \qquad t \in \mathbb{R}_+.$$

For each $n = 1, 2, \ldots$ define the function $a$ by the rule:

$$a(t) = \begin{cases} -n, & \textit{if} \quad t \in [n-1, n-2/3]; \\ n, & \textit{if} \quad n \in (n-2/3, n). \end{cases}$$

Let $\varepsilon > 0$. By the definition of the function $a$ it is easy to find $l(\varepsilon)$ such that for all $(t,s) \in \Delta$, satisfying the condition $t - s > l$, the inequality $\int_s^t a(\tau)\,d\tau \geq -\ln \varepsilon$ holds. Therefore,

$$0 < C(t,s) = \exp\left(-\int\limits_s^t a(\tau)\,d\tau\right) \leq \varepsilon.$$

Consequently, if $t - s \to \infty$ then the function $C(t,s)$ tends to zero uniformly with respect to $s$.

On the other hand, for any $n = 1, 2, \ldots$ the equality $C(n - 2/3, n - 1) = e^{n/3}$ holds, i.e. the Cauchy function is not bounded with respect to $(t,s)$.

Thus, the demand for uniform stability cannot be eliminated from the definition of uniform asymptotic stability.

**Remark 1.3.** If for some $K > 0$ we have $|A(t)| \leq K$, $t \in \mathbb{R}_+$, then for every $l > 0$ we have $\sup_{0 \leq t-s \leq l} |C(t,s)| \leq e^{Kl}$. Therefore, in this case the condition of uniform stability can be excluded from the definition of uniform asymptotic stability.

Now we can put in order the different types of stability of equation (1.1). Exponential and uniform asymptotic stabilities imply uniform stability and asymptotic stability and, consequently, Lyapunov stability.

**Theorem 1.4.** *For equation* (1.1) *uniform asymptotic stability is equivalent to exponential stability.*

*Proof.* Obviously, uniform asymptotic stability follows from exponential stability.

Suppose equation (1.1) is uniformly asymptotically stable. We use induction principle. Fix an arbitrary $\varepsilon \in (0,1)$ and find $l(\varepsilon) > 0$ such that for all $(t,s) \in \Delta$ satisfying $t - s \geq l$ the estimate $|C(t,s)| \leq \varepsilon$ holds. Assume that if $t - s \geq nl$ then $|C(t,s)| \leq \varepsilon^n$, and consider the case $t - s \geq (n+1)l$. By the semigroup property of the Cauchy function we obtain

$$|C(t,s)| \leq |C(t,s+l)||C(s+l,s)| \leq \varepsilon^{n+1}.$$

To complete the proof it remains to note that by virtue of the uniform stability of equation (1.1) there is $K > 0$ such that if $t < s + l$ then $|C(t,s)| \leq K$. □

**Remark 1.5.** The properties of uniform asymptotic and exponential stability are closely connected with the well-known result by Massera and Schäffer [6], from which it follows that the exponential stability of equation (1.1) is equivalent to the existence of a continuous positive function $\sigma \colon \mathbb{R}_+ \to (0,\infty)$ such that $\inf_{t \in \mathbb{R}_+} \sigma(t) < 1$ and $|C(t,s)| \leq \sigma(t-s)$ for all $(t,s) \in \Delta$.

If equation (1.1) is uniformly asymptotically stable, then the existence of such a function $\sigma$ is obvious. However, the proof of Theorem 1.4 shows that it is due to the semigroup property that exponential stability follows from uniform asymptotic stability.

## 2 Uniform asymptotic stability and exponential stability of linear functional differential equations

Consider an equation (more precisely, the family of equations depending on the parameter $s \in \mathbb{R}_+$)

$$\dot{x}(t) + \int\limits_s^t d_\tau R(t,\tau)x(\tau) = f(t), \qquad t \geq s \geq 0, \tag{2.1}$$

where $f\colon \mathbb{R}_+ \to \mathbb{C}^r$ is a vector function with locally integrable components. The integral in (2.1) is understood in the Riemann–Stieltjes sense. The matrix function $R\colon \Delta \to \mathbb{C}^{r\times r}$ has the following properties:

- the matrix function $R(\cdot,\tau)$ is locally integrable;
- $R(t,\cdot)$ is a matrix function of bounded variation, and the function $\rho(t) = \operatorname{var}_0^t R(t,\cdot)$ is locally integrable.

By a *solution* of equation (2.1) we mean a locally absolutely continuous vector function $x\colon [s,\infty) \to \mathbb{C}^r$ that satisfies the equality (2.1) almost everywhere.

Equation (2.1) is a significant generalization of equation (1.1). It is a linear functional differential equation of retarded type. Equation (2.1) includes [1, pp. 8–11], [2, pp. 1–6] ordinary differential equations, differential equations with concentrated aftereffect, and Volterra integro-differential equations, as special cases. If a linear delay differential equation is written in the traditional form [3], then it is necessary to specify an initial function $x(\tau) = \varphi(\tau)$, $\tau < s$. It is shown in [1] that the initial function can be rearranged to the right-hand part and the equation can be turned into the form (2.1).

It is known [1] that equation (2.1) with an initial condition given is uniquely solvable, and its solution can be represented in the form

$$x(t) = C(t,s)x(s) + \int_s^t C(t,\tau)f(\tau)\,d\tau.$$

Here components of the matrix function $C\colon \Delta \to \mathbb{C}^{r\times r}$ are locally absolutely continuous with respect to the argument $t$, and $C$ is a solution of the initial problem

$$\frac{\partial C(t,s)}{\partial t} + \int_s^t d_\xi R(t,\xi)C(\xi,s) = \Theta, \qquad C(s,s) = E, \qquad t \ge s,$$

for every fixed $s \in \mathbb{R}_+$.

The function $C$ is called the *Cauchy function* of equation (2.1). The above representation of a solution makes it possible to apply the definition of stability, given for equation (1.1), to equation (2.1), and to use the Cauchy function as a central object in the study of stability.

However, it should be noted that the Cauchy function of the functional differential equation (2.1) has a significantly more complicated structure in comparison with the Cauchy function of equation (1.1). In particular, it does not possess the remarkable semigroup property (1.2): it is proved in [4] that equality (1.2) is satisfied if and only if equation (2.1) degenerates into (1.1).

Nevertheless, for the Cauchy function of equation (2.1) the following similar property [2] is valid. For all $(t,s) \in \Delta$ and all $\xi$ such that $s \le \xi \le t$ there holds the equality:

$$C(t,s) = C(t,\xi)C(\xi,s) + \int_\xi^t C(t,\theta)\int_s^\xi d_\eta R(\theta,\eta)C(\eta,s)\,d\theta. \tag{2.2}$$

Let us introduce some auxiliary functions to describe the properties of the function $R\colon \Delta \to \mathbb{C}^{r\times r}$:

- the function $h\colon \mathbb{R}_+ \to \mathbb{R}_+$, putting $h(t) = \inf\{s \mid R(t,s) \neq \Theta\}$;

- the function $\mu\colon \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$, putting $\mu(t) = \sup\{\tau \mid h(\tau) \le t\}$;
- the function $\nu\colon \mathbb{R}_+ \to \mathbb{R}_+$, putting $\nu(t) = \inf\{\tau \mid \mu(t) \ge t\}$.

Now the integration limits in equality (2.2) can be set more precisely.

**Lemma 2.1.** *For all $(t,s) \in \Delta$ and all $\xi$ such that $s \le \xi \le t$ for the Cauchy function of equation (2.1) there holds the equality*

$$C(t,s) = C(t,\xi)C(\xi,s) + \int\limits_{\xi}^{\mu(\xi)} C(t,\theta) \int\limits_{\nu(\xi)}^{\xi} d_\eta R(\theta,\eta)C(\eta,s)\,d\theta. \tag{2.3}$$

*Proof.* If $\theta > \mu(\eta)$, i.e. $\theta > \sup\{t \mid \inf\{s \mid R(t,s) \ne \Theta\} \le \eta\}$, then $\inf\{s \mid R(t,s) \ne \Theta\} > \eta$, so $R(\theta,\eta) = \Theta$.

The condition $\theta > \mu(\eta)$ in equality (2.2) will be established, if $\theta > \mu(\xi)$ (because the function $\mu$ is non-decreasing) or if $\eta < \nu(\xi) = \inf\{s \mid \mu(s) \ge \xi\}$. Thus, for all $\theta > \mu(\xi)$ and for all $\eta < \nu(\xi)$ we have $R(\theta,\eta) = \Theta$ in equality (2.2), i.e. (2.2) turns into (2.3). □

**Remark 2.2.** If the function $h$ satisfies $\sup_{\xi\in\mathbb{R}_+}(\xi - h(\xi)) = \delta < \infty$, then, in terms of [2], the function $R$ satisfies the '$\delta$-condition'. In this case equation (2.2) is often called an *equation with bounded delay*. If the function satisfies the $\delta$-condition, then equality (2.3) holds for $\mu(\xi) = \xi + \delta$ and $\nu(\xi) = \xi - \delta$.

Now we investigate the relation between the uniform asymptotic stability and exponential stability of equation (2.1). The next example shows that these two types of stability are not equivalent.

**Example 2.3.** Consider a scalar equation of the form

$$\dot{x}(t) = -x(t) + \frac{x(0)}{(t+1)^2}, \qquad t \ge 0.$$

Let us construct its Cauchy function. If $s = 0$, we have $C(t,0) = \exp(-t) + \int_0^t \frac{\exp(-(t-\tau))}{(\tau+1)^2}\,d\tau$, i.e. $\lim_{t\to\infty} |C(t,0)| = 0$, but at the same time $C(t,0) \ge \frac{1}{(t+1)^2}$. For all $s > 0$ we get $C(t,s) = \exp(-(t-s))$.

Thus, the equation is uniformly asymptotically stable but not exponentially stable.

The possibility to construct such an example is provided by the fact that the asymptotic behavior of a solution depends essentially on its value at the point $t = 0$. Below we introduce restrictions on parameters of equation (2.1) to make it "forget the history" of its solutions as the argument increases.

**Lemma 2.4.** *If there holds the condition*

$$\sup_{t\in\mathbb{R}_+} \int\limits_{h(t)}^{t} \rho(\tau)\,d\tau = V < \infty, \tag{2.4}$$

*where $\rho(t) = \operatorname{var}_0^t R(t,\cdot)$, then for each $t \in \mathbb{R}_+$ there also holds the inequality $\int_t^{\mu(t)} \rho(\tau)\,d\tau \le V$.*

*Proof.* Suppose that there is $t_0 \in \mathbb{R}_+$ such that $\mu(t_0) < \infty$ and $\int_{t_0}^{\mu(t_0)} \rho(\tau)\,d\tau = V + \varepsilon > V$. By the definition of the function $\mu$, one can find $t_1 \le \mu(t_0)$ such that $h(t_1) \le t_0$ and $\int_{t_1}^{\mu(t_0)} \rho(\tau)\,d\tau < \varepsilon$. But then

$$V \ge \int\limits_{h(t_1)}^{t_1} \rho(\tau)\,d\tau \ge \int\limits_{t_0}^{\mu(t_0)} \rho(\tau)\,d\tau - \int\limits_{t_1}^{\mu(t_0)} \rho(\tau)\,d\tau > V,$$

which is impossible.

Suppose now that there is $t_0 \in \mathbb{R}_+$ such that $\mu(t_0) = \infty$ and $\int_{t_0}^{\infty} \rho(\tau)\,d\tau > V$. Then there exists $t_1 > t_0$ such that for all $\xi > t_1$ the inequality $\int_{t_0}^{\xi} \rho(\tau)\,d\tau > V$ holds. Since $\mu(t_0) = \infty$, there exists $t^* > t_1$ such that $h(t^*) \le t_0$. Again, we get a contradiction:

$$V \ge \int\limits_{h(t^*)}^{t^*} \rho(\tau)\,d\tau \ge \int\limits_{t}^{t^*} \rho(\tau)\,d\tau > V. \qquad \square$$

**Lemma 2.5.** *If condition* (2.4) *holds and equation* (2.1) *is uniformly asymptotically stable then* $\sup_{t\in\mathbb{R}_+}(t - h(t)) < \infty$.

*Proof.* Choose $l > 0$ such that for all $s \in \mathbb{R}_+$ and $t \ge s + l$ the inequality $|C(t,s)| < 1/2$ holds. Then $|C(s,s) - C(s+l,s)| > 1/2$.

On the other hand, by virtue of the uniform stability of the equation there exists $K > 0$ such that for all $(t,s) \in \Delta$ the estimate $|C(t,s)| < K$ is valid. From the definition of the Cauchy function we have

$$|C(s,s) - C(s+l,s)| \le \left| \int\limits_{s}^{s+l} \int\limits_{s}^{t} d_\tau R(t,\tau) C(\tau,s)\,dt \right| \le K \int\limits_{s}^{s+l} \rho(t)\,dt.$$

Thus for every $s \in \mathbb{R}_+$ the inequality $\int_s^{s+l} \rho(t)\,dt > \frac{1}{2K}$ holds. Let $l^* > 2KlV + l$. Then for all $s \ge l^*$ we have $\int_{s-l^*}^{s} \rho(t)\,dt > V$.

If $\sup_{t\in\mathbb{R}_+}(t - h(t)) = \infty$, then there exists $t^* > l^*$ such that $t^* - h(t^*) \ge l^*$. But this leads to a contradiction:

$$V \ge \int\limits_{h(t^*)}^{t^*} \rho(t)\,dt \ge \int\limits_{t^*-l^*}^{t^*} \rho(t)\,dt > V. \qquad \square$$

**Theorem 2.6.** *Suppose condition* (2.4) *holds. Then for equation* (2.1) *uniform asymptotic stability is equivalent to exponential stability.*

*Proof.* Uniform asymptotic stability follows from exponential stability.

Let equation (2.1) be uniformly asymptotically stable. Fix $\varepsilon > 0$ so that $\varepsilon(1+V) < 1$, and find $l > 0$ such that for all $(t,s) \in \Delta$, where $t - s \ge l$, the inequality $|C(t,s)| \le \varepsilon$ holds.

By Lemma 2.5, we have $\sup_{t\in\mathbb{R}_+}(\xi - h(\xi)) = \delta < \infty$. It is easily seen that $\sup_{t\in\mathbb{R}_+}(\xi - \nu(\xi)) = \delta$. We prove by induction on $n \in \mathbb{N}$ that if $t - s \ge nl + 2(n-1)\delta$ then the inequality $|C(t,s)| \le \varepsilon^n (1+V)^{n-1}$ holds .

Let $t - s \ge (n+1)l + 2n\delta$. Suppose $\xi = s + l + \delta$. By Lemma 2.1, equality (2.2) can be written in the form (2.3). In equality (2.3) we have $\nu(\xi) = \xi - \delta$, $t - \xi \ge nl + 2(n-1)\delta$,

$t-\theta \geq nl + 2(n-1)\delta$, $\xi - s \geq l$, $\eta - s \geq l$. By the induction hypothesis, the properties of Stieltjes integral, and Lemma 2.4, we have:

$$|C(t,s)| \leq |C(t,\xi)||C(\xi,s)| + \int\limits_{\xi}^{\mu(\xi)} |C(t,\theta)| \left| \int\limits_{\xi-\delta}^{\xi} d_\eta R(\theta,\eta) C(\eta,s) \right| d\theta \leq$$

$$\leq \varepsilon^n (1+V)^{n-1} \left( \varepsilon + \sup_{\xi-\delta\leq\eta\leq\xi} |C(\eta,s)| \int\limits_{\xi}^{\mu(\xi)} \rho(\theta)\, d\theta \right) \leq \varepsilon^{n+1}(1+V)^n.$$

To complete the proof it remains to note that by virtue of the uniform stability of equation (2.1), there is $K > 0$ such that if $t < s + l$, then $|C(t,s)| \leq K$. □

**Remark 2.7.** It is obvious that Theorem 1.4 is the simplest special case of Theorem 2.6, because for equation (1.1) we have $h(t) = t$, i.e. (2.4) is satisfied automatically.

Results associating uniform asymptotic and exponential stability for equation (2.1) were obtained in [3] and [5] under the assumption that the $\delta$-condition and the Massera condition

$$\sup_{t\in\mathbb{R}_+} \int\limits_{t}^{t+1} \rho(s)\, ds < \infty$$

are satisfied. As is shown by Lemma 2.5, the condition (2.4) and uniform asymptotic stability taken together provide the $\delta$-condition, and (2.4) follows from the $\delta$-condition and the Massera condition. Thus, the mentioned results of [3] and [5] follow from Theorem 2.6. It is easy to see that the converse is not true.

**Remark 2.8.** Note that the delay is unbounded in Example 2.3. The reviewer of the paper suggested us the following question, which we suppose to be an interesting open problem. Suppose delay is bounded, while condition (2.4) and, consequently, the Massera condition are not satisfied. In this case, is there an equation of the form (2.1) that is uniformly asymptotically stable and is not exponentially stable?

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## References

[1] N. V. Azbelev, V. P. Maksimov, L. F. Rakhmatullina, *Introduction to the theory of linear functional differential equations*, World Federation Publisher Company Inc., Atlanta, 1995. MR1422013

[2] N. V. Azbelev, P. M. Simonov, *Stability of differential equations with aftereffect*, Stability and control: theory, methods and applications, Vol. 20, Taylor & Francis, London, 2003. MR1965019

[3] J. K. Hale, *Theory of functional differential equations*, Applied Mathematical Sciences, Vol. 3, Springer-Verlag, New York, 1977. MR0508721

[4] V. P. Maksimov, *Voprosy obshey teorii funktsional'no-differentsial'nykh uravneniĭ* (in Russian) [Questions of the common theory of functional differential equations], PGU Press, Perm, Russia, 2003.

[5] V. V. Malygina, Some criteria for stability of equations with retarded argument, *Differential Equations* **28**(1992), No. 10, 1398–1405. MR1208401

[6] J. L. Massera, J. J. Schäffer, Linear differential equations and functional analysis. I, *Ann. of Math. (2)* **67**(1958), No. 3, 517–573. MR0096985