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Gevrey class regularity for analytic differential-delay equations

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Abstract. This paper considers differential-delay equations of the form

$$x'(t) = p(t)x(t-1),$$

where the coefficient function $p\colon \mathbb{R} \to \mathbb{C}$ is analytic and not bounded on any δ-neighborhood of the intervals $(-\infty, \gamma]$, $\gamma \in \mathbb{R}$. For these equations, we cannot apply the known results regarding the analyticity of the bounded solutions $x\colon (-\infty, \gamma] \to \mathbb{C}$. We prove Gevrey class regularity for such solutions.

Keywords: delay equation, analyticity, Gevrey class.

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1 Introduction

The analyticity of globally defined bounded solutions of autonomous analytic delay equations was studied first in [6]. The result of [6] was generalized to the nonautonomous case in [4]. Paper [4] verifies that if $\gamma \in \mathbb{R}$, $x\colon (-\infty, \gamma] \to \mathbb{C}^n$ is a bounded, uniformly continuous solution of

$$x'(t) = f(t, x_t)$$

on the interval $(-\infty, \gamma]$, and f is analytic and bounded on a δ-neighborhood of the set $\{(t, x_t) : t \in (-\infty, \gamma]\}$, then x is real analytic, i.e., there exists an open neighborhood V of $(-\infty, \gamma]$ and a complex analytic map $\hat{x}\colon V \to \mathbb{C}^n$ such that $\hat{x}|_{(-\infty,\gamma]} = x$. It is an interesting question whether the condition regarding the boundedness of f can be relaxed.

The result of [4] is not applicable to equations of the form

$$x'(t) = p(t)x(t-1) \tag{1.1}$$

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if p is analytic but not bounded on any δ-neighborhood of $(-\infty, \gamma]$. Typical examples of such coefficient functions are $p(t) = e^{it^q}$ and $p(t) = \sin(t^q)$ with an integer $q \geq 2$. In this paper we investigate the case when

$$p(t) = \sum_{m \in F} A_m e^{im\omega t^q}, \qquad t \in \mathbb{R},$$

F is a finite set of integers, $A_m \in \mathbb{C}$ for $m \in F$, $\omega > 0$, $i = \sqrt{-1}$, $A_0 = 0$ and $q \geq 2$ is an integer. We know from [5] that for such coefficient functions and for any $c \in \mathbb{C}\backslash\{0\}$, there exists a C^∞ function x such that x satisfies the equation (1.1) for all $t \in \mathbb{R}$ and $\lim_{t\to-\infty} x(t) = c$. Is this solution analytic at any $t_0 \in \mathbb{R}$? We conjecture that the answer is negative. We can prove that x is of Gevrey class.

Gevrey classes are intermediate spaces between the spaces of C^∞ functions and real analytic functions. Let J be a nonempty, open subset of $\mathbb{R}$. Let $q > 1$. We say that $x\colon J \to \mathbb{C}$ is of Gevrey class q in J if for each compact set $K \subset J$, there exists a constant C_K such that

$$\left|x^{(n)}(t)\right| \leq C_K^{n+1}(n!)^q$$

for all $t \in K$ and for all nonnegative integer n [1]. In this work $J = \mathbb{R}$.

Gevrey classes play a prominent role in the theory of partial differential equations; but, to the best of our knowledge, they have not previously been studied in connection with differential-delay equations. Our results below suggest that further work in this direction may be appropriate.

Theorem 1.1. *Let*

$$p(t) = \sum_{m \in F} A_m e^{im\omega t^q}, \qquad t \in \mathbb{R},$$

where F is a finite set of integers, $A_m \in \mathbb{C}$ for $m \in F$, $\omega > 0$, $i = \sqrt{-1}$, $A_0 = 0$ and $q \geq 2$ is an integer. Suppose that $x\colon \mathbb{R} \to \mathbb{C}$ satisfies (1.1) for each $t \in \mathbb{R}$, and x has a nonzero limit as $t \to -\infty$. Then x is of Gevrey class q in $\mathbb{R}$.

The question of analyticity is even more interesting for delay equations with time-dependent or state-dependent delays. Mallet-Paret and Nussbaum have constructed a time-dependent delay equation in [3] such that a given solution is analytic at certain points of its domain and nonanalytic at others. Krisztin has shown analyticity for a particular class of equations with state-dependent delay in [2]. As far as we know, this is the only positive result in the state-dependent delay case.

We close the paper by showing that in general we cannot expect the nonanalytic solutions of analytic equations to admit Gevrey regularity. We consider the linear inhomogeneous equation

$$x'(t) = a(t)x(t) + b(t)x(\eta(t)) + h(t) \tag{1.2}$$

from [3], where a, b, h and η are analytic in t in a neighborhood of $t = t_0 \in \mathbb{R}$. We assume that t_0 is an expansive fixed point of η, i.e., $\eta(t_0) = t_0$ and $|\eta'(t_0)| > 1$. Let $x_0 \in \mathbb{R}$ be given. The paper [3] gives a mild technical condition under which equation (1.2) with initial value $x(t_0) = x_0$ has no analytic solution in any open neighborhood of $t = t_0$. Using the results of [3], we easily show at the end of this paper that such solutions are not of Gevrey class q for any $q > 1$ either.

2 The proof of Theorem 1.1

The proof of the theorem relies on two lemmas and estimates on the derivatives of the coefficient function p.

Recall that by the product rule,

$$(f_1(t)f_2(t))^{(n)} = \sum_{i=0}^{n} \binom{n}{i} f_1^{(n-i)}(t) f_2^{(i)}(t)$$

for all $n \in \mathbb{N} = \{0,1,2,\dots\}$, $f_1 \in C^n(\mathbb{R},\mathbb{C})$, $f_2 \in C^n(\mathbb{R},\mathbb{C})$ and $t \in \mathbb{R}$. Hence for any solution $x\colon \mathbb{R} \to \mathbb{C}$ of equation (1.1), $t \in \mathbb{R}$ and $n \in \mathbb{N}$ with $n \geq 1$,

$$x^{(n)}(t) = \sum_{i=0}^{n-1} \binom{n-1}{i} p^{(n-1-i)}(t) x^{(i)}(t-1). \tag{2.1}$$

We use this observation to express $x^{(n)}(t)$, $n \geq 1$, $t \in \mathbb{R}$, as a function of the values $x(t-k)$, $k \in \{1,\dots,n\}$, and the derivatives of p at $t-l$, where $l \in \{0,\dots,n-1\}$.

For all $n \geq 1$ and $1 \leq k \leq n$, let $\sum_{(n,k)}$ denote the sum taken over the elements of the set

$$S_{n,k} = \Big\{(j_0, j_1, \dots, j_k) \in \mathbb{N}^{k+1} : n = j_0 > j_1 > \cdots > j_k = 0\Big\}. \tag{2.2}$$

Lemma 2.1. *Assume that $x\colon \mathbb{R} \to \mathbb{C}$ satisfies equation (1.1) on $\mathbb{R}$. Then for all $t \in \mathbb{R}$ and $n \geq 1$,*

$$x^{(n)}(t) = \sum_{k=1}^{n} q_{n,k}(t) x(t-k),$$

where

$$q_{n,k}(t) = n! \sum\nolimits_{(n,k)} \prod_{l=0}^{k-1} \frac{p^{(j_l - 1 - j_{l+1})}(t-l)}{j_l (j_l - 1 - j_{l+1})!} \tag{2.3}$$

for all $t \in \mathbb{R}$, $n \geq 1$ and $1 \leq k \leq n$.

Note that by Lemma 2.1,

$$q_{n,1}(t) = p^{(n-1)}(t) \quad \text{and} \quad q_{n,n}(t) = \prod_{l=0}^{n-1} p(t-l) \quad \text{for } t \in \mathbb{R} \text{ and } n \geq 1.$$

Proof. It is clear that $x^{(n)}(t)$ exists for all $t \in \mathbb{R}$ and $n \geq 1$.

The proof goes by induction on n. By definition, $q_{11}(t) = p(t)$ for all real t, hence the assertion holds for all $t \in \mathbb{R}$ and $n = 1$. Let $n \geq 2$ and suppose the lemma holds for all $t \in \mathbb{R}$ and $i \in \mathbb{N}$ with $1 \leq i < n$. Then applying (2.1) and our induction hypothesis, we deduce that

$$\begin{aligned}
x^{(n)}(t) &= \sum_{i=0}^{n-1} \binom{n-1}{i} p^{(n-1-i)}(t) x^{(i)}(t-1) \\
&= p^{(n-1)}(t)x(t-1) + \sum_{i=1}^{n-1} \binom{n-1}{i} p^{(n-1-i)}(t) \sum_{k=1}^{i} q_{i,k}(t-1)x(t-1-k) \\
&= p^{(n-1)}(t)x(t-1) + \sum_{i=1}^{n-1} \binom{n-1}{i} p^{(n-1-i)}(t) \sum_{k'=2}^{i+1} q_{i,k'-1}(t-1)x(t-k') \\
&= p^{(n-1)}(t)x(t-1) + \sum_{k=2}^{n} \sum_{i=k-1}^{n-1} \binom{n-1}{i} p^{(n-1-i)}(t) q_{i,k-1}(t-1)x(t-k).
\end{aligned}$$

Let

$$\hat{q}_{n,k}(t) = \sum_{i=k-1}^{n-1} \binom{n-1}{i} p^{(n-1-i)}(t) q_{i,k-1}(t-1), \qquad t \in \mathbb{R},\ k \in \{2,\dots,n\}. \tag{2.4}$$

As $p^{(n-1)} \equiv q_{n,1}$ by definition, we need to show that $\hat{q}_{n,k} \equiv q_{n,k}$ for all $k \in \{2,\dots,n\}$.

Formula (2.3) and the substitutions $j'_l = j_{l-1}$, $l \in \{1,\dots,k\}$, give that

$$q_{i,k-1}(t-1) = i! \sum_{(j'_1,\dots,j'_k)\in S_{i,k-1}} \prod_{l=1}^{k-1} \frac{p^{(j'_l-1-j'_{l+1})}(t-l)}{j'_l(j'_l-1-j'_{l+1})!}.$$

Recall that $j'_1 = i$ in the above expression. Substituting i for j'_1 in (2.4), we see that

$$\hat{q}_{n,k}(t) = n! \sum_{j'_1=k-1}^{n-1} \frac{p^{(n-1-j'_1)}(t)}{n(n-1-j'_1)!} \sum_{(j'_1,\dots,j'_k)\in S_{i,k-1}} \prod_{l=1}^{k-1} \frac{p^{(j'_l-1-j'_{l+1})}(t-l)}{j'_l(j'_l-1-j'_{l+1})!}. \tag{2.5}$$

It is clear that $(n, j'_1,\dots,j'_k) \in S_{n,k}$ if and only if

$$k-1 \le j'_1 \le n-1 \quad \text{and} \quad (j'_1,\dots,j'_k) \in S_{i,k-1}.$$

Writing j_l instead of j'_l, this means that

$$\hat{q}_{n,k}(t) = n! \sum\nolimits_{(n,k)} \prod_{l=0}^{k-1} \frac{p^{(j_l-1-j_{l+1})}(t-l)}{j_l(j_l-1-j_{l+1})!} = q_{n,k}(t)$$

for all $k \in \{2,\dots,n\}$ and $t \in \mathbb{R}$, and the proof is complete. □

We obtain the following as a consequence.

Lemma 2.2. *Suppose there exist $q > 1$, $C \ge 1$ and $t_0 \in \mathbb{R}$ such that*

$$\left|p^{(n)}(t)\right| \le C^{n+1}(\max(|t|,1))^{(q-1)n} n! \quad \text{for } t \le t_0 \text{ and } n \in \mathbb{N}. \tag{2.6}$$

Let $x\colon \mathbb{R} \to \mathbb{C}$ be a solution of equation (1.1) on $\mathbb{R}$ such that $|x(t)| \le M$ for all $t \le t_0$. Then

$$\left|x^{(n)}(t)\right| \le M(2C)^n(|t|+n)^{(q-1)n} n! \quad \text{for all } t \le t_0 \text{ and } n \in \mathbb{N}.$$

Proof. By assumption we have $|x(t)| \le M(2C)^0(\max(|t|,1))^{(q-1)0}0!$ for all $t \le t_0$.

Fix $n \ge 1$ and $t \le t_0$. According to Lemma 2.1,

$$x^{(n)}(t) = \sum_{k=1}^{n} q_{n,k}(t) x(t-k),$$

where the coefficient functions $q_{n,k}$, $k \in \{1,\dots,n\}$, are defined by (2.2) and (2.3). The estimate (2.6) implies that

$$|q_{n,k}(t)| \le n! \sum\nolimits_{(n,k)} \prod_{l=0}^{k-1} \frac{C^{j_l-j_{l+1}}(\max(|t-l|,1))^{(q-1)(j_l-1-j_{l+1})}}{j_l}$$

for all $1 \le k \le n$. Notice that

$$\max(|t-l|,1) \le |t|+k \quad \text{for any } k \ge 1 \text{ and } 0 \le l \le k-1.$$

Observe that

$$|S_{n,k}| = \binom{n-1}{k-1} \quad \text{for all } 1 \le k \le n,$$

moreover,

$$\sum_{l=0}^{k-1} j_l - 1 - j_{l+1} = j_0 - j_k - k = n - k \quad \text{and} \quad \prod_{l=0}^{k-1} j_l \ge n(k-1)!$$

hold for all $1 \le k \le n$ and $(j_0, j_1, \dots, j_k) \in S_{n,k}$.

Hence

$$\begin{aligned} |q_{n,k}(t)| &\le (n-1)! \sum_{(n,k)} \frac{1}{(k-1)!} C^n (|t|+k)^{(q-1)(n-k)} \\ &= (n-1)! |S_{n,k}| \frac{1}{(k-1)!} C^n (|t|+k)^{(q-1)(n-k)} \\ &= (n-1)! \binom{n-1}{k-1} \frac{1}{(k-1)!} C^n (|t|+k)^{(q-1)(n-k)} \end{aligned}$$

for all $1 \le k \le n$, and

$$\left|x^{(n)}(t)\right| \le \sum_{k=1}^{n} |q_{n,k}(t)||x(t-k)| \le MC^n (n-1)! \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{(|t|+k)^{(q-1)(n-k)}}{(k-1)!}. \tag{2.7}$$

If we note that

$$\binom{n-1}{k-1} \frac{1}{(k-1)!} \le \binom{n-1}{k-1} \le 2^{n-1}$$

and $(|t|+k)^{(q-1)(n-k)} \le (|t|+n)^{(q-1)n}$ for $1 \le k \le n$, we obtain from (2.7) that

$$\left|x^{(n)}(t)\right| \le MC^n (n-1)! n (2^{n-1}) (|t|+n)^{(q-1)n} \le M(2C)^n n! (|t|+n)^{(q-1)n}.$$

□

Remark 2.3. One might hope that by a more careful exploitation of inequality (2.7), one could improve the estimate in Lemma 2.2 for $x^{(n)}(t)$, but this does not seem to be true. Let $\varepsilon \in (0, 1/2)$. If n is large enough, one can give a lower estimate for the sum

$$MC^n (n-1)! \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{(|t|+k)^{(q-1)(n-k)}}{(k-1)!} \tag{2.8}$$

by considering only the k^{th} term

$$MC^n (n-1)! \binom{n-1}{k-1} \frac{(|t|+k)^{(q-1)(n-k)}}{(k-1)!} = MC^n n! \frac{k^2}{n^2} \frac{n!}{(k!)^2 (n-k)!} (|t|+k)^{(q-1)(n-k)}, \tag{2.9}$$

where $1 \le (\varepsilon/2)n \le k \le \varepsilon n$. Recall that if m is a positive integer, Stirling's formula asserts that there is a real number $\lambda(m) \in (0,1)$ such that

$$m! = \sqrt{2\pi m} \left(\frac{m}{e}\right)^m e^{\frac{\lambda(m)}{12m}}.$$

Using this, we see that

$$\frac{n!}{(k!)^2 (n-k)!} = \frac{\sqrt{n}}{2\pi k \sqrt{n-k}} \frac{n^n}{k^{2k} (n-k)^{n-k}} \exp\left(k + \frac{\lambda(n)}{12n} - \frac{\lambda(k)}{6k} - \frac{\lambda(n-k)}{12(n-k)}\right).$$

As $(\varepsilon/2)n \le k \le \varepsilon n$ and $\lambda(m) \in (0,1)$ for all $m \ge 1$, there exist a constant $C^* > 0$ independent of n and k such that

$$\frac{n!}{(k!)^2(n-k)!} \ge C^* \frac{\sqrt{n}}{\varepsilon n\sqrt{n(1-\varepsilon/2)}} \frac{n^n}{(\varepsilon n)^{2\varepsilon n}(n(1-\varepsilon/2))^{n(1-\varepsilon/2)}}$$
$$= C^* \frac{1}{n^{1+3\varepsilon n/2}} \frac{1}{\varepsilon^{1+2\varepsilon n}} \Big(1-\frac{\varepsilon}{2}\Big)^{-1/2-n(1-\varepsilon/2)}.$$

In addition,

$$\frac{k^2}{n^2} \ge \Big(\frac{\varepsilon}{2}\Big)^2 \quad \text{and} \quad (|t|+k)^{(q-1)(n-k)} \ge \Big(\frac{\varepsilon}{2}\Big)^{(q-1)(1-\varepsilon/2)n} n^{(q-1)(1-\varepsilon)n}.$$

We conclude that there are constants $C_1 > 0$ and $C_2 > 0$ independent of n such that the expression (2.9) (and thus (2.8)) is not smaller than

$$MC^n n! C_1 (C_2)^n n^{(q-1)(1-\varepsilon)n-1-3\varepsilon n/2}$$

for each $n \in \mathbb{N}$ and $t \le t_0$.

Consider the case when $p(t) = e^{it^q}$ for all real t with an integer $q \ge 2$. Our next objective is to give a formula for $p^{(n)}(t)$ for each $n \in \mathbb{N}$ and $t \in \mathbb{R}$.

For each $u \in \mathbb{C}$,

$$u^q - t^q = \prod_{k=0}^{q-1}(u-\eta_k t), \quad \text{where } \eta_k = e^{\frac{2\pi i k}{q}},\ k \in \{0,1,\dots,q-1\} \text{ and } i = \sqrt{-1}.$$

It follows that

$$e^{i(u^q-t^q)} = \sum_{j=0}^{\infty} \frac{(i(u^q-t^q))^j}{j!} = \sum_{j=0}^{\infty} \frac{i^j}{j!} \prod_{k=0}^{q-1}(u-\eta_k t)^j. \tag{2.10}$$

For each $j \ge 0$, define a set $R_{n,q,j}$ of q-tuples as

$$R_{n,q,j} = \Big\{ (l_0, l_1, \dots, l_{q-1}) \in \mathbb{N}^q : \sum_{k=0}^{q-1} l_k = n,\ l_0 = j,\ 0 \le l_k \le j \text{ for } 1 \le k \le q-1 \Big\}.$$

Let $\sum^{(n,q,j)}$ denote the sum taken over the elements of $R_{n,q,j}$. Let D_t^n denote the n-fold differentiation with respect to t.

Note that $\eta_0 = 1$ and $\eta_k \ne 1$ if $1 \le k \le q-1$. This observation and the product rule for higher order derivatives together give that

$$D_u^n \prod_{k=0}^{q-1}(u-\eta_k t)^j|_{u=t} = \sum^{(n,q,j)} \frac{n!}{l_0! l_1! \dots l_{q-1}!} \prod_{k=0}^{q-1} \frac{j!}{(j-l_k)!}(t-\eta_k t)^{j-l_k}$$
$$= n! t^{qj-n} \sum^{(n,q,j)} \prod_{k=0}^{q-1} \binom{j}{l_k} (1-\eta_k)^{j-l_k}.$$

As $l_k \le j$ for all $0 \le k \le q-1$, we see that $n \le qj$. The above sum is nonempty if and only if

$$\frac{n}{q} \le j \le n.$$

Substituting into equation (2.10), we deduce that

$$\begin{aligned} D_u^n e^{iu^q}|_{u=t} &= e^{it^q} D_u^n e^{i(u^q-t^q)}|_{u=t} \\ &= e^{it^q} \sum_{\frac{n}{q}\le j\le n} \frac{i^j}{j!} n! t^{qj-n} \sum\nolimits^{(n,q,j)} \prod_{k=0}^{q-1} \binom{j}{l_k} (1-\eta_k)^{j-l_k}. \end{aligned}$$

Actually we eventually shall need a formula for $D_t^n e^{i\alpha t^q}$, where $\alpha \in \mathbb{R}$ is a constant. However, such a formula follows easily from the above formula for $D_t^n e^{it^q}$. Select $\beta \in \mathbb{C}$ such that $\beta^q = \alpha$ and write $u = \beta t$. Then

$$D_t^n e^{i(\beta t)^q} = \beta^n D_u^n e^{iu^q}|_{u=\beta t}.$$

By the above equation for $D_u^n e^{iu^q}$,

$$\begin{aligned} D_t^n e^{i\alpha t^q} &= \beta^n D_u^n e^{iu^q}|_{u=\beta t} \\ &= e^{i\alpha t^q} \sum_{\frac{n}{q}\le j\le n} \frac{(i\alpha)^j}{j!} n! t^{qj-n} \sum\nolimits^{(n,q,j)} \prod_{k=0}^{q-1} \binom{j}{l_k} (1-\eta_k)^{j-l_k}. \end{aligned}$$

Next we obtain upper estimates for $\left|D_t^n e^{i\alpha t^q}\right|$ for each $t \in \mathbb{R}$ and $n \in \mathbb{N}$ when $\alpha \in \mathbb{R}$ and $q \in \mathbb{N}$ with $q \ge 2$.

Assume that $n/q \le j \le n$. If $|t| \ge 1$, then $|t|^{qj-n} \le |t|^{qn-n} = |t|^{(q-1)n}$. If $|t| \le 1$, then $|t|^{qj-n} \le 1$. Thus for all $t \in \mathbb{R}$, we have

$$|t|^{qj-n} \le (\max(|t|, 1))^{(q-1)n}.$$

Since

$$\binom{j}{l_k} \le 2^j \quad \text{for } 1 \le k \le q-1 \quad \text{and} \quad \binom{j}{l_0} = 1,$$

we see that

$$\prod_{k=0}^{q-1} \binom{j}{l_k} \le 2^{(q-1)j}.$$

As $|1-\eta_k| \le 2$ and $0 \le l_k \le j$,

$$|1-\eta_k|^{j-l_k} \le 2^{j-l_k} \qquad \text{and} \qquad \left|\prod_{k=0}^{q-1} (1-\eta_k)^{j-l_k}\right| \le 2^{qj-n},$$

where we have used that $\sum_{k=0}^{q-1} l_k = n$. It follows that

$$\left|\prod_{k=0}^{q-1} \binom{j}{l_k} (1-\eta_k)^{j-l_k}\right| \le 2^{(q-1)j} 2^{qj-n} = 2^{(2q-1)j-n}.$$

It is an elementary combinatorial result that the number of ordered $(q-1)$-tuples of non-negative integers $(l_1, \ldots, l_{q-1})$ such that $\sum_{k=1}^{q-1} l_k = n - j$ is

$$\binom{n-j+q-2}{q-2}.$$

This estimate does not take into account that $l_k \leq j$ for $1 \leq k \leq q-1$. It follows that for $n/q \leq j \leq n$,

$$\left|\sum\nolimits^{(n,q,j)} \prod_{k=0}^{q-1} \binom{j}{l_k} (1-\eta_k)^{j-l_k}\right| \leq \binom{n-j+q-2}{q-2} 2^{(2q-1)j-n} \leq 2^{n-j+q-2} 2^{(2q-1)j-n} = 2^{2(q-1)j+q-2}.$$

Our estimates imply that

$$\left|D_t^n e^{i\alpha t^q}\right| \leq \sum_{\frac{n}{q}\leq j\leq n} |\alpha|^j \frac{n!}{j!(n-j)!}(n-j)!(\max(|t|,1))^{(q-1)n} 2^{2(q-1)j+q-2}$$
$$\leq 2^{q-2}(\max(|t|,1))^{(q-1)n}(n-j_*)! \sum_{\frac{n}{q}\leq j\leq n} \binom{n}{j} \left(|\alpha| 2^{2(q-1)}\right)^j,$$

where j_* denotes the smallest positive integer j such that $n/q \leq j \leq n$. By the binomial theorem,

$$\sum_{\frac{n}{q}\leq j\leq n} \binom{n}{j} \left(|\alpha| 2^{2(q-1)}\right)^j \leq \sum_{0\leq j\leq n} \binom{n}{j} \left(|\alpha| 2^{2(q-1)}\right)^j = \left(1+|\alpha| 2^{2(q-1)}\right)^n,$$

so

$$\left|D_t^n e^{i\alpha t^q}\right| \leq 2^{q-2}(\max(|t|,1))^{(q-1)n}\left(1+|\alpha| 4^{(q-1)}\right)^n (n-j_*)!. \tag{2.11}$$

We conclude that there exists a constant $C \geq 1$ such that for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\left|D_t^n e^{i\alpha t^q}\right| \leq C^{n+1}(\max(|t|,1))^{(q-1)n}(n-j_*)! \leq C^{n+1}(\max(|t|,1))^{(q-1)n} n!.$$

As a consequence we can verify the theorem.

Proof of Theorem 1.1. Let $p(t) = \sum_{m\in F} A_m e^{im\omega t^q}$ for all $t \in \mathbb{R}$, where F is a finite set of integers, $A_m \in \mathbb{C}$ for $m \in F$, $\omega > 0$, $A_0 = 0$ and $q \geq 2$ is an integer. Our calculations above show that p satisfies inequality (2.6) in Lemma 2.2. The boundedness of x on intervals of the form $(-\infty, t_0]$ is clear because $\lim_{t\to-\infty} x(t)$ exists and is finite. If one applies Lemma 2.2 and uses Stirling's formula, the theorem follows. □

Remark 2.4. In fact, with the aid of the advanced calculus form of Stirling's formula, one can replace $(n-j_*)!$ in (2.11) with $(n!)^{1-1/q}$.

It is obvious that $(n-j_*)! = 1$ for $n = 1, 2$ because $q \geq 2$. Thus we can assume that $n \geq 3$.

Note that $n/q \leq j_* < n/q + 1$, so if we choose $q_* > 0$ such that $j_* = n/q_*$, then

$$\frac{1}{q} \leq \frac{1}{q_*} < \frac{1}{q} + \frac{1}{n} \quad \text{and} \quad n - \frac{n}{q} \geq n - \frac{n}{q_*} > n - \frac{n}{q} - 1.$$

Also, since $n \geq 3$ and $q \geq 2$, it is true that $n - n/q > 1$.

By Stirling's formula,

$$(n-j_*)! = \frac{\left(n\left(1-\frac{1}{q_*}\right)\right)^{n\left(1-\frac{1}{q_*}\right)}}{e^{n\left(1-\frac{1}{q_*}\right)}} \sqrt{2\pi n\left(1-\frac{1}{q_*}\right)} \exp\left(\frac{\lambda\left(n\left(1-\frac{1}{q_*}\right)\right)}{12n\left(1-\frac{1}{q_*}\right)}\right).$$

Since $n(1-1/q) > 1$ and $n(1-1/q) \geq n(1-1/q_*)$,

$$\left(n\left(1-\frac{1}{q_*}\right)\right)^{n\left(1-\frac{1}{q_*}\right)}\sqrt{2\pi n\left(1-\frac{1}{q_*}\right)} \leq \left(n\left(1-\frac{1}{q}\right)\right)^{n\left(1-\frac{1}{q}\right)}\sqrt{2\pi n\left(1-\frac{1}{q}\right)}.$$

Since

$$e^{n\left(1-\frac{1}{q_*}\right)} > e^{n\left(1-\frac{1}{q}\right)-1},$$

we conclude that

$$(n-j_*)! \leq e\left(\frac{n}{e}\right)^{n\left(1-\frac{1}{q}\right)}\left[\left(\frac{q-1}{q}\right)^{\left(\frac{q-1}{q}\right)}\right]^n\sqrt{2\pi n\left(1-\frac{1}{q}\right)}\exp\left(\frac{\lambda\left(n\left(1-\frac{1}{q_*}\right)\right)}{12n\left(1-\frac{1}{q_*}\right)}\right).$$

Stirling's formula for $n!$ gives that

$$\left(\frac{n}{e}\right)^{n\left(1-\frac{1}{q}\right)} = (n!)^{1-\frac{1}{q}}(2\pi n)^{-\frac{1}{2}\left(1-\frac{1}{q}\right)}\exp\left(\frac{-\lambda(n)\left(1-\frac{1}{q}\right)}{12n}\right).$$

Substituting for $(n/e)^{n\left(1-\frac{1}{q}\right)}$ gives that

$$(n-j_*)! \leq \left[\left(\frac{q-1}{q}\right)^{\left(\frac{q-1}{q}\right)}\right]^n(n!)^{1-\frac{1}{q}}(2\pi n)^{\frac{1}{2q}}\exp\left(1+\frac{\lambda\left(n\left(1-\frac{1}{q_*}\right)\right)}{12n\left(1-\frac{1}{q_*}\right)}\right).$$

It follows, using our previous estimates for $\left|D_t^n e^{i\alpha t^q}\right|$, that there exists a constant $C_* \geq 1$ such that for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\left|D_t^n e^{i\alpha t^q}\right| \leq C_*^{n+1}(\max(|t|,1))^{(q-1)n}(n!)^{1-\frac{1}{q}}.$$

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