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Existence and controllability for stochastic evolution inclusions of Clarke's subdifferential type

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Abstract. In this paper, we investigate a class of stochastic evolution inclusions of Clarke's subdifferential type in Hilbert spaces. The existence of mild solutions and controllability results are given and proved by using stochastic analysis techniques, semigroup of operators theory, a fixed point theorem of multivalued maps and properties of generalized Clarke subdifferential. An example is included to illustrate the applicability of the main results.

Keywords: existence of mild solution, controllability, stochastic evolution equations, generalized Clarke subdifferential.

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1 Introduction

It is well known that controllability plays a significant role in the concept of control theory and engineering. Currently, fruitful achievements have been obtained on controllability of stochastic systems and inclusion problems, see e.g. Bashirov and Mahmudov [1], Mahmudov [20], Obukhovski and Zecca [24] and Rykaczewski [27] and the references therein. In addition, the controllability problems for stochastic differential equations have become a field of increasing interest due to its applications in economics, ecology and finance. More precisely, Klamka [6–10] studied stochastic controllability systems with different kind of delays. Lin and Hu [11] considered the existence results of stochastic inclusions with nonlocal initial conditions. Sakthivel et al. [29,30] obtained the approximate controllability of semilinear fractional differential systems in Hilbert spaces. Ren et al. [26] studied the controllability of impulsive neutral stochastic differential inclusions with infinite delay.

Recently, many researchers have paid increasingly attention to the evolution inclusions with Clarke's subdifferential type which have have been studied in many papers, we refer the readers to [12–18,22,23,32,33] and the references therein. In fact, Clarke's subdifferential has important applications in mechanics and engineering, especially in nonsmooth analysis and

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optimization [2,23]. At present, although some significant results have been obtained for the solvability and control problems of evolution inclusions of generalized Clarke subdifferential, it seems that there are still many interesting ideas and unanswered questions. However, the study of the controllability of the systems described by stochastic evolution inclusions of generalized Clarke subdifferential in Hilbert spaces has not been investigated yet and the investigation on this topic has not been appreciated well enough.

Motivated by the above consideration, we will study the existence of mild solutions and controllability of the following stochastic evolution inclusions of generalized Clarke's subdifferential type with nonlocal initial conditions:

$$\begin{cases} dx(t) \in (Ax(t) + Bu(t))dt + \sigma(t, x(t))\, dw(t) + \partial F(t, x(t))\, dt, \quad t \in J = [0, b], \\ x(0) = x_0 + g(x), \end{cases} \tag{1.1}$$

where $x(\cdot)$ takes the value in the separable Hilbert space H, $A\colon D(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup $T(t)$ $(t \geq 0)$ on H. The control function $u(\cdot)$ takes values in a separable Hilbert space U and B is a bounded linear operator from U into H. The notation ∂F stands for the generalized Clarke subdifferential (cf. [2]) of a locally Lipschitz function $F(t, \cdot)\colon H \to R$; σ and g are given appropriate functions to be specified later; w is a Q-Wiener process on a complete probability space $(\Omega, \Gamma, \mathbb{P})$ and x_0 is Γ_0 measurable H-valued random variable independent of w. If the operator A is monotone, there are a lot of results in this direction (cf. [31]).

The rest of this paper is organized as follows. In Section 2, we will recall some useful preliminary facts. In Section 3, the existence of mild solutions of the system (1.1) is established and proved by applying stochastic analysis techniques, semigroup of operators theory, a fixed point theorem of multivalued maps and properties of generalized Clarke subdifferential. In Section 4, the controllability of the system (1.1) is formulated and proved mainly by using a fixed point technique. Finally, an example is given to illustrate our main results in Section 5.

2 Preliminaries

Let $(\Omega, \Gamma, \{\Gamma_t,\ t \geq 0\}, \mathbb{P})$ be a complete probability space equipped with a normal filtration $\{\Gamma_t,\ t \geq 0\}$ satisfying that Γ_0 contains all $\mathbb{P}$-null sets of Γ. $E(\cdot)$ denotes the expectation of a random variable or the Lebesgue integral with respect to the probability measure $\mathbb{P}$. Let H, U be separable Hilbert spaces and $\{w(t),\ t \geq 0\}$ be a Wiener process with the linear bounded covariance operator Q such that $\operatorname{tr} Q < \infty$.

We assume that there exist a complete orthonormal system $\{e_k\}_{k\geq 1}$ in H, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k$ $(k = 1, 2, \dots)$ and a sequence of independent Brownian motions $\{\beta_k\}_{k\geq 1}$ such that

$$\langle w(t), e\rangle = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\langle e_k, e\rangle \beta_k(t), \qquad e \in H, \qquad t \geq 0$$

and $\Gamma_t = \Gamma_t^w$, where Γ_t^w is the σ-algebra generated by $\{w(s) : 0 \leq s \leq t\}$. Let $L_0^2 = L^2(Q^{\frac{1}{2}}H, H)$ be a space of all Hilbert–Schmidt operators from $Q^{\frac{1}{2}}H$ to H with the inner product $\langle \phi, \varphi\rangle_{L_0^2} = \operatorname{tr}[\phi Q \varphi^*]$, $L^2(\Gamma_b, H)$ be a Banach space of all Γ_b-measurable square integrable random variables with values in the Hilbert space H. $L^2(\Gamma, H) = L^2(\Omega, \Gamma, \mathbb{P}, H)$ denotes a Hilbert space of strongly Γ-measurable, H valued random variables x satisfying $E\|x\|_H^2 < \infty$. Since for each

$t \geq 0$ the sub-σ-algebra Γ_t is complete, $L^2(\Gamma_t, H)$ is a closed subspace of $L^2(\Gamma, H)$, and hence $L^2(\Gamma_t, H)$ is a Hilbert space. $C(J, L^2(\Gamma, H))$ denotes the Banach space of all mean square continuous maps x from J into $L^2(\Gamma, H)$ with the norm $\|x\| = (\sup_{t\in J} E\|x(t)\|^2)^{\frac{1}{2}} < \infty$. $L^2_\Gamma(J, H)$ will denote the Hilbert space of all Γ_t-adapted measurable random processes defined on $[0, b]$ with values in H and the norm $\|x\|_{L^2_\Gamma(J,H)} = \left(E\int_0^b \|x(t)\|^2_H dt\right)^{1/2} < \infty$. The space $L^2_\Gamma(J, U)$ has the same definition as $L^2_\Gamma(J, H)$ with the norm $\|u\|_{L^2_\Gamma(J,U)} = \left(\int_0^b E\|u(t)\|^2_U\right)^{1/2}$. For details, we refer the reader to [3,28] and references therein.

Next, we introduce some basic definitions on multivalued maps, for more details, please refer to the books [4,5].

For a Banach space X with the norm $\|\cdot\|$, X^* denotes its dual and $\langle\cdot,\cdot\rangle$ the duality pairing of X and X^*. For convenience, we use the following notations:

$$P_{f(c)}(X) = \{\Omega \subseteq X : \Omega \text{ is nonempty, closed (convex)}\},$$
$$P_{(w)k(c)}(X) = \{\Omega \subseteq X : \Omega \text{ is nonempty, (weakly) compact (convex)}\}.$$

Definition 2.1. Given a Banach space X and a multivalued map $G\colon X \to 2^X \setminus \emptyset = P(X)$, we say

(i) G is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$.

(ii) G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e. $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

(iii) G is upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set U of X containing $G(x_0)$, there exists an open neighborhood V of x_0 such that $G(V) \subseteq U$.

(iv) G is completely continuous if $G(B)$ is relatively compact for every bounded subset $B \in P(X)$.

(v) G has a fixed point if there is a $x \in X$ such that $x \in G(x)$.

Now, recall the definition of the generalized gradient of Clarke for a locally Lipschitzian functional $F\colon X \to R$. From [2], we denote by $F^0(x; v)$ the Clarke generalized directional derivative of F at x in the direction v, that is

$$F^0(x; v) = \limsup_{x'\to x\ \lambda\to 0^+} \frac{F(x' + \lambda v) - F(x')}{\lambda}$$

and we denote by ∂F, which is a subset of X^* given by

$$\partial F(x) = \{x^* \in X^* : F^0(x; v) \geq \langle x^*, v\rangle, \text{ for all } v \in X\}$$

the generalized gradient of F at x (the Clarke subdifferential).

The following basic properties play important roles in our main results.

Lemma 2.2 (Proposition 3.23 of [2]). *If $F\colon \Omega \to R$ is a locally Lipschitz function on an open set Ω of X, then*

(i) for every $v \in X$, one has $F^0(x; v) = \max\{\langle x^*, v\rangle : \text{ for all } x^* \in \partial F(x)\}$;

(ii) for every $x \in \Omega$, the gradient $\partial F(x)$ is a nonempty, convex, weak *-compact subset of X^* and $\|x^*\|_{X^*} \leq \Lambda$ for any $x^* \in \partial F(x)$ (where $\Lambda > 0$ is the Lipschitz constant of F near x);

(iii) the graph of the generalized gradient ∂F is closed in $\Omega \times X^*_{w^*}$ topology, i.e., if $\{x_n\} \subset \Omega$ and $\{x_n^*\} \subset X$ are sequences such that $x_n^* \in \partial F(x_n)$ and $x_n \to x$ in X, $x_n^* \to x^*$ weakly* in X^*, then $x^* \in \partial F(x)$ (where $X^*_{w^*}$ denotes the Banach space X^* furnished with the w^*-topology);

(iv) the multifunction $\Omega \ni x \to \partial F(x) \subseteq X^*$ is u.s.c. from Ω into $X^*_{w^*}$.

Lemma 2.3 (Proposition 3.44 of [23]). *Let X be a separable reflexive Banach space, $0 < b < \infty$ and $h\colon (0,b) \times X \to R$ be a function such that $h(\cdot, x)$ is measurable for all $x \in X$ and $h(t,\cdot)$ is locally Lipschitz on X for all $t \in (0,b)$. Then the multifunction $(0,b) \times X \ni (t,x) \to \partial h(t,x) \subset X^*$ is measurable, where ∂h denotes the Clarke generalized gradient of $h(t,\cdot)$.*

Lemma 2.4 (Theorem 2.2.1 of [5]). *If (Ω, Σ) is a measurable space, X is a Polish space (i.e., separable completely metric space) and $F\colon \Omega \to \mathcal{P}_f(X)$ is measurable, then $F(\cdot)$ admits a measurable selection (i.e., there exists $f\colon \Omega \to X$ measurable such that for every $x \in \Omega$, $f(x) \in F(x)$).*

Lemma 2.5 (Proposition 3.16 of [23]). *Let (Ω, Σ, μ) be a σ-finite measure space, E be a Banach space and $1 \le p < \infty$. If f_n, $f \in L^p(\Omega, E)$, $f_n \to f$ weakly in $L^p(\Omega, E)$ and $f_n(x) \in G(x)$ for μ-a.e. $x \in \Omega$ and all $n \in N$ where $G(x) \in \mathcal{P}_{wk}(E)$ for μ-a.e. $x \in \Omega$, then*

$$f(x) \in \overline{\text{conv}}\Big(w\text{-}\limsup\{f_n(x)\}_{n\in N}\Big) \quad \text{for } \mu\text{-a.e. on } x \in \Omega,$$

where $\overline{\text{conv}}$ denotes the closed convex hull of a set.

At the end of this section, we present the following lemma and fixed point theorem that are the key tools in our main results.

Lemma 2.6 ([3]). *Let $G\colon [0,b] \times \Omega \to L_0^2$ be a strongly measurable mapping such that $\int_0^b E\|G(t)\|^p_{L_0^2} dt < \infty$. Then*

$$E\left\| \int_0^t G(s)\, dw(s) \right\|^p \le L_G \int_0^t E\|G(s)\|^p_{L_0^2}\, ds$$

for all $0 \le t \le b$ and $p \ge 2$, where L_G is the constant involving p and b.

Theorem 2.7 ([19]). *Let X be a locally convex Banach space and $\mathcal{F}\colon X \to 2^X$ be a compact convex valued, u.s.c. multivalued map such that there exists a closed neighborhood V of 0 for which $\mathcal{F}(V)$ is a relatively compact set. If the set*

$$\Omega = \{x \in X : \lambda x \in \mathcal{F}(x) \text{ for some } \lambda > 1\}$$

is bounded, then $\mathcal{F}$ has a fixed point.

3 Existence of mild solutions

In this section, we study the existence of mild solutions for the system (1.1). Firstly, according to the book [25], we may define a mild solution of problem (1.1) as follows.

Definition 3.1. For each $u \in L^2_\Gamma(J, U)$, a Γ_t-adapted stochastic process $x \in C(J, L^2(\Gamma, H))$ is a mild solution of the control system (1.1) if $x(0) = x_0 \in H$ and there exists $f \in L^2_\Gamma(J, H)$ such that $f(t) \in \partial F(t, x(t))$ for a.e. $t \in J$ and

$$x(t) = T(t)(x_0 + g(x)) + \int_0^t T(t-s)[f(s) + Bu(s)]\, ds + \int_0^t T(t-s)\sigma(s, x(s))\, dw(s), \qquad t \in J.$$

In the following, we impose the following hypotheses.

(H1) $A\colon D(A) \subseteq H \to H$ is the infinitesimal generator of a C_0-semigroup $T(t)(t \geq 0)$ and the semigroup $T(t)$ is compact for $t > 0$.

By Theorem 1.2.2 of [25], there exist constants $\varpi \geq 0$ and $\overline{M} \geq 1$ such that

$$\|T(t)\| \leq \overline{M}e^{\varpi t} \leq \overline{M}e^{\varpi b} := M, \qquad \forall t \in J.$$

(H2) $F\colon J \times H \to R$ satisfies the following assumptions:

(i) $F(\cdot, x)$ is measurable for all $x \in H$;

(ii) $F(t, \cdot)$ is locally Lipschitz continuous for a.e. $t \in J$;

(iii) there exist a function $a \in L^1(J, R^+)$ and a constant $c \geq 0$ such that

$$\|\partial F(t,x)\|^2 = \sup\{\|f(t)\|^2 : f(t) \in \partial F(t,x)\} \leq a(t) + c\|x\|^2,$$

for a.e. $t \in J$ and all $x \in H$.

(H3) $\sigma\colon J \times H \to L_0^2$ is continuous in the second variable for a.e. $t \in J$ and there exist a function $\eta \in L^2(J, R^+)$ and a constant $d \geq 0$ such that

$$\|\sigma(t,x)\|^2 \leq \eta(t) + d\|x\|^2.$$

(H4) $g\colon C(J,H) \to H$ is continuous and there exists a constant $e \geq 0$ such that

$$\|g(x)\|^2 \leq e(1 + \|x\|^2).$$

(H5) The linear operator $W\colon L^2_\Gamma(J,U) \to H$, defined by

$$Wu = \int_0^b T(b-s)Bu(s)\,ds$$

has an inverse operator W^{-1} which takes value $L^2_\Gamma(J,H)/\ker W$ and there exist two positive constants $M_1,\ M_2 > 0$ such that

$$\|B\| \leq M_1 \quad \text{and} \quad \|W^{-1}\| \leq M_2.$$

Next, we define an operator $\mathcal{N}\colon L^2_\Gamma(J,H) \to 2^{L^2_\Gamma(J,H)}$ as follows

$$\mathcal{N}(x) = \{f \in L^2_\Gamma(J,H) : f(t) \in \partial F(t, x(t)) \text{ a.e. } t \in J \text{ for } x \in L^2_\Gamma(J,H)\}.$$

To obtain our main results, we also need the following lemmas.

Lemma 3.2. *If the assumption (H2) holds, then for each $x \in L^2_\Gamma(J,H)$, the set $\mathcal{N}(x)$ has nonempty, convex and weakly compact values.*

Proof. The main idea of the proof comes from Lemma 5.3 of [23] and Lemma 2.6 of [16].

Firstly, from Lemma 2.2 (ii), $\partial F(t,x)$ is nonempty, convex and weakly compact in the Hilbert H and ∂F is $P_{wkc}(H)$-valued. Thus $\mathcal{N}(x)$ has convex and weakly compact values.

Next, we will prove that $\mathcal{N}(x)$ is nonempty. Let $x \in L^2_\Gamma(J,H)$, then there exists a sequence $\{\varphi_n\} \subseteq L^2_\Gamma(J,H)$ of simple functions such that

$$\varphi_n(t) \to x(t) \quad \text{in } L^2_\Gamma(J,H) \text{ for a.e. } t \in J. \tag{3.1}$$

From hypotheses (H2) (i)–(ii), and Lemma 2.3, $t \to \partial F(t,\varphi_n(t))$ is measurable from J into $P_{fc}(H)$. By Lemma 2.4, for every $n \geq 1$, there exists a measurable function $\zeta_n\colon J \to H$ such that $\zeta_n(t) \in \partial F(t,\varphi_n(t))$ a.e. $t \in J$. Next, from (H2) (iii), we get

$$\|\zeta_n\|^2_{L^2_\Gamma(J,H)} \leq \|a\|_{L^1(J,R^+)} + c\|\varphi_n\|^2_{L^2_\Gamma(J,H)}.$$

Hence, $\{\zeta_n\}$ remains in a bounded subset of $L^2_\Gamma(J,H)$. Thus, we can suppose that $\zeta_n \to \zeta$ weakly in $L^2_\Gamma(J,H)$ with $\zeta \in L^2_\Gamma(J,H)$. Then from Lemma 2.5,

$$\zeta(t) \in \overline{\text{conv}}\big(w\text{-}\limsup\{\zeta_n(t)\}_{n\geq 1}\big) \quad \text{a.e. } t \in J. \tag{3.2}$$

Moreover, (H2) (iii) and Lemma 2.2 (iv) imply that $x \to \partial F(t,x)$ is u.s.c. Recalling that the graph of an u.s.c. multifunction with closed values is closed (cf. Proposition 3.12 of [23]), we obtain that for a.e. $t \in J$, if $f_n \in \partial F(t,\zeta_n)$, $f_n \in H$, $f_n \to f$ weakly in H, $\zeta_n \in L^2_\Gamma(J,H)$, $\zeta_n \to \zeta$ in $L^2_\Gamma(J,H)$, then $f \in \partial F(t,\zeta)$. Hence by (3.2), we have

$$w\text{-}\limsup \partial F(t,\varphi_n(t)) \subset \partial F(t,x(t)) \quad \text{a.e. } t \in J, \tag{3.3}$$

where the Kuratowski upper limit (cf. Definition 3.14 of [23]) of set $\partial F(t,\varphi_n(t))$ is given by

$$w\text{-}\limsup \partial F(t,\varphi_n(t)) = \{\zeta \in H : \zeta = w\text{-}\lim \zeta_{n_k},\ \zeta_{n_k} \in \partial F(t,\varphi_n(t)),\ n_1 < \cdots < n_k < \cdots\}.$$

Further, by (3.2) and (3.3), we get

$$\begin{aligned}\zeta(t) \in \overline{\text{conv}}(w\text{-}\limsup\{\zeta_n(t)\}_{n\geq 1}) &\subset \overline{\text{conv}}(w\text{-}\limsup \partial F(t,\varphi_n(t)) \\ &\subset \partial F(t,x(t)) \quad \text{a.e. } t \in J.\end{aligned}$$

Since $\zeta \in L^2_\Gamma(J,H)$ and $\zeta(t) \in \partial F(t,x(t))$ a.e. $t \in J$, thus $\zeta \in \mathcal{N}(x)$ which implies that $\mathcal{N}(x)$ is nonempty. The proof is completed. □

Lemma 3.3 (Lemma 11 of [22]). *If (H2) holds, the operator $\mathcal{N}$ satisfies: if $x_n \to x$ in $L^2_\Gamma(J,H)$, $w_n \to w$ weakly in $L^2_\Gamma(J,H)$ and $w_n \in N(x_n)$, then we have $w \in \mathcal{N}(x)$.*

Now, we study the existence of mild solutions for the system (1.1).

Theorem 3.4. *For each $u \in L^2_\Gamma(J,U)$, if the hypotheses (H1)–(H4) are satisfied, then the system (1.1) has a mild solution on J provided that*

$$K = 5M^2[e + b(bc+d)] < 1.$$

Proof. Firstly, for any $x \in C(J,L^2(\Gamma,H)) \subset L^2(J,H)$, from Lemma 3.2, we can consider the multivalued map $\mathcal{F}\colon C(J,L^2(\Gamma,H)) \to 2^{C(J,L^2(\Gamma,H))}$ defined by

$$\mathcal{F}(x) = \left\{ \begin{aligned} h \in C(J,L^2(\Gamma,H)) : h(t) &= T(t)(x_0 + g(x)) \\ &\quad + \int_0^t T(t-s)f(s)\,ds + \int_0^t T(t-s)Bu(s)\,ds \\ &\quad + \int_0^t T(t-s)\sigma(s,x(s))\,dw(s), \qquad f \in \mathcal{N}(x) \end{aligned} \right\}.$$

It is clear that problem (1.1) is reduced to find a fixed point of $\mathcal{F}$. We will show that the operator $\mathcal{F}$ satisfies all the conditions of Theorem 2.7. Next, to complete the proof, we divide the proof into six steps.

Step 1: $\mathcal{F}(x)$ is convex for each $x \in C(J, L^2(\Gamma, H))$.

By Lemma 3.2, $\mathcal{N}(x)$ has convex values. So if $f_1, f_2 \in \mathcal{N}(x)$, then $af_1 + (1-a)f_2 \in \mathcal{N}(x)$ for all $a \in [0,1]$, which implies clearly that $\mathcal{F}(x)$ is convex.

Step 2: The operator $\mathcal{F}$ is bounded on bounded subset of $C(J, L^2(\Gamma, H))$.

For $\forall r > 0$, let $\mathcal{B}_r = \{x \in C(J, L^2(\Gamma, H)) : \|x\|^2 \leqslant r\}$. Obviously, $\mathcal{B}_r$ is a bounded, closed and convex set of $C(J, L^2(\Gamma, H))$. We claim that there exists a positive number ℓ such that for each $\varphi \in \mathcal{F}(x),\ x \in \mathcal{B}_r,\ \|\varphi\|^2 \leq \ell$.

In fact, if $\varphi \in \mathcal{F}(x)$, then there exists a $f \in \mathcal{N}(x)$ such that

$$\begin{aligned}\varphi(t) = {} & T(t)x_0 + T(t)g(x) + \int_0^t T(t-s)f(s)\,ds \\ & + \int_0^t T(t-s)Bu(s)\,ds + \int_0^t T(t-s)\sigma(s,x(s))\,dw(s), \qquad t \in J. \end{aligned} \tag{3.4}$$

From (H1)–(H4), the Hölder inequality and Lemma 2.6, for $t \in J$, we have

$$\begin{aligned} E\|\varphi(t)\|^2 &\leq 5\bigg(E\|T(t)x_0\|^2 + E\|T(t)g(x)\|^2 + b\int_0^t E\|T(t-s)f(s)\|^2\,ds \\ &\quad + b\int_0^t E\|T(t-s)Bu(s)\|^2\,ds + \int_0^t E\|T(t-s)\sigma(s,x(s))\|^2\,dw(s)\bigg) \\ &\leq 5M^2\bigg[E\|x_0\|^2 + e(1+\|x\|^2) + b\int_0^t (a(s) + cE\|x(s)\|^2)\,ds \\ &\quad + b\int_0^t \|B\|^2 E\|u(s)\|_U^2\,ds + \int_0^t (\eta(s) + dE\|x(s)\|^2)\,ds\bigg] \\ &\leq 5M^2\bigg[E\|x_0\|^2 + e(1+r) + b(\|a\|_{L^1(J,R^+)} + bcr) \\ &\quad + bM_1^2\|u\|^2_{L^2_{\mathcal{F}}(J,U)} + \sqrt{b}\|\eta\|_{L^2(J,R^+)} + bdr\bigg] =: \ell. \end{aligned}$$

Thus, $\mathcal{F}(\mathcal{B}_r)$ is bounded in $C(J, L^2(\Gamma, H))$.

Step 3: $\{\mathcal{F}(x) : x \in \mathcal{B}_r\}$ is equicontinuous.

Firstly, for $\forall x \in \mathcal{B}_r,\ \varphi \in \mathcal{F}(x)$, there exists an $f \in \mathcal{N}(x)$ such that (3.4) holds for each $t \in J$. Next, for $0 < \tau_1 < \tau_2 \leq b$, we get

$$\begin{aligned} E\|\varphi(\tau_2) - \varphi(\tau_1)\|^2 &\leq 5\|T(\tau_2) - T(\tau_1)\|^2 E\|x_0\|^2 + \|T(\tau_2) - T(\tau_1)\|^2 E\|g(x)\|^2 \\ &\quad + 5E\bigg\|\int_0^{\tau_2} T(\tau_2 - s)f(s)\,ds - \int_0^{\tau_1}[T(\tau_1 - s)f(s)\,ds\bigg\|^2 \\ &\quad + 5E\bigg\|\int_0^{\tau_2} T(\tau_2 - s)Bu(s)\,ds - \int_0^{\tau_1}[T(\tau_1 - s)Bu(s)\,ds\bigg\|^2 \\ &\quad + 5E\bigg\|\int_0^{\tau_2} T(\tau_2 - s)\sigma(s,x(s))\,dw(s) - \int_0^{\tau_1} T(\tau_1 - s)\sigma(s,x(s))\,dw(s)\bigg\|^2. \\ &= 5\|T(\tau_2) - T(\tau_1)\|^2 E\|x_0\|^2 + D_1 + D_2 + D_3 + D_4. \end{aligned}$$

Then, we have

$$\begin{aligned}
D_1 &\le 5e(1+r)\|T(\tau_2)-T(\tau_1)\|^2,\\
D_2 &\le 5\Bigg(\tau_1\int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|^2[a(s)+E\|x(s)\|^2]\,ds\\
&\quad+(\tau_2-\tau_1)\int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|^2[a(s)+E\|x(s)\|^2]\,ds\Bigg)\\
&\le 5\tau_1(\|a\|_{L^1(J,R^+)}+\tau_1 r)\sup_{s\in[0,\tau_1]}\|T(\tau_2-s)-T(\tau_1-s)\|^2\\
&\quad+5M^2[\|a\|_{L^1(J,R^+)}+r(\tau_2-\tau_1)](\tau_2-\tau_1).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
D_3 &\le 5M^2M_1^2\sup_{s\in[0,\tau_1]}\|T(\tau_2-s)-T(\tau_1-s)\|^2+5M^2M_1^2\|u\|^2_{L^2_\Gamma(J,U)}(\tau_2-\tau_1),\\
D_4 &\le 5\Bigg(\int_0^{\tau_1}\|T(\tau_2-s)-T(\tau_1-s)\|^2[\eta(s)+E\|x(s)\|^2]\,ds\\
&\quad+\int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|^2[\eta(s)+E\|x(s)\|^2]\,ds\Bigg)\\
&\le 5(\sqrt{b}\|\eta\|_{L^2(J,R^+)}+\tau_1 r)\sup_{s\in[0,\tau_1]}\|T(\tau_2-s)-T(\tau_1-s)\|^2\\
&\quad+5M^2[\|\eta\|_{L^2(J,R^+)}\sqrt{\tau_2-\tau_1}+r(\tau_2-\tau_1)].
\end{aligned}$$

Hence, using the compactness of $T(t)$ $(t>0)$, we conclude that the right-hand side of the above inequalities tends to zero as $\tau_2-\tau_1\to 0$. Thus we conclude $\mathcal{F}(x)(t)$ is continuous from the right in $(0,b]$. Similarly, for $\tau_1=0$ and $0<\tau_2\le b$, we may prove that $E\|\varphi(\tau_2)-x_0\|^2$ tends to zero independently of $x\in\mathcal{B}_r$ as $\tau_2\to 0$.

Hence, by the above arguments, we can deduce that $\{\mathcal{F}(x):x\in\mathcal{B}_r\}$ is an equicontinuous family of functions in $C(J,L^2(\Gamma,H))$.

Step 4: $\mathcal{F}$ is completely continuous.

According to Definition 2.1 (iv), we will show the set $\Pi(t)=\{\varphi(t):\varphi\in\mathcal{F}(\mathcal{B}_r)\}$ is relatively compact in H for $t\in J$ be fixed. To this end, taking account Steps 2–3 and making use of Ascoli–Arzelà theorem, we have to prove that the set $\Pi(t)=\{\varphi(t):\varphi\in\mathcal{F}(\mathcal{B}_r)\}$ is relatively compact in H.

Clearly, $\Pi(0)=\{x_0\}$ is compact. So we consider $t>0$. Let $0<t\le b$ be fixed. For any $x\in\mathcal{B}_r$, $\varphi\in\mathcal{F}(x)$, there exists an $f\in\mathcal{N}(x)$ such that (3.4) holds for each $t\in J$. For each $\varepsilon\in(0,t)$, $t\in(0,b]$ and any $x\in\mathcal{B}_r$, we define

$$\varphi^\varepsilon(t)=T(t)(x_0+g(x))+\int_0^{t-\varepsilon}T(t-s)[f(s)+Bu(s)]\,ds+\int_0^{t-\varepsilon}T(t-s)\sigma(s,x(s))\,dw(s).$$

From the boundedness of $\int_0^{t-\varepsilon}T(t-s)[f(s)+Bu(s)]\,ds$, $\int_0^{t-\varepsilon}T(t-s)\sigma(s,x(s))\,dw(s)$ and the compactness of $T(t)(t>0)$, we obtain that the set $\Pi_\varepsilon(t)=\{\varphi^\varepsilon(t):\varphi^\varepsilon\in\mathcal{F}(\mathcal{B}_r)\}$ is relatively compact in H. Moreover, we have

$$\begin{aligned}
E\|\varphi(t)-\varphi^\varepsilon(t)\|^2 &\le 3M^2\left[\varepsilon\int_{t-\varepsilon}^t(a(s)+cr)\,ds+\varepsilon M_1^2\|u\|^2_{L^2_\Gamma(J,U)}+\int_{t-\varepsilon}^t(\eta(s)+dr)\,ds\right]\\
&\le 3M^2\left[\varepsilon\|a\|_{L^2(J,R^+)}+\varepsilon M_1\|u\|^2_{L^2_\Gamma(J,U)}+\sqrt{\varepsilon}\|\eta\|_{L^2(J,R^+)}+(c\varepsilon+d)r\varepsilon\right],
\end{aligned}$$

which implies the set $\Pi(t)$ $(t > 0)$ is totally bounded. In view of Step 3, it is relatively compact in H, which completes the proof of Step 4.

Step 5: $\mathcal{F}$ has a closed graph.

Let $x_n \to x_*$ in $C(J, L^2(\Gamma, H))$, $\varphi_n \in \mathcal{F}(x_n)$ and $\varphi_n \to \varphi_*$ in $C(J, L^2(\Gamma, H))$. We will show that $\varphi_* \in \mathcal{F}(x_*)$. Indeed, $\varphi_n \in \mathcal{F}(x_n)$ means that there exists a $f_n \in \mathcal{N}(x_n)$ such that

$$\varphi_n(t) = T(t)(x_0 + g(x_n)) + \int_0^t T(t-s)[f_n(s) + Bu(s)]\,ds + \int_0^t T(t-s)\sigma(s, x_n(s))\,dw(s). \quad (3.5)$$

From (H2)–(H4), it is not difficult to show that $\{g(x_n), f_n, \sigma(\cdot, x_n)\}_{n\geq 1} \subseteq H \times L^2_\Gamma(J, H) \times L^2_0$ is bounded. Hence, passing to a subsequence if necessary,

$$(g(x_n), f_n, \sigma(\cdot, x_n)) \to (g(x_*), f_*, \sigma(\cdot, x_*)) \quad \text{weakly in } H \times L^2_\Gamma(J, H) \times L^2_0. \quad (3.6)$$

It follows from (3.5), (3.6) and the compactness of the operator $T(t)$ that

$$\varphi_n(t) \to T(t)(x_0 + g(x_*)) + \int_0^t T(t-s)[f_*(s) + Bu(s)]\,ds + \int_0^t T(t-s)\sigma(s, x_*(s))\,dw(s). \quad (3.7)$$

Note that $\varphi_n \to \varphi_*$ in $C(J, L^2(\Gamma, H))$ and $f_n \in \mathcal{N}(x_n)$. From Lemma 3.3 and (3.7), we obtain $f_* \in \mathcal{N}(x_*)$. Thus we have shown that $\varphi_* \in \mathcal{F}(x_*)$, which implies that $\mathcal{F}$ has a closed graph. By Proposition 3.3.12 (2) of [23], $\mathcal{F}$ is u.s.c.

Step 6: A priori estimate.

By Steps 1–5, we have obtained that $\mathcal{F}$ is compact convex valued and u.s.c., $\mathcal{F}(\mathcal{B}_r)$ is a relatively compact set. According to Theorem 2.7, it remains to prove the set

$$\Omega = \{x \in C(J, L^2(\Gamma, H)) : \lambda x \in \mathcal{F}(x),\ \lambda > 1\} \text{ is bounded.}$$

Let $x \in \Omega$ and suppose that there exists a $f \in \mathcal{N}(x)$ such that

$$\begin{aligned} x(t) &= \lambda^{-1}T(t)(x_0 + g(x)) + \lambda^{-1}\int_0^t T(t-s)f(s)\,ds \\ &\quad + \lambda^{-1}\int_0^t T(t-s)Bu(s)\,ds + \lambda^{-1}\int_0^t T(t-s)\sigma(s, x(s))\,dw(s). \end{aligned}$$

Then by the assumptions (H1), (H2) (iii), (H3) and (H4), we obtain

$$\begin{aligned} E\|x(t)\|^2 &\leq 5\bigg(E\|T(t)x_0\|^2 + E\|g(x))\|^2 + E\left\|\int_0^t T(t-s)f(s)\,ds\right\|^2 \\ &\quad + E\left\|\int_0^t T(t-s)Bu(s)\,ds\right\|^2 + E\left\|\int_0^t T(t-s)\sigma(s, x(s))\,dw(s)\right\|^2\bigg) \\ &\leq 5M^2E\|x_0\|^2 + 5M^2e(1 + \|x\|^2) + 5bM^2(\|a\|_{L^1(J,R^+)} + bc\|x\|^2) \\ &\quad + 5bM^2M_1^2\|u\|^2_{L^2_\Gamma(J,U)} + 5M^2(\|\eta\|_{L^2(J,R^+)}\sqrt{b} + bd\|x\|^2) \\ &\leq \rho + 5M^2[e + b(bc + d)]\|x\|^2, \end{aligned} \quad (3.8)$$

where

$$\rho = 5M^2\Big[E\|x_0\|^2 + e + 5bM^2\|a\|_{L^1(J,R^+)} + bM_1^2\|u\|^2_{L^2_\Gamma(J,U)} + \|\eta\|_{L^2(J,R^+)}\sqrt{b}\Big].$$

Since $K < 1$, from (3.8), we obtain

$$\|x\|^2 = \sup_{t\in J} E\|x(t)\|^2 \leq \rho + K\|x(t)\|^2, \quad \text{thus} \quad \|x\|^2 \leq \frac{\rho}{1-K}.$$

Hence, the set Ω is bounded. By Theorem 2.7, $\mathcal{F}$ has a fixed point. The proof is completed. □

4 Controllability results

In this section, we mainly investigate the complete controllability of the system (1.1). The following definition of the controllability is standard. We state it here for the sake of convenience.

Definition 4.1 (Complete controllability)**.** The system (1.1) is said to be completely controllable on the interval J if, for every $x_0, x_1 \in H$, there exists a stochastic control $u \in L^p_\Gamma(J, U)(p > 1)$ which is adapted to the filtration $\{\Gamma_t\}_{t\geq 0}$ such that a mild solution x of system (1.1) satisfies $x(b) = x_1$.

Theorem 4.2. *Suppose that the assumptions (H1)–(H5) are satisfied. Then the system* (1.1) *is completely controllable on J provided that*

$$K_1 = 5M^2(1 + 5bM_1^2M_2^2M^2)[e + b(bc + d)] < 1.$$

Proof. Firstly, for any $x \in C(J, L^2(\Gamma, H)) \subset L^2(J, H)$ and $x_1 \in L^2_\Gamma(\Gamma_b, H)$, from Lemma 3.2, we can define a multivalued map $\mathcal{F}_u : C(J; L^2(\Gamma, H)) \to 2^{C(J,L^2(\Gamma,H))}$ by

$$\mathcal{F}_u(x) = \left\{ \begin{array}{l} h \in C(J; L^2(\Gamma, H)) : h(t) = T(t)x_0 + T(t)g(x) \\ \qquad\qquad + \int_0^t T(t-s)f(s)\,ds + \int_0^t T(t-s)Bu_\alpha(s)\,ds \\ \qquad\qquad + \int_0^t T(t-s)\sigma(s, x(s))\,dw(s), f \in \mathcal{N}(x) \end{array} \right\},$$

where

$$u_\alpha(t) = W^{-1}\bigg(x_1 - T(b)x_0 - T(b)g(x) - \int_0^b T(b-s)f(s)\,ds \\ - \int_0^b T(b-s)\sigma(s, x(s))\,dw(s)\bigg). \quad (4.1)$$

Using the control u_α and the assumptions, it is easy to see that the multivalued map $\mathcal{F}_u$ is well defined and $x_1 \in (\mathcal{F}_u x)(b)$. Thus to obtain the complete controllability, we only need to prove that $\mathcal{F}_u$ has a fixed point.

The proof is similar to Theorem 3.4. To complete the proof, a simple version of proof is given.

Step 1: Clearly, for $\forall x \in C(J, L^2(\Gamma, H))$, $\mathcal{F}_u$ is convex by the convexity of $\mathcal{N}(x)$.

Step 2: The operator $\mathcal{F}$ is bounded on bounded subset of $C(J, L^2(\Gamma, H))$.

Let $\mathcal{B}_\zeta = \{x \in C(J, L^2(\Gamma, H)) : \|x\|^2 \leq \zeta\}$. In fact, it is enough to show that there exists a positive constant ℓ_0 such that for each $\varphi \in \mathcal{F}_u(x),\ x \in \mathcal{B}_\zeta,\ \|\varphi\|^2 \leq \ell_0$. If $\varphi \in \mathcal{F}_u(x)$, then there exists a $f \in \mathcal{N}(x)$ such that for $t \in J$

$$\varphi(t) = T(t)(x_0 + g(x)) + \int_0^t T(t-s)f(s + Bu_\alpha(s)]\,ds \\ + \int_0^t T(t-s)Bu_\alpha(s)\,ds + \int_0^t T(t-s)\sigma(s, x(s))\,dw(s). \quad (4.2)$$

where u_α is given by (4.1). Then notice that

$$E\|u_\alpha(t)\|^2 \leq 5M_2^2\bigg[E\|x_1\|^2 + M^2\Big(E\|x_0\|^2 + e(1+\zeta) + b\|a\|_{L^1(J,R^+)} \\ + \sqrt{b}\|\eta\|_{L^2(J,R^+)} + (bc+d)b\zeta\Big)\bigg] =: \Lambda$$

and

$$E\|\varphi(t)\|^2 \le 5M^2\Big[E\|x_0\|^2 + e(1+\zeta) + b(\|a\|_{L^1(J,R^+)} + bc\zeta) + bM_1^2\Lambda + \sqrt{b}\|\eta\|_{L^2(J,R^+)} + bd\zeta\Big] =: \ell_0.$$

Thus, $\mathcal{F}_u(\mathcal{B}_\zeta)$ is bounded in $C(J, L^2(\Gamma, H))$.

Step 3: $\{\mathcal{F}_u(x) : x \in \mathcal{B}_\zeta\}$ is equicontinuous.

For $\forall x \in \mathcal{B}_\zeta$, $\varphi \in \mathcal{F}_u(x)$, there exists a $f \in \mathcal{N}(x)$ such that for each $t \in J$, we have φ as (4.2). Using the estimation on $E\|u_\alpha(t)\|^2$ similarly to Step 3 of Theorem 3.4, we know that $\{\mathcal{F}_u(x) : x \in \mathcal{B}_\zeta\}$ is equicontinuous family of functions in $C(J, L^2(\Gamma, H))$.

Step 4: $\mathcal{F}_u$ is completely continuous.

Let $t \in J$ be fixed. We show that the set $\Pi(t) = \{\varphi(t) : \varphi \in \mathcal{F}_u(\mathcal{B}_\zeta)\}$ is relatively compact in H. Clearly, $\Pi(0) = \{x_0\}$ is compact. So it is sufficient to consider $t > 0$. Let $0 < t \le b$ be fixed. For any $x \in \mathcal{B}_\zeta$, $\varphi \in \mathcal{F}_u(x)$, there exists $f \in \mathcal{N}(x)$ such that $\varphi(t)$ satisfies (4.2). For each $\epsilon \in (0, t)$, $t \in (0, b]$ and any $x \in \mathcal{B}_\zeta$, we can use the way in Theorem 3.4 to prove that the set $\Pi(t) = \{\varphi(t) : \varphi \in \mathcal{F}_u(B_\zeta)\}$ is totally bounded.

Taking account Steps 2–3 and making use of Ascoli–Arzelà theorem, we obtain that $\mathcal{F}_u$ is completely continuous.

Step 5: $\mathcal{F}_u$ has a closed graph.

Let $x_n \to x_*$ in $C(J, L^2(\Gamma, H))$, $\varphi_n \in \mathcal{F}_u(x_n)$ and $\varphi_n \to \varphi_*$ in $C(J, L^2(\Gamma, H))$. We will show that $\varphi_* \in \mathcal{F}_u(x_*)$. Indeed, $\varphi_n \in \mathcal{F}_u(x_n)$ means that there exists a $f_n \in \mathcal{N}(x_n)$ such that

$$\begin{aligned}\varphi_n(t) = {} & T(t)x_0 + T(t)g(x_n) + \int_0^t T(t-s)f_n(s)\,ds + \int_0^t T(t-s)\sigma(s, x_n(s)\,dw(s) \\ & + \int_0^t T(t-s)BW^{-1}\Big(x_1 - T(b)x_0 - T(b)g(x_n) - \int_0^b T(b-\tau)f_n(\tau)\,d\tau \\ & - \int_0^b T(b-\tau)\sigma(\tau, x_n(\tau)\,dw(\tau)\Big)ds. \end{aligned} \tag{4.3}$$

From (H2)–(H4), it is not difficult to show that $\{g(x_n), f_n, \sigma(\cdot, x_n)\}_{n\ge1} \subseteq H \times L^2_\Gamma(J, H) \times L^2_0$ is bounded. Hence, passing to a subsequence if necessary,

$$(g(x_n), f_n, \sigma(\cdot, x_n)) \to (g(x_*), f_*, \sigma(\cdot, x_*)) \quad \text{weakly in } H \times L^2_\Gamma(J, H) \times L^2_0. \tag{4.4}$$

From the compactness of $T(t)$, (4.3) and (4.4), we obtain

$$\begin{aligned}\varphi_n(t) \quad \to \quad & T(t)x_0 + T(t)g(x_*) + \int_0^t T(t-s)f_*(s)\,ds + \int_0^t T(t-s)\sigma(s, x_*(s)\,dw(s) \\ & + \int_0^t T(t-s)BW^{-1}\Big(x_1 - T(b)x_0 - T(t)g(x_*) - \int_0^b T(b-\tau)f_*(\tau)\,d\tau \\ & + \int_0^b T(b-s)\sigma(s, x_*(s)\,dw(s)\Big)ds. \end{aligned} \tag{4.5}$$

Note that $\varphi_n \to \varphi_*$ in $C(J, L^2(\Gamma, H))$ and $f_n \in \mathcal{N}(x_n)$. From Lemma 3.3 and (4.5), we obtain $f_* \in \mathcal{N}(x_*)$. Hence, we have proved that $\varphi_* \in \mathcal{F}_u(x_*)$, which implies that $\mathcal{F}_u$ has a closed graph. It follows from Proposition 3.3.12 (2) of [23] that $\mathcal{F}_u$ is u.s.c.

Step 6: A priori estimate.

From Steps 1–5, $\mathcal{F}_u$ is compact convex valued and u.s.c., and $\mathcal{F}_u(B_\zeta)$ is a relatively compact set. According to Theorem 2.7, it remains to prove that the set

$$\Omega = \{x \in C(J, L^2(\Gamma, H)) : \lambda x \in \mathcal{F}_u(x),\ \lambda > 1\} \quad \text{is bounded.}$$

Let $x \in \Omega$ and assume that there exists $f \in \mathcal{N}(x)$ such that

$$\begin{aligned} x(t) &= \lambda^{-1}T(t)x_0 + \lambda^{-1}T(t)g(x) + \lambda^{-1}\int_0^t T(t-s)f(s)\,ds \\ &\quad + \lambda^{-1}\int_0^t T(t-s)\sigma(s,x(s))\,dw(s) + \lambda^{-1}\int_0^t T(t-s)BW^{-1}\Big(x_1 - T(b)x_0 \\ &\quad - T(b)g(x) - \int_0^b T(b-\tau)f(\tau)\,d\tau - \int_0^b T(b-\tau)\sigma(\tau,x(\tau))\,dw(\tau)\Big)ds. \end{aligned}$$

Then by the assumptions (H1)–(H5), we obtain

$$\begin{aligned} E\|x(t)\|^2 &\le 5M^2\Big[E\|x_0\|^2 + e(1+\|x\|^2) + b(\|a\|_{L^1(J,R^+)} + bc\|x\|^2) + \sqrt{b}\|\eta\|_{L^2(J,R^+)} \\ &\quad + bd\|x\|^2 + 5bM_1^2M_2^2\Big(E\|x_1\|^2 + M^2\Big[E\|x_0\|^2 + e(1+\|x\|^2) \\ &\quad + b(\|a\|_{L^1(J,R^+)} + bc\|x\|^2) + \sqrt{b}\|\eta\|_{L^2(J,R^+)} + bd\|x\|^2\Big]\Big)\Big] \\ &\le \varrho + K_1\|x\|^2, \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} \varrho &= 5M^2\Big[E\|x_0\|^2 + e + b\|a\|_{L^1(J,R^+)} + \sqrt{b}\|\eta\|_{L^2(J,R^+)} \\ &\quad + 5bM_1^2M_2^2\Big(E\|x_1\|^2 + M^2\big[E\|x_0\|^2 + e + b\|a\|_{L^1(J,R^+)} + \sqrt{b}\|\eta\|_{L^2(J,R^+)}\big]\Big)\Big]. \end{aligned}$$

and

$$K_1 = 5M^2(1 + 5bM_1^2M_2^2M^2)[e + b(bc + d)].$$

Therefore, by the hypothesis $K_1 < 1$ and the formula (4.6), it is easy to see that

$$\|x\|^2 = \sup_{t\in J} E\|x(t)\|^2 \le \varrho + K_1\|x\|^2, \quad \text{thus } \|x\|^2 \le \frac{\varrho}{1-K_1} =: \widehat{\omega}.$$

Hence, the set Ω is bounded. By Theorem 2.7, we obtain that $\mathcal{F}_u$ has a fixed point which completes the proof. □

Remark 4.3. We refer the readers to [30] where a linear stochastic control system is given to illustrate the compactness assumption on semigroup $T(t)$ is not necessary. For more results on complete controllability without the compactness assumption on semigroup of infinite-dimensional control linear systems, see [21]. Therefore, there are sufficient but not necessary conditions for complete controllability in Theorem 4.2.

5 An example

As an application of the main result, we consider the following control system described by evolution inclusions of Clarke subdifferential:

$$\begin{cases} dx(t,z) \in [x_{zz}(t,z) + Bu(t,z)]\,dt + \partial F(t,z,x(t,z))\,dt + \sigma(t,z,x(t,z))\,dw(t), \\ \qquad\qquad 0 < t < b, \quad 0 < z < \pi, \\ x(t,0) = x(t,\pi) = 0, \quad t \in (0,b), \\ x(0,z) = x_0(z), \quad z \in (0,\pi), \end{cases} \tag{5.1}$$

where $x(t,z)$ represents the temperature at the point $z \in (0,\pi)$ and time $t \in (0,b)$, $w(t)$ is a two sided and standard one dimensional Brownian motion defined on the filtered probability space $(\Omega, \Gamma, \mathbb{P})$. Here $F = F(t,z,\nu)$ is a locally Lipschitz energy function which is generally nonsmooth and nonconvex. ∂F denotes the generalized Clarke's gradient in the third variable ν (cf. [2]). A simple example of the function F which satisfies hypotheses (H2) is $F(\nu) = \min\{h_1(\nu), h_2(\nu)\}$, where $h_i\colon R \to R (i = 1,2)$ are convex quadratic functions (cf. [23]).

Next, to write the above system (5.1) into the abstract form of (1.1), let $H = U = L^2[0,\pi]$. Define an operator $A\colon L^2[0,b] \to L^2[0,b]$ by $Ax = x''$ with domain

$$D(A) = \{x \in H : x,\ x' \text{ are absolutely continuous},\ x'' \in H,\ x(0) = x(\pi) = 0\}.$$

$$Ax = \sum_{n=1}^{\infty} n^2 \langle x, e_n\rangle e_n,\ x \in D(A),$$

where $e_n(y) = \sqrt{2}\sin(ny)$ $(n = 1,2,\dots)$ is an orthonormal set of eigenvectors in A. It is well known that A generates a compact, analytic semigroup $\{T(t), t \geq 0\}$ in H and

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}\langle x, e_n\rangle e_n, \quad x \in H \qquad \text{and} \qquad \|T(t)\| \leq e^{-t} \quad \text{for all } t \geq 0.$$

Define $x(t)(z) = x(t,z)$ and $\sigma(t,x(t))(z) = \sigma(t,x(t,z))$ which satisfy assumption (H3). Assume that the infinite dimensional space U defined by

$$U = \left\{ u : u = \sum_{n=2}^{\infty} u_n e_n \quad \text{with} \quad \sum_{n=2}^{\infty} u_n^2 < \infty \right\},$$

with the norm defined by $\|u\|_U = (\sum_{n=2}^{\infty} u_n^2)^{1/2}$. Define a mapping $B \in \mathcal{L}(U,H)$ as follows:

$$Bu = 2u_2 e_1 + \sum_{n=2}^{\infty} u_n e_n \quad \text{for} \quad u = \sum_{n=2}^{\infty} u_n e_n \in U.$$

Under the above assumptions, we know that the system (5.1) can be written in the abstract form (1.1) and all the conditions of Theorem 4.2 are satisfied. Therefore, by Theorem 4.2, stochastic control system (5.1) is completely controllable on $J = [0,b]$.

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References

[1] A. E. Bashirov, N. I. Mahmudov, On concepts of controllability for deterministic and stochastic systems, *SIAM J. Control Optim.* **37**(1999), No. 6, 1808–1821. MR1720139; url

[2] F. H. Clarke, *Optimization and nonsmooth analysis*, Wiley, New York, 1983. MR0709590

[3] G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions*, Encyclopedia of Mathematics and its Applications, Vol. 152, Cambridge University Press, Cambridge, 1992. MR3236753; url

[4] Z. Denkowski, S. Migórski, N. S. Papageorgiou, *An introduction to nonlinear analysis: theory*, Kluwer Academic/Plenum Publishers, Boston, London, New York, 2003. MR2024162

[5] S. Hu, N. S. Papageorgiou, *Handbook of multivalued analysis Vol. I. Theory*, Kluwer Academic Publishers, Dordrecht, London, 1997. MR1485775

[6] J. Klamka, Stochastic controllability of systems with multiple delays in control, *Int. J. Appl. Math. Comput. Sci.* **19**(2009), 39–47. MR2515022

[7] J. Klamka, Stochastic controllability of systems with variable delay in control, *Bull. Pol. Acad. Sci., Tech. Sci.* **56**(2008), No. 3, 279–284. url

[8] J. Klamka, Stochastic controllability and minimum energy control of systems with multiple delays in control. *Appl. Math. Comput.* **206**(2008), No. 2, 704–715. MR2483043; url

[9] J. Klamka, Stochastic controllability of linear systems with delay in control, *Bull. Pol. Acad. Sci., Tech. Sci.* **55**(2007), No. 1, 23–29. url

[10] J. Klamka, Stochastic controllability of linear systems with state delays, *Int. J. Appl. Math. Comput. Sci.* **17**(2007), 5–13. MR2310791

[11] A. Lin, L. Hu. Existence results for impulsive neutral stochastic functional integrodifferential inclusions with nonlocal initial conditions, *Comput. Math. Appl.* **59**(2010), No. 1, 64–73. url

[12] Z. H. Liu, Browder–Tikhonov regularization of non-coercive evolution hemivariational inequalities, *Inverse Probl.* **21**(2005), 13–20. MR2146161

[13] Z. H. Liu, Existence results for quasilinear parabolic hemivariational inequalities, *J. Differential Equations* **244**(2008), 1395–1409. MR2396503; url

[14] Z. H. Liu, Anti-periodic solutions to nonlinear evolution equations, *J. Funct. Anal.* **258**(2010), No. 6, 2026–2033. MR2578462; url

[15] Z. H. Liu, B. Zeng, Existence and controllability for fractional evolution inclusions of Clarke's subdifferential type, *Appl. Math. Comput.* **257**(2015), 178–189. MR3320658; url

[16] Z. H. Liu, X. W. Li, Approximate controllability for a class of hemivariational inequalities, *Nonlinear Anal. Real World Appl.* **22**(2015), 581–591. MR3280853; url

[17] Z. H. Liu, X. W. Li, Approximate controllability of fractional evolution systems with Riemann–Liouville fractional derivatives, *SIAM J. Control Optim.* **53**(2015), No. 4, 1920–1933. MR3369987; url

[18] L. Lu, Z. H. Liu, Existence and controllability results for stochastic fractional evolution hemivariational inequalities, *Appl. Math. Comput.* **268**(2015), 1164–1176. url

[19] T. W. Ma, Topological degrees for set-valued compact vector fields in locally convex spaces, *Dissertationes Math. Rozprawy Mat.* **92**(1972), 43 pp. MR0309103

[20] N. I. Mahmudov, Approximate controllability of evolution systems with nonlocal conditions, *Nonlinear Anal.* **68**(2008), 536–546. MR2372363

[21] N. I. Mahmudov, Controllability of linear stochastic systems in Hilbert spaces, *J. Math. Anal. Appl.* **259**(2001), 64–82. url

[22] S. Migórski, A. Ochal, Quasi-static hemivariational inequality via vanishing acceleration approach, *SIAM J. Math. Anal.* **41**(2009), 1415–1435. MR2540272

[23] S. Migórski, A. Ochal, M. Sofonea, *Nonlinear inclusions and hemivariational inequalities. Models and analysis of contact problems*, Advances in Mechanics and Mathematics, Vol. 26, Springer, New York, 2013. MR2976197; url

[24] V. Obukhovski, P. Zecca, Controllability for systems governed by semilinear differential inclusions in a Banach space with a noncompact semigroup, *Nonlinear Anal.* **70**(2009), No. 9, 3424–3436. MR2503088

[25] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Springer-Verlag, New York, 1983. MR0710486

[26] Y. Ren, L. Hu, R. Sakthivel, Controllability of impulsive neutral stochastic functional differential inclusions with infinite delay, *J. Comput. Appl. Math.* **235**(2011), No. 8, 2603–2614.

[27] K. Rykaczewski, Approximate controllability of differential inclusions in Hilbert spaces, *Nonlinear Anal.* **75**(2012), 2701–2712. MR2878467; url

[28] K. Sobczyk, *Stochastic differential equations with applications to physics and engineering*, Kluwer Academic Publishers, London, 1991.

[29] R. Sakthivel, Y. Ren, N. I. Mahmudov, On the approximate controllability of semilinear fractional differential systems, *Comput. Math. Appl.* **62**(2011), 1451–1459. url

[30] R. Sakthivel, S. Suganya, S. M. Anthoni, Approximate controllability of fractional stochastic evolution equations, *Comput. Math. Appl.* **63**(2012), 660–668. url

[31] L. Simon, *Application of monotone type operators to nonlinear PDEs*, Prime Rate Kft., Budapest, 2013.

[32] A. A. Tolstonogov, Control systems of subdifferential type depending on a parameter, *Izv. Math.* **72**(2008), No. 5, 985–1022. MR2473775

[33] A. A. Tolstonogov, Relaxation in nonconvex optimal control problems with subdifferential operators, *J. Math. Sci.* **140**(2007), No. 6, 850–872. MR2179455