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Existence and stability of periodic solutions for a delayed prey–predator model with diffusion effects

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Abstract. Existence and stability of spatially periodic solutions for a delay prey–predator diffusion system are concerned in this work. We obtain that the system can generate the spatially nonhomogeneous periodic solutions when the diffusive rates are suitably small. This result demonstrates that the diffusion plays an important role in deriving the complex spatiotemporal dynamics. Meanwhile, the stability of the spatially periodic solutions is also studied. Finally, in order to verify our theoretical results, some numerical simulations are also included.

Keywords: existence, stability, spatially periodic solutions.

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1 Introduction

In recent years, the interactions between two species have attracted much attention due to their theoretical and practical significance since the pioneering theoretical works by Lotka [22] and Volterra [28], see [4, 8, 10, 19, 30, 32]. It is well known that the interactions between two species have mainly three kinds of fundamental forms such as competition, cooperation and prey-predation in population biology. Among these interactions, extreme attention has been payed to the prey–predation mechanism because it possesses a very significant function as a kind of restriction factor in the process of evolvement of biology [6,9,17,23,25]. Understanding the dynamics of predator–prey models will be very helpful for investigating multiple species interactions. In [1], Beretta and Kuang have explored the dynamics of the following delayed Leslie–Gower model.

$$\begin{cases} \dfrac{du(t)}{dt} = r_1 u(t)\left[1 - \dfrac{u(t)}{K}\right] - mu(t)v(t), & t > 0, \\ \dfrac{dv(t)}{dt} = r_2 v(t)\left[1 - \dfrac{v(t-\tau)}{ru(t-\tau)}\right], & t > 0, \end{cases} \tag{1.1}$$

where $u(t), v(t)$ are the population densities of the prey and the predator, respectively; $r_1 > 0$, $r_2 > 0$ denote the intrinsic growth rates of the prey and the predator, respectively. $K > 0$ is the

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carrying capacity of the prey and ru takes on the role of a prey-dependent carrying capacity for the predator. The parameter $r > 0$ is a measure of the quality of the prey as food for the predator. They presented some results on the boundedness of solutions permanence, global stability of the boundary equilibrium and local stability results of the positive equilibrium. Following this work, Song et al. [27] considered the properties of the local Hopf bifurcation and the global continuation of the local Hopf bifurcation for model (1.1).

In fact, the distribution of species is generally spatially inhomogeneous and therefore the species always tend to migrate toward regions of lower population density to improve the possibility of survival [29]. Therefore, spatial diffusion should be considered in modelling biological interactions, see [2, 3, 12, 13, 16, 20, 21, 26, 31]. Thus, the dynamics behavior of two species to model (1.1) should be described by the following model

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = d_1 \triangle u(t,x) + r_1 u(t,x)\left[1 - \frac{u(t,x)}{K}\right] - mu(t,x)v(t,x), & t > 0,\ x \in \Omega \\ \frac{\partial v(t,x)}{\partial t} = d_2 \triangle v(t,x) + r_2 v(t,x)\left[1 - \frac{v(t-\tau,x)}{ru(t-\tau,x)}\right], & t > 0,\ x \in \Omega, \\ u(t,x) = \phi(t,x) \geq 0,\ v(t,x) = \psi(t,x) \geq 0, & (t,x) \in [-\tau, 0] \times \Omega, \end{cases} \tag{1.2}$$

with Neumann boundary conditions

$$\frac{\partial u(t,x)}{\partial \nu} = \frac{\partial v(t,x)}{\partial \nu} = 0, \qquad x \in \partial\Omega,\ t \geq 0. \tag{1.3}$$

$\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$; ν is the unit outward normal vector on the boundary of Ω and the Neumann boundary conditions in (1.3) imply that two species have zero flux across the domain boundary $\partial\Omega$; $d_1 > 0$, $d_2 > 0$ denote the diffusion coefficients of two species; $(\phi, \psi) \in C = C([-\tau, 0], X)$, X is defined by

$$X = \left\{ (u,v) : u, v \in W^{2,2}(\Omega) : \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0,\ x \in \partial\Omega \right\},$$

with the inner product $\langle \cdot, \cdot \rangle$.

The main goals of the present paper are to consider the existence and stability of spatially periodic solutions of system (1.2). By regarding the time delay τ as the bifurcation parameter and analyzing the associated characteristic equation, we find that an increase of τ can lead to the occurrence of spatially nonhomogeneous periodic solutions at (u^*, v^*). Moreover, the stability of the spatially nonhomogeneous periodic solutions is studied.

The remaining parts of this paper are organized as follows. In Section 2, the existence of spatially nonhomogeneous periodic solutions is investigated. In Section 3, we derive conditions for determining the stability of the spatially nonhomogeneous periodic solutions on the center manifold. Finally, some conclusions and numerical simulations are presented in Section 4. Throughout the paper, we denote by $\mathbb{N}$ the set of all positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2 Existence of spatially periodic solutions

In this section, we focus on investigating the local stability and the existence of spatially periodic solutions of the positive constant steady-state of system (1.2). It is easy to see that system (1.2) has two feasible boundary equilibria $(0, 0)$, $(K, 0)$ and a unique positive constant steady-state $E^*(u^*, v^*)$, where

$$u^* = \frac{1}{r} v^*, \qquad v^* = \frac{K r_1 r}{r_1 + Kmr}.$$

Let $\overline{u}(t,x) = u(t,x) - u^*$, $\overline{v}(t,x) = v(t,x) - v^*$, for convenience, we still use u and v to denote $\overline{u}$ and $\overline{v}$. Then system (1.2) can be transformed into the following reaction-diffusion system when Ω is restricted to the one-dimensional spatial domain $(0,\pi)$:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = d_1 \frac{\partial^2 u(t,x)}{\partial x^2} + r_1(u(t,x)+u^*)\left[1 - \frac{u(t,x)+u^*}{K}\right] - m(u(t,x)+u^*)(v(t,x)+v^*), \\ \frac{\partial v(t,x)}{\partial t} = d_2 \frac{\partial^2 v(t,x)}{\partial x^2} + r_2(v(t,x)+v^*)\left[1 - \frac{v(t-\tau,x)+v^*}{r(u(t-\tau,x)+u^*)}\right], \quad t>0,\ x\in\Omega, \\ \frac{\partial u(t,x)}{\partial x} = \frac{\partial v(t,x)}{\partial x} = 0, \quad t \geq 0,\ x \in \partial\Omega, \\ u(t,x) = \phi(t,x) - u^*,\ v(t,x) = \psi(t,x) - v^*, \quad (t,x) \in [-\tau, 0]\times\Omega. \end{cases} \tag{2.1}$$

Thus, the positive constant steady state $E^*(u^*,v^*)$ of system (1.2) is transformed into the zero steady state of system (2.1).

By virtue of the Taylor expansions, system (2.1) can be rewritten as the following system

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = d_1 \triangle u(t,x) + \beta_{11}u(t,x) + \beta_{12}v(t,x) + \beta_{13}u^2(t,x) + \beta_{14}u(t,x)v(t,x), \\ \frac{\partial v(t,x)}{\partial t} = d_2 \triangle v(t,x) + \beta_{21}u(t-\tau,x) + \beta_{22}v(t-\tau,x) \\ \qquad + \sum\limits_{i+j+k\geq 2} \frac{1}{i!j!k!} f_{ijk} u^i(t-\tau,x)v^j(t-\tau,x)v^k(t,x), \end{cases} \tag{2.2}$$

where

$$\beta_{11} = \frac{-r_1}{K}u^* < 0, \qquad \beta_{12} = -mu^* < 0, \qquad \beta_{13} = \frac{-r_1}{K} < 0, \qquad \beta_{14} = -m < 0,$$

$$\beta_{21} = rr_2 > 0, \qquad \beta_{22} = -r_2 < 0$$

$$f_{ijk} = \frac{\partial^{i+j+k} f(0,0)}{\partial u^i \partial v^j \partial v_1^k}, \qquad f(u,v) = r_2 v_1(t,x)\left(1 - \frac{v(t,x)}{ru(t,x)}\right).$$

Let $u_1(t) = u(t,\cdot)$, $u_2(t) = v(t,\cdot)$, and $U(t) = (u_1(t), u_2(t))^T$. According to [11, 12], then system (2.2) can be rewritten as a delay differential equation in the phase space $C = C([-\tau,0],X)$

$$\frac{d}{dt}U(t) = d\triangle U(t) + L(U_t) + F(U_t), \tag{2.3}$$

where

$$d = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \qquad \triangle = \begin{pmatrix} \frac{\partial^2}{\partial x^2} & 0 \\ 0 & \frac{\partial^2}{\partial x^2} \end{pmatrix},$$

$U_t(\theta) = U(t+\theta)$, $-\tau \leq \theta \leq 0$, $L : C \to X$ and $F : C \to X$ are given, respectively, by

$$L(\varphi) = \begin{pmatrix} \beta_{11}\varphi_1(0) + \beta_{12}\varphi_2(0) \\ \beta_{21}\varphi_1(-\tau) + \beta_{22}\varphi_2(-\tau) \end{pmatrix},$$

$$F(\varphi) = \begin{pmatrix} \beta_{13}\varphi_1^2(0) + \beta_{14}\varphi_1(0)\varphi_2(0) \\ \sum_{i+j+k\geq 2} \frac{1}{i!j!k!} f_{ijk}\varphi_1^i(-\tau)\varphi_2^j(-\tau)\varphi_2^k(0) \end{pmatrix},$$

where $\varphi(\theta) = U_t(\theta)$, $-\tau \leq \theta \leq 0$, $\varphi = (\varphi_1, \varphi_2)^T \in C$.

Linearizing (2.3) at $(0,0)$ gives the linear equation

$$\frac{d}{dt}U(t) = d\triangle U(t) + L(U_t). \tag{2.4}$$

The characteristic equation for the linearized equation (2.4) is

$$\lambda y - d\Delta y - L(e^{\lambda \cdot} y) = 0, \tag{2.5}$$

where $y \in \mathrm{dom}(\Delta)\backslash\{0\}$ and $\mathrm{dom}(\Delta) \subset X$.

It is well known that the linear operator Δ on $(0,\pi)$ with homogeneous Neumann boundary conditions has the eigenvalues $-k^2$ ($k \in \mathbb{N}_0$) and the corresponding eigenfunctions are

$$\beta_k^1 = \begin{pmatrix} \gamma_k \\ 0 \end{pmatrix}, \qquad \beta_k^2 = \begin{pmatrix} 0 \\ \gamma_k \end{pmatrix}, \qquad \gamma_k = \frac{\cos(kx)}{\|\cos(kx)\|_{2,2}}, \qquad k \in \mathbb{N}_0.$$

Notice that $(\beta_k^1, \beta_k^2)_{k=0}^{\infty}$ construct an orthogonal basis of the Banach space X (see [12]). Therefore $L(\beta_k^1, \beta_k^2) \subset \mathrm{span}\{\beta_k^1, \beta_k^2\}$ and thus any element y in X can be expanded a Fourier series in the form

$$\begin{aligned} y &= \sum_{k=0}^{\infty} \left(\langle y, \beta_k^1 \rangle \beta_k^1 + \langle y, \beta_k^2 \rangle \beta_k^2 \right) \\ &= \sum_{k=0}^{\infty} \left(\langle y, \beta_k^1 \rangle, \langle y, \beta_k^2 \rangle \right) \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix}. \end{aligned} \tag{2.6}$$

In addition, some easy computations can show that

$$L\left(\varphi^T \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix} \right) = [L(\varphi)]^T \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix}, \tag{2.7}$$

where $\varphi = (\varphi_1, \varphi_2)^T \in C$.

From (2.6) and (2.7), (2.5) is equivalent to

$$\sum_{k=0}^{\infty} \left(\langle y, \beta_k^1 \rangle, \langle y, \beta_k^2 \rangle \right) \left[\begin{pmatrix} \lambda + d_1 k^2 & 0 \\ 0 & \lambda + d_2 k^2 \end{pmatrix} - \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} e^{-\lambda\tau} & \beta_{22} e^{-\lambda\tau} \end{pmatrix} \right] \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix} = 0.$$

Hence, we conclude that the characteristic equation (2.4) is equivalent to the sequence of the characteristic equations

$$\begin{aligned} &\lambda^2 + [(d_1+d_2)k^2 - \beta_{11}]\lambda + [d_1 d_2 k^4 - d_2 \beta_{11} k^2] \\ &\qquad + [-\beta_{22}\lambda - d_1 \beta_{22} k^2 + \beta_{11}\beta_{22} - \beta_{12}\beta_{21}] e^{-\lambda\tau} = 0, \qquad k \in \mathbb{N}_0. \end{aligned} \tag{2.8}$$

It is obvious that equation (2.8) has no zero roots since $\beta_{11} < 0$, $\beta_{12} < 0$, $\beta_{21} > 0$, $\beta_{22} < 0$, $d_1 > 0$, $d_2 > 0$.

When $\tau = 0$, (2.8) reduces to the following quadratic equation with respect to λ

$$\begin{aligned} &\lambda^2 + [(d_1+d_2)k^2 - \beta_{11} - \beta_{22}]\lambda \\ &\qquad + [d_1 d_2 k^4 - d_2 \beta_{11} k^2 - d_1 \beta_{22} k^2 + \beta_{11}\beta_{22} - \beta_{12}\beta_{21}] = 0, \qquad k \in \mathbb{N}_0, \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} [(d_1+d_2)k^2 - \beta_{11} - \beta_{22}] &> 0, \\ [d_1 d_2 k^4 - d_2 \beta_{11} k^2 - d_1 \beta_{22} k^2 + \beta_{11}\beta_{22} - \beta_{12}\beta_{21}] &> 0. \end{aligned}$$

Consequently, all roots of equations (2.9) have negative real parts. Therefore, the positive steady state $E(u^*, v^*)$ of system (1.2) is locally asymptotically stable in the absence of delay.

When $d_1 = d_2 = 0$ and $\tau = 0$, system (1.2) becomes an ordinary differential equation, we know that roots of the characteristic equation of ordinary differential equations have negative real parts. This indicates that the diffusion coefficients d_1, d_2 have no effect on the stability of the positive steady state $E(u^*, v^*)$ in the absence of delay.

Denote

(H) $d_1\beta_{22} - d_2\beta_{11} > 0$, and $d_1 d_2 + (d_1\beta_{22} - d_2\beta_{11}) + (\beta_{12}\beta_{21} - \beta_{11}\beta_{22}) > 0$.

Lemma 2.1. *Assume that the condition* **(H)** *holds. If*

$$(d_1^2 + d_2^2) - 2\beta_{11}d_1 < \beta_{22}^2 - \beta_{11}^2 < 16(d_1^2 + d_2^2) - 8\beta_{11}d_1, \tag{2.10}$$

$$\left[A_1^2 - 2B_1^2 - \beta_{22}^2\right]^2 - 4(B_1^2 - C_1^2) > 0, \tag{2.11}$$

then (2.8) *with* $k = 1$ *has purely imaginary roots* $\pm i\omega_1$, *where*

$$\omega_1 = \sqrt{\frac{-(A_1^2 - 2B_1 - \beta_{22}^2) \pm \sqrt{(A_1^2 - 2B_1 - \beta_{22}^2)^2 - 4(B_1^2 - C_1^2)}}{2}}.$$

Proof. Assuming $i\omega$ $(\omega > 0)$ is a solution of (2.8) with $k \geq 1$, then substituting $i\omega$ into equation (2.8) and separating the real and imaginary parts, one can get that

$$-\omega^2 + B_k - \beta_{22}\omega\sin(\omega\tau) + C_k\cos(\omega\tau) = 0, \tag{2.12}$$

$$A_k\omega - \beta_{22}\omega\cos(\omega\tau) - C_k\sin(\omega\tau) = 0, \tag{2.13}$$

where

$$A_k = (d_1 + d_2)k^2 - \beta_{11} > 0, \qquad B_k = d_1 d_2 k^4 - d_2\beta_{11}k^2 \geq 0,$$
$$C_k = -d_1\beta_{22}k^2 + \beta_{11}\beta_{22} - \beta_{12}\beta_{21} > 0, \qquad k \in \mathbb{N}_0.$$

From (2.12) and (2.13), it is easy to see that

$$\omega^4 + (A_k^2 - 2B_k - \beta_{22}^2)\omega^2 + B_k^2 - C_k^2 = 0, \qquad k \in \mathbb{N}_0. \tag{2.14}$$

By computing, we have $B_k - C_k = d_1 d_2 k^4 + (d_1\beta_{22} - d_2\beta_{11})k^2 + \beta_{12}\beta_{21} - \beta_{11}\beta_{22}$. It is clear that $d_1 d_2 k^4 + (d_1\beta_{22} - d_2\beta_{11})k^2 + \beta_{12}\beta_{21} - \beta_{11}\beta_{22} \geq d_1 d_2 + (d_1\beta_{22} - d_2\beta_{11}) + \beta_{12}\beta_{21} - \beta_{11}\beta_{22}$ when $d_1\beta_{22} - d_2\beta_{11} > 0$ $(k \geq 1)$. According to $B_k \geq 0, C_k > 0$, if the condition **(H)** holds, we can get $B_k^2 > C_k^2$ when $k \geq 1$. Obviously, $A_k^2 - 2B_k - \beta_{22}^2 = (d_1^2 + d_2^2)k^4 - 2d_1\beta_{11}k^2 + \beta_{11}^2 - \beta_{22}^2$; if $16(d_1^2 + d_2^2) - 8d_1\beta_{11} + a_{11}^2 - \beta_{22}^2 > 0$, that is, $\beta_{22}^2 - \beta_{11}^2 < 16(d_1^2 + d_2^2) - 8d_1\beta_{11}$, then (2.14) with $k \geq 2$ has no positive roots.

Clearly, if $d_1^2 + d_2^2 - 2d_1\beta_{11} + \beta_{11}^2 - \beta_{22}^2 < 0$, that is, $\beta_{22}^2 - \beta_{11}^2 > d_1^2 + d_2^2 - 2d_1\beta_{11}$, and $\left[A_1^2 - 2B_1^2 - \beta_{22}^2\right]^2 - 4(B_1^2 - C_1^2) \geq 0$, then (2.14) with $k = 1$ has at least one positive root ω_1. From (2.10), (2.11), **(H)** and (2.14), we have

$$\omega_1 = \sqrt{\frac{-(A_1^2 - 2B_1 - \beta_{22}^2) \pm \sqrt{(A_1^2 - 2B_1 - \beta_{22}^2)^2 - 4(B_1^2 - C_1^2)}}{2}}. \tag{2.15}$$

That is, it has ω_1 such that (2.8) with $k = 1$ has purely imaginary eigenvalues $\pm i\omega_1$. Thus the proof is complete. □

According to (2.12),(2.13) and Lemma 2.1, we get

$$\tau_j = \frac{1}{\omega_1}\left(\arccos\frac{(A_1\beta_{22}+C_1)\omega_1^2 - B_1C_1}{C_1^2+\beta_{22}^2\omega_1^2} + 2j\pi\right), \qquad j = 0,1,\dots \tag{2.16}$$

Lemma 2.2. *Let $\lambda(\tau) = \mu(\tau) \pm i\omega(\tau)$ be the root of (2.8) with $k = 1$ near $\tau = \tau_j$ satisfying $\mu(\tau_j) = 0,\ \omega(\tau_j) = \omega_1,\ j = 0,1,\dots$ Then, the following transversality condition holds*

$$\operatorname{sign}\operatorname{Re}\left[\frac{d\lambda}{d\tau}\right]_{\tau=\tau_j} \neq 0.$$

Proof. Taking the derivative for equation (2.8) with respect to τ at τ_j, we have

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{(2\lambda+A_1)e^{\lambda\tau} - \beta_{11}}{(C_1-\beta_{11}\lambda)\lambda} - \frac{\tau}{\lambda}. \tag{2.17}$$

From (2.17), we get

$$\begin{aligned}
\operatorname{sign}\operatorname{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_j} &= \operatorname{sign}\operatorname{Re}\left[\frac{(2\lambda+A_1)e^{\lambda\tau}-\beta_{22}}{(C_1-\beta_{22}\lambda)\lambda} - \frac{\tau}{\lambda}\right]_{\tau=\tau_j} \\
&= \frac{(A_1\cos\omega_1\tau_j - 2\omega_1\sin\omega_1\tau_j - \beta_{22})\beta_{11}\omega_1^2}{\left(\beta_{22}^2\omega_1^2+C_1^2\right)\omega_1^2} \\
&\quad + \frac{(A_1\sin\omega_1\tau_j + 2\omega_1\cos\omega_1\tau_j)C_1\omega_1}{\left(\beta_{22}^2\omega_1^2+C_1^2\right)\omega_1^2} \\
&= \frac{1}{\beta_{22}^2\omega_1^2+C_1^2}\left[A_1^2 - 2B_1 - \beta_{22}^2 + 2\omega_1^2\right] \\
&= \frac{1}{\beta_{22}^2\omega_1^2+C_1^2}\left[\sqrt{(A_1^2-2B_1-\beta_{22}^2)^2 - 4[B_1^2-C_1^2]}\right] \neq 0,
\end{aligned}$$

then

$$\operatorname{sign}\operatorname{Re}\left[\frac{d\lambda}{d\tau}\right]_{\tau=\tau_j} \neq 0.$$

Thus the proof is complete. □

Therefore, we have the following conclusions,

Theorem 2.3. *Suppose that the conditions in Lemma 2.1 are satisfied. Let τ_j be defined as in (2.16).*

(i) If $\tau \in [0,\tau_0)$, then the positive constant steady-state solution $E^* = (u^*, v^*)$ of system (1.2) is stable and unstable when $\tau > \tau_0$.

(ii) System (1.2) can have spatially nonhomogeneous periodic solutions at the positive constant steady-state solution $E^* = (u^*, v^*)$ when $\tau = \tau_j$.

3 Stability of spatially periodic solutions

In the previous section, we have obtained the existence of spatially periodic solutions of system (1.2) when the parameter τ crosses through the critical value τ_j $(j = 0,1,2,\dots)$. In this section, we shall study the stability of periodic solutions by applying the normal form theory of partial functional differential equations developed by [15,29].

Normalizing the delay τ in system (2.2) by the time-scaling $t \to \frac{t}{\tau}$, (2.2) is transformed into

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \tau\{d_1 \triangle u + \beta_{11}u(t) + \beta_{12}v(t) + \beta_{13}u^2(t) + \beta_{14}u(t)v(t), \\ \frac{\partial v(t,x)}{\partial t} = \tau\Big\{d_2 \triangle v + \beta_{21}u(t-1) + \beta_{22}v(t-1) + \sum\limits_{i+j+k\geq 2} \frac{1}{i!j!k!} f_{ijk} u^i(t-1)v^j(t-1)v^k\Big\}, \end{cases} \tag{3.1}$$

where f_{ijk} is defined by (2.2). Let $\tau = \tau_j + \alpha$, then, (3.1) can be written in abstract form in $\mathcal{C} = C([-1,0] : X)$ as

$$\frac{d}{dt}U(t) = (\tau_j + \alpha)d\Delta U(t) + L(\tau_j)(U_t) + F(U_t, \alpha), \tag{3.2}$$

where $d = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$, $L(\alpha)(\cdot) : \mathcal{C} \to X$, $F(\cdot, \alpha) : \mathcal{C} \to X$ are given by

$$L(\alpha)(\varphi) = (\tau_j + \alpha) \begin{pmatrix} \beta_{11}\varphi_1(0) + \beta_{12}\varphi_2(0) \\ \beta_{21}\varphi_1(-1) + \beta_{22}\varphi_2(-1) \end{pmatrix},$$

$$F(\varphi, \alpha) = \alpha\Delta\varphi(0) + L(\alpha)\varphi + f(\varphi, \alpha),$$

and

$$f(\varphi, \alpha) = (\tau_j + \alpha) \begin{pmatrix} \beta_{13}\varphi_1^2(0) + \beta_{14}\varphi_1(0)\varphi_2(0), \\ \sum_{i+j+k\geq 2} \frac{1}{ijk} f_{ijk}\varphi_1^i(-1)\varphi_2^j(-1)\varphi_2^k(0) \end{pmatrix},$$

for $\varphi = (\varphi_1, \varphi_2)^T \in \mathcal{C}$.

Linearizing (3.2) at $(0,0)$ leads to the following linear equation

$$\frac{d}{dt}U(t) = \tau_j d\Delta U(t) + L(\tau_j)(U_t). \tag{3.3}$$

It is easy to see from the discussions in the previous section that (2.8) has two purely imaginary eigenvalues $\pm i\omega_1$ (ω_1 is defined by (2.15)).

Let $\Lambda_1 = \{-i\omega_1, i\omega_1\}$, consider the following FDE on $C([-1,0], \mathbb{R}^2)$

$$\dot{z}(t) = L(\tau_j)(z_t), \tag{3.4}$$

that is,

$$\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = (\tau_j + \alpha)\left\{\begin{pmatrix} \beta_{11} & \beta_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \beta_{21} & \beta_{22} \end{pmatrix}\begin{pmatrix} z_1(t-1) \\ z_2(t-1) \end{pmatrix}\right\}.$$

As is well known, $L(\tau_j)$ is a continuous linear function mapping $C([-1,0], \mathbb{R}^2)$ into $\mathbb{R}^2$. According to the Riesz representation theorem, there exists a 2×2 matrix function $\eta(\theta, \tau)$ $(-1 \leq \theta \leq 0)$, whose elements are of bounded variation such that

$$L(\tau_j)(\phi) = \int_{-1}^{0} d\eta(\theta, \tau_j)\,\phi(\theta) \quad \text{for } \phi \in C. \tag{3.5}$$

Thus, we can choose

$$\eta(\theta, \tau_j) = (\tau_j + \alpha)\begin{pmatrix} \beta_{11} & \beta_{12} \\ 0 & 0 \end{pmatrix}\delta(\theta) - (\tau_j + \alpha)\begin{pmatrix} 0 & 0 \\ \beta_{21} & \beta_{22} \end{pmatrix}\delta(\theta + 1), \tag{3.6}$$

where $\delta(0) = 1$, $\delta(\theta) = 0$, $-1 \le \theta < 0$, then (3.5) is satisfied.

If ϕ is any given function in $C([-1,0],\mathbb{R}^2)$ and $u(\phi)$ is the unique solution of the linear equation (3.3) with the initial function ϕ at zero, then the solution operator

$$T(t) : C([-1,0],\mathbb{R}^2) \to C([-1,0],\mathbb{R}^2)$$

is defined by

$$T(t)\phi = u_t(\phi), \qquad t \ge 0.$$

Let $A(\tau_j)$ denote the infinitesimal generator of the strongly continuous semigroup, according to [14], then,

$$A\left(\tau_j\right)\phi\left(\theta\right) = \begin{cases} d\phi\left(\theta\right)/d\theta, & \theta \in [-1,0)\,, \\ L(\tau_j)\left(\phi\right) \overset{\text{def}}{=} \int_{-1}^{0} d\eta\left(t,\tau_j\right)\phi\left(t\right), & \theta = 0, \end{cases} \tag{3.7}$$

where $\phi \in C^1\left([-1,0]\,,\mathbb{R}^2\right)$.

For $\psi \in C^1\left([0,1],(\mathbb{R}^2)^*\right)$, define

$$A^*\psi\left(s\right) = \begin{cases} -d\psi\left(s\right)/ds, & s \in (0,1]\,, \\ \int_{-1}^{0}\psi\left(-t\right)d\eta\left(t,\tau_j\right), & s = 0, \end{cases} \tag{3.8}$$

and a bilinear inner product of the Sobolev space $W^{2,2}(0,\pi)$.

$$\begin{aligned} \left(\psi\left(s\right),\phi\left(\theta\right)\right) &= \psi\left(0\right)\phi\left(0\right) - \int_{-1}^{0}\int_{\xi=0}^{\theta}\psi(\xi-\theta)d\eta(\theta)d\xi \\ &= \psi\left(0\right)\phi\left(0\right) - \tau_j\int_{-1}^{0}\psi\left(s+1\right)\begin{pmatrix} 0 & 0 \\ \beta_{21} & \beta_{22}\end{pmatrix}\phi\left(s\right)ds, \end{aligned}$$

where $\eta\left(\theta\right) = \eta\left(\theta,\tau_j\right)$ and A^* is the formal adjoint of $A\left(\tau_j\right)$.

Obviously, the characteristic equation of the linear operator $A\left(\tau_j\right)$ is (2.8) with $k = 1$. So, it is easy to see from Section 2 that $A(\tau_j)$ has a pair of simple purely imaginary eigenvalues $\pm i\omega_1$ and they are also eigenvalues of A^* since $A\left(\tau_j\right)$ and A^* are adjoint operators. Let $\mathcal{P}$ and $\mathcal{P}^*$ be the center spaces, that is, the generalized eigenspaces, of $A\left(\tau_j\right)$ and A^* associated with Λ_1, respectively, then $\mathcal{P}^*$ is the adjoint space of $\mathcal{P}$ and $\dim\mathcal{P} = \dim\mathcal{P}^*$=2.

In addition, according to [11,27], a few simple calculations, we can choose Φ and Ψ be the bases for $\mathcal{P}$ and $\mathcal{P}^*$, respectively. It is known that $\dot{\Phi} = \Phi B$, where B is the 2×2 diagonal matrix $B = \left(\begin{smallmatrix} i\omega_1 & 0 \\ 0 & i\omega_1\end{smallmatrix}\right)$.

Let $\Phi = (\Phi_1,\Phi_2)$ and $\Psi = (\Psi_1,\Psi_2)^T$, where

$$\Phi_1(\theta) = e^{i\omega_1\theta}\left(1, \frac{i\omega_1 + d_1 - \beta_{11}}{\beta_{12}}\right)^{\mathrm{T}}, \qquad \Phi_2(\theta) = \overline{\Phi_1(\theta)}, \qquad -1 \le \theta \le 0,$$

$$\Psi_1(s) = \frac{1}{\rho}\left(1, -\frac{i\omega_1 - d_1 + \beta_{11}}{\beta_{21}e^{i\omega_1\tau_j}}\right)^{\mathrm{T}} e^{-i\omega_1 s}, \qquad \Psi_2(s) = \overline{\Psi_1(s)}, \qquad 0 \le s \le 1,$$

$$\overline{\rho} = \frac{1}{1 + \varrho v - \tau_j(-d_1 + \beta_{11} + \beta_{21}\varrho + \beta_{12}v + (\beta_{22} - d_2)\varrho v},$$

where $\varrho = \frac{i\omega_1+d_1-\beta_{11}}{\beta_{12}}$, $v = -\frac{i\omega_1-d_1+\beta_{11}}{\beta_{21}e^{i\omega_1\tau_j}}$. Let $f_1 = (\beta_1^1, \beta_1^2)$, $c \cdot f_1$ be defined by $c \cdot f_1 = c_1\beta_1^1 + c_2\beta_1^2$ for $c = (c_1, c_2)^T \in \mathbb{R}^2$ and $(\psi \cdot f_1)(\theta) = \psi(\theta) \cdot f_1$ for $\theta \in [-1, 0]$. Then the center space of linear equation (3.3) is given by $\mathcal{P}_{CN}\mathcal{C}$, where

$$P_{CN}\varphi = \Phi\left(\Psi, \langle\varphi, f_1\rangle\right) \cdot f_1, \qquad \varphi \in \mathcal{C}, \tag{3.9}$$

and $\mathcal{C} = \mathcal{P}_{CN}\mathcal{C} \oplus \mathcal{P}_Q\mathcal{C}$, here $\mathcal{P}_Q\mathcal{C}$ denotes the complementary subspace of $\mathcal{P}_{CN}\mathcal{C}$ in $\mathcal{C}$.

Let A_{τ_j} be defined by:

$$A_{\tau_j}\varphi(\theta) = \dot{\varphi}(\theta) + X_1(\theta)\left[\tau_j\Delta\varphi(0) + L_*(\tau_j)(\varphi(\theta)) - \dot{\varphi}(0)\right], \qquad \varphi \in \mathcal{C},$$

where X_1: $[-1, 0] \to B(X, X)$ is given by

$$X_1 = \begin{cases} 0, & \theta \in [-1, 0), \\ I, & \theta = 0. \end{cases}$$

Then the infinitesimal generator A_{τ_j} induced by the solution of (3.3) and (3.2) can be rewritten as the following operator differential equation

$$\dot{U}_t = A_{\tau_j}U_t + X_1F(U_t, \alpha). \tag{3.10}$$

Using the decomposition $\mathcal{C} = P_{CN} \oplus \mathcal{P}_Q\mathcal{C}$ and (3.9), the solution of (3.1) can be written as

$$U_t = \Phi\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_1 + h(x_1, x_2, \alpha),$$

where $(x_1, x_2)^T = (\Psi, \langle U_t, f_1\rangle)$, and $h(x_1, x_2, \alpha) \in \mathcal{P}_Q\mathcal{C}$ with $h(0, 0, 0) = Dh(0, 0, 0) = 0$.

Thus, the flow on the center manifold for (3.2) can be described as

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & \omega_1 \\ -\omega_1 & 0 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \Psi(0)F(0, x_1(t), x_2(t)),$$

where

$$F(0, x_1(t), x_2(t)) = \left\langle f\left(\alpha, \Phi\left(\begin{smallmatrix} x_1(t) \\ x_2(t) \end{smallmatrix}\right) \cdot f_1 + h(\alpha, x_1(t), x_2(t))\right),\ f_1\right\rangle.$$

Let $z = x_1 - ix_2$, and $\Psi(0) = (\Psi_1(0), \Psi_2(0))^T$, when $\alpha = 0$, then z satisfies

$$\dot{z} = i\omega_1 z + g(z, \bar{z}), \tag{3.11}$$

where

$$g(z, \bar{z}) = (\Psi_1(0) - i\Psi_2(0))\left\langle f\left(0, \frac{1}{2}\Phi\left(\begin{smallmatrix} z+\bar{z} \\ (z-\bar{z})i \end{smallmatrix}\right) \cdot f_1 + w(z, \bar{z})\right),\ f_1\right\rangle,$$

$$w(z, \bar{z}) = h\left(0, \frac{z+\bar{z}}{2}, \frac{(z-\bar{z})i}{2}\right), \tag{3.12}$$

$$w(z, \bar{z}) = w_{20}\frac{z^2}{2} + w_{11}z\bar{z} + w_{02}\frac{\bar{z}^2}{2} + w_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{3.13}$$

Noticing that $p_1 = \Phi_1 + i\Phi_2$, $p_2 = \overline{p}_1$, therefore, solutions of (3.10) can be rewritten as

$$U_t = \frac{1}{2}\Phi\left(\begin{smallmatrix} (z+\bar{z}) \\ i(z-\bar{z}) \end{smallmatrix}\right) \cdot f_1 + w(z, \bar{z}) = \frac{1}{2}(p_1 z + p_2\bar{z}) \cdot f_1 + w(z, \bar{z}). \tag{3.14}$$

In addition, (3.11) can be rewritten as the following form

$$\dot{z} = i\omega_1 z + g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \tag{3.15}$$

Let

$$g(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots.$$

From (3.12), we have

$$\begin{aligned}
&\langle F(U_t,0), f_1\rangle \\
&= \frac{\tau_j}{4}\begin{pmatrix} \left(\varrho\beta_{14} + \frac{1}{2}\beta_{13}\right)z^2 \\ e^{-2i\omega_1\tau_j}(\varrho f_{110} + e^{i\omega_1\tau_j}\varrho f_{101} + e^{i\omega_1\tau_j}\varrho^2 f_{011}^{(2)} + \frac{1}{2}f_{200} + \frac{1}{2}\varrho^2 f_{020})z^2 \end{pmatrix} \\
&\quad + \frac{\tau_j}{4}\begin{pmatrix} [(\overline{\varrho}+\varrho)\beta_{14} + \beta_{13}]\, z\overline{z} \\ \left[(\overline{\varrho}+\varrho)f_{110} + e^{-i\omega_1\tau_j}\overline{\varrho}(f_{101} + \varrho f_{011}) + e^{i\omega_1\tau_j}\varrho(f_{101} + \overline{\varrho}f_{011}) + f_{200} + \varrho\overline{\varrho}f_{020}\right] z\overline{z} \end{pmatrix} \\
&\quad + \frac{\tau_j}{4}\begin{pmatrix} (\overline{\varrho}\beta_{14} + \beta_{13})\,\overline{z}^2 \\ e^{2i\omega_1\tau_j}(\overline{\varrho}f_{110} + e^{-i\omega_1\tau_j}\overline{\varrho}f_{101} + e^{-i\omega_1\tau_j}\overline{\varrho}^2 f_{011} + \frac{1}{2}f_{200} + \frac{1}{2}\overline{\varrho}^2 f_{020})\overline{z}^2 \end{pmatrix} \\
&\quad + \frac{\tau_j}{2}\begin{pmatrix} \left\langle \beta_{14}\left(w_{11}^2(0) + \frac{w_{20}^2(0)}{2} + w_{11}^1(0)\varrho + \frac{w_{20}^1(0)}{2}\overline{\varrho}\right) + \beta_{13}\left(w_{11}^1(0) + \frac{w_{20}^1(0)}{2}\right), 1\right\rangle \\
\Bigg\langle f_{110}e^{-i\omega_1\tau_j}\left(w_{11}^2(-1) + e^{2i\omega_1\tau_j}\frac{w_{20}^2(-1)}{2} + w_{11}^1(-1)\varrho + e^{2i\omega_1\tau_j}\frac{w_{20}^1(-1)}{2}\overline{\varrho}\right) \\
\quad + f_{101}\left(e^{-i\omega_1\tau_j}w_{11}^2(0) + e^{i\omega_1\tau_j}\frac{w_{20}^2(0)}{2} + w_{11}^1(-1)\varrho + \frac{w_{20}^1(-1)}{2}\overline{\varrho}\right) \\
\quad + f_{011}\left(e^{-i\omega_1\tau_j}w_{11}^2(0)\varrho + e^{i\omega_1\tau_j}\frac{w_{20}^2(0)}{2}\overline{\varrho} + w_{11}^2(-1)\varrho + \frac{w_{20}^2(-1)}{2}\overline{\varrho}\right) \\
\quad + \frac{1}{2}f_{200}\left(2e^{-i\omega_1\tau_j}w_{11}^1(-1) + e^{i\omega_1\tau_j}w_{20}^1(-1)\right) \\
\quad + \frac{1}{2}f_{020}\left(2e^{-i\omega_1\tau_j}w_{11}^2(-1)\varrho + e^{i\omega_1\tau_j}w_{20}^2(-1)\varrho\right),\ 1\Bigg\rangle \end{pmatrix} z^2\overline{z} \\
&\quad + \cdots,
\end{aligned}$$

where

$$\left\langle w_{ij}^n(\theta), 1\right\rangle = \frac{1}{\pi}\int_0^\pi w_{ij}^n(\theta)(x)dx, \qquad i+j=2,\ n=1,2.$$

Noting that $\Psi_1(0) - i\Psi_2(0) = \frac{2(1-i\omega_1)}{(1+\omega_1^2)(1+\varrho v)}(1,v)$. Therefore,

$$\begin{aligned}
g_{20} &= \frac{\tau_j(1-i\omega_1)}{(1+\omega_1^2)(1+\varrho v)} \\
&\quad \times \left[\left(\varrho\beta_{14} + \frac{1}{2}\beta_{13}\right) + e^{-2i\omega_1\tau_j}\left(\varrho f_{110} + e^{i\omega_1\tau_j}\varrho f_{101} + e^{i\omega_1\tau_j}\varrho^2 f_{011} + \frac{1}{2}f_{200} + \frac{1}{2}\varrho^2 f_{020}\right)v\right], \\
g_{11} &= \frac{\tau_j(1-i\omega_1)}{(1+\omega_1^2)(1+\varrho v)} \\
&\quad \times \Big\{[(\overline{\varrho}+\varrho)\beta_{14} + \beta_{13}] \\
&\qquad + \left[(\overline{\varrho}+\varrho)f_{110} + e^{-i\omega_1\tau_j}\overline{\varrho}(f_{101} + \varrho f_{011}) + e^{i\omega_1\tau_j}\varrho(f_{101}^{(2)} + \overline{\varrho}f_{011}) + f_{200} + \varrho\overline{\varrho}f_{020}\right]v\Big\}, \\
g_{02} &= \overline{g_{20}},
\end{aligned}$$

$$
\begin{aligned}
g_{21} = {} & \frac{2\tau_j(1-i\omega_1)}{(1+\omega_1^2)(1+\varrho v)} \\
& \times \Bigg[\left\langle \beta_{14}\left(w_{11}^2(0)+\frac{w_{20}^2(0)}{2}+w_{11}^1(0)\varrho+\frac{w_{20}^1(0)}{2}\overline{\varrho}\right)+\beta_{13}\left(w_{11}^1(0)+\frac{w_{20}^1(0)}{2}\right),\ 1\right\rangle \\
& \quad + \Bigg\langle f_{110}e^{-i\omega_1\tau_j}\left(w_{11}^2(-1)+e^{2i\omega_1\tau_j}\frac{w_{20}^2(-1)}{2}+w_{11}^1(-1)\varrho+e^{2i\omega_1\tau_j}\frac{w_{20}^1(-1)}{2}\overline{\varrho}\right) \\
& \qquad + f_{101}\left(e^{-i\omega_1\tau_j}w_{11}^2(0)+e^{i\omega_1\tau_j}\frac{w_{20}^2(0)}{2}+w_{11}^1(-1)\varrho+\frac{w_{20}^1(-1)}{2}\overline{\varrho}\right) \\
& \qquad + f_{011}\left(e^{-i\omega_1\tau_j}w_{11}^2(0)\varrho+e^{i\omega_1\tau_j}\frac{w_{20}^2(0)}{2}\overline{\varrho}+w_{11}^2(-1)\varrho+\frac{w_{20}^2(-1)}{2}\overline{\varrho}\right) \\
& \qquad + \frac{1}{2}f_{200}\left(2e^{-i\omega_1\tau_j}w_{11}^1(-1)+e^{i\omega_1\tau_j}w_{20}^1(-1)\right) \\
& \qquad + \frac{1}{2}f_{020}\left(2e^{-i\omega_1\tau_j}w_{11}^2(-1)\varrho+e^{i\omega_1\tau_j}w_{20}^2(-1)\varrho\right),\ 1\Bigg\rangle v\Bigg].
\end{aligned}
$$

To determine the properties of the Hopf bifurcation, we need to compute w_{ij}, $i+j=2$, since $w_{20}(\theta)$ and $w_{11}(\theta)$ for ($\theta \in [-1,0]$) appear in g_{21}.

In addition, we can rewrite (3.12) as

$$\dot{w}\,(z,\overline{z}) = w_{20}z\dot{z} + w_{11}(\dot{z}\overline{z}+z\dot{\overline{z}}) + w_{02}\overline{z}\dot{\overline{z}} + \cdots \tag{3.16}$$

and

$$A_{\tau_j}w = A_{\tau_j}w_{20}\frac{z^2}{2} + A_{\tau_j}w_{11}z\overline{z} + A_{\tau_j}w_{02}\frac{\overline{z}^2}{2} + \cdots. \tag{3.17}$$

According to [29], we can know

$$\dot{w} = A_{\tau_j}w + H(z,\overline{z}), \tag{3.18}$$

where

$$H(z,\overline{z}) = H_{20}\frac{z^2}{2} + H_{11}z\overline{z} + H_{02}\frac{\overline{z}^2}{2} + \cdots \tag{3.19}$$

and $H_{ij} \in \mathcal{P}_Q\mathcal{C}, i+j=2$.

Thus, by using the chain rule

$$\dot{w} = \frac{\partial w(z,\overline{z})}{\partial z}\dot{z} + \frac{\partial w(z,\overline{z})}{\partial \overline{z}}\dot{\overline{z}}.$$

And according to (3.14) and (3.18), we can obtain

$$\begin{cases} (2i\omega_1 - A_{\tau_j})w_{20} = H_{20}, \\ -A_{\tau_j}w_{11} = H_{11}, \\ (-2i\omega_1 - A_{\tau_j})w_{02} = H_{02}. \end{cases} \tag{3.20}$$

Noticing that A_{τ_j} has only two eigenvalues $\pm i\omega_1$, therefore, (3.20) has the unique solution w_{ij} $(i+j=2)$ in $\mathcal{P}_Q\mathcal{C}$ and

$$\begin{cases} w_{20} = (2i\omega_1 - A_{\tau_j})^{-1}H_{20}, \\ w_{11} = -A_{\tau_j}^{-1}H_{11}, \\ w_{02} = (-2i\omega_1 - A_{\tau_j})^{-1}H_{02}. \end{cases} \tag{3.21}$$

From (3.19), then, for $-1 \le \theta < 0$,

$$\begin{aligned} H(z,\overline{z}) &= -\Phi(\theta)\Psi(0)\left\langle f\left(U_t,0\right),f_1\right\rangle \cdot f_1 \\ &= -\left(\frac{p_1(\theta)+p_2(\theta)}{2},\frac{p_1(\theta)-p_2(\theta)}{2i}\right)\begin{pmatrix}\Psi_1(0)\\ \Psi_2(0)\end{pmatrix}\left\langle f\left(U_t,0\right),f_1\right\rangle \cdot f_1 \\ &= -\frac{1}{2}\left[p_1(\theta)\left(\Psi_1(0)-i\Psi_2(0)\right)+p_2(\theta)\left(\Psi_1(0)+i\Psi_2(0)\right)\right]\left\langle f\left(U_t,0\right),f_1\right\rangle \cdot f_1 \\ &= -\frac{1}{4}\left[g_{20}p_1(\theta)+\overline{g}_{02}p_2(\theta)\right]z^2\cdot f_1+\cdots. \end{aligned}$$

So, for $-1 \le \theta < 0$,

$$H_{20}(\theta) = -\frac{1}{2}\left[g_{20}p_1(\theta)+\overline{g}_{02}p_2(\theta)\right]\cdot f_1, \qquad H_{11}(\theta)=0.$$

When $\theta = 0$,

$$\begin{aligned} H_{20}(0) &= \frac{i\tau_j}{2}\begin{pmatrix}\varrho\beta_{14}+\frac{1}{2}\beta_{13}\\ e^{-2i\omega_1\tau_j}\left(\varrho f_{110}+e^{i\omega_1\tau_j}\varrho f_{101}+e^{i\omega\tau_j}\varrho^2 f_{011}+\frac{1}{2}f_{200}+\frac{1}{2}\varrho^2 f_{020}\right)\end{pmatrix} \\ &\quad -\frac{1}{2}\left[g_{20}p_1(0)+\overline{g}_{02}p_2(0)\right]\cdot f_1, \\ H_{11}(0) &= 0. \end{aligned}$$

Using the definition of A_{τ_j}, for $-1 \le \theta < 0$, we have

$$\dot{w}_{20}(\theta) = 2i\omega_1 w_{20}(\theta)+\frac{1}{2}\left[g_{20}p_1(\theta)+\overline{g}_{02}p_2(\theta)\right]\cdot f_1, \qquad -1\le\theta<0.$$

Note that $p_1(\theta) = p_1(0)e^{i\omega_1\theta}$, $-1 \le \theta \le 0$, hence

$$w_{20}(\theta) = \frac{i}{2}\left[\frac{g_{20}}{\omega_1}p_1(\theta)+\frac{\overline{g}_{02}}{3\omega_1}p_2(\theta)\right]\cdot f_1 + e^{2i\omega_1\theta}E, \qquad -1\le\theta<0, \tag{3.22}$$

and

$$E = w_{20}(0)-\frac{i}{2}\left[\frac{g_{20}}{\omega_1}p_1(0)+\frac{\overline{g}_{02}}{3\omega_1}p_2(0)\right]\cdot f_1. \tag{3.23}$$

By the definition of A_{τ_j} again, and combining (3.17) and (3.20), we have

$$\begin{aligned} &2i\omega_1\left[\frac{ig_{20}}{2\omega_1}p_1(0)\cdot f_1+\frac{i\overline{g}_{02}}{6\omega_1}p_2(0)\cdot f_1+E\right] \\ &\quad -\tau_j\Delta\left[\frac{ig_{20}}{2\omega_1}p_1(0)\cdot f_1+\frac{i\overline{g}_{02}}{6\omega_1}p_2(0)\cdot f_1+E\right] \\ &\quad -L^*(\tau_j)\left[\frac{ig_{20}}{2\omega_1}p_1(\theta)\cdot f_1+\frac{i\overline{g}_{02}}{6\omega_1}p_2(\theta)\cdot f_1+Ee^{2i\omega_1\theta}\right] \\ &= \frac{i\tau_j}{2}\begin{pmatrix}\varrho\beta_{14}+\frac{1}{2}\beta_{13}\\ e^{-2i\omega_1\tau_j}\left(\varrho f_{110}+e^{i\omega_1\tau_j}\varrho f_{101}+e^{i\omega_1\tau_j}\varrho^2 f_{011}+\frac{1}{2}f_{200}+\frac{1}{2}\varrho^2 f_{020}\right)\end{pmatrix} \\ &\quad -\frac{1}{2}\left[g_{20}p_1(0)+\overline{g}_{02}p_2(0)\right]\cdot f_1. \end{aligned}$$

For

$$\tau_j\Delta p_1(0)\cdot f_1 + L_*(\widetilde{\tau})(p_1(\theta)\cdot f_1) = i\omega_1 p_1(0)\cdot f_1,$$

$$\tau_j \Delta p_2(0) \cdot f_1 + L_*(\tau_j)(p_2(\theta) \cdot f_1) = -i\omega_1 p_2(0) \cdot f_1,$$

then

$$\begin{aligned} & 2i\omega_1 E - \tau_j \Delta E - L_*(\tau_j)(E e^{2i\omega_1 \theta}) \\ & \quad = \frac{i\tau_j}{2} \begin{pmatrix} \varrho\beta_{14} + \frac{1}{2}\beta_{13} \\ e^{-2i\omega_1\tau_j}\left(\varrho f_{110} + e^{i\omega_1\tau_j}\varrho f_{101} + e^{i\omega_1\tau_j}\varrho^2 f_{011} + \frac{1}{2} f_{200} + \frac{1}{2}\varrho^2 f_{020}\right) \end{pmatrix} \end{aligned}$$

From the above expression, we can see easily that

$$\begin{aligned} E &= \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \\ &= \frac{i}{2} \begin{pmatrix} -2i\omega_1 + \beta_{11} & \beta_{12} \\ \beta_{21} & -2i\omega_1 + \beta_{22} \end{pmatrix}^{-1} \\ &\quad \times \begin{pmatrix} \varrho\beta_{14} + \frac{1}{2}\beta_{13} \\ e^{-2i\omega_1\tau_j}\left(\varrho f_{110} + e^{i\omega_1\tau_j}\varrho f_{101} + e^{i\omega_1\tau_j}\varrho^2 f_{011} + \frac{1}{2} f_{200} + \frac{1}{2}\varrho^2 f_{020}\right) \end{pmatrix}. \end{aligned}$$

Thus, from the definition of g_{20} and g_{02} we see easily that

$$w_{20}(0) = \frac{i}{2}\left(\frac{g_{20}}{\omega_1} + \frac{\overline{g}_{02}}{3\omega_1}\right)\begin{pmatrix} 1 \\ \varrho \end{pmatrix} + E$$

and

$$w_{20}(-1) = \frac{1}{2}\left(\frac{g_{20}}{\omega_1} - \frac{\overline{g}_{02}}{3\omega_1}\right)\begin{pmatrix} 1 \\ \varrho \end{pmatrix} - E.$$

Therefore, g_{21} can be determined by the parameters.

In fact, by a transformation

$$z = \vartheta + a_{20}\frac{\vartheta^2}{2} + a_{11}\vartheta\overline{\vartheta} + a_{20}\frac{\overline{\vartheta}^2}{2} + \cdots,$$

where $a_{20} = \frac{g_{20}}{i\omega_1}, a_{11} = \frac{ig_{11}}{\omega_1}, a_{02} = \frac{ig_{02}}{3\omega_1}$. Then, (3.15) can be written as Poincaré normal form

$$\dot{\vartheta} = i\omega_1\vartheta + c_1(0)\vartheta|\vartheta|^2 + O(|\vartheta|^5),$$

where

$$c_1(0) = \frac{i}{2\omega_1}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}^2|}{3}\right) + \frac{g_{21}}{2}.$$

Thus, we can compute $\epsilon_2 = 2\,\mathrm{Re}(c_1(0))$. Hence, we have the following result.

Theorem 3.1. *If $\epsilon_2 < 0$, then the spatially periodic solutions are stable; if $\epsilon_2 > 0$, then the spatially periodic solutions are unstable.*

4 Conclusions and numerical simulations

In this paper, by studying the existence and stability of spatially periodic solutions for a delay Leslie–Gower diffusion system, we obtain that the system can generate the spatially nonhomogeneous periodic solutions when the diffusive rates are suitably small. We discover that system (1.2) have more abundant dynamic behavior than system (1.1), this indicates that diffusion plays a fundamental role in classifying the rich dynamics. In addition, we considered the stability of periodic solutions by applying the normal form theory of partial functional differential equations.

To illustrate the analytical results obtained, we give some numerical simulations and consider the following case of model (1.2)

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = 10\triangle u(t,x) + u(t,x)\left[1 - u(t,x)\right] - u(t,x)v(t,x), & t > 0,\ x \in (0,\pi) \\ \frac{\partial v(t,x)}{\partial t} = 0.01\triangle v(t,x) + v(t,x)\left[1 - \frac{v(t-\tau,x)}{2u(t-\tau,x)}\right], & t > 0,\ x \in (0,\pi), \\ u(t,x) = 0.2,\ v(t,x) = 0.2, & (t,x) \in [-\tau,0] \times (0,\pi), \end{cases} \tag{4.1}$$

which has a positive equilibrium $E^* = (0.3333, 0.6667)$. From (2.15), (2.16) and system (4.1), we know $\omega_1 = 1.0638$, $\tau_0 = 1.4796$. Therefore, we know from Theorem 2.3 that the positive equilibrium $E^* = (0.3333, 0.6667)$ is asymptotically stable when $\tau \in [0, 1.4796)$. These properties are illustrated by the numerical simulation in Figs. 1–10

When τ passes through the critical value τ_0, $E^* = (0.3333, 0.6667)$ loss its stability, a family of periodic solution bifurcates from equilibrium $E^* = (0.3333, 0.6667)$ which is depicted by the numerical simulation in Figs. 11–12. From Theorem 3.1 and system (4.1), we can compute $c_1(0) = -2.0788e + 0.03 - 6.9946e + 0.02i$ by the software package *Matlab R2009b*, and $\epsilon_2 = 2\,\mathrm{Re}(c_1(0)) < 0$. Therefore, the bifurcated periodic solutions are orbitally asymptotically stable on the center manifold. These properties are depicted by the numerical simulation in Figs. 11–12.

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Figures

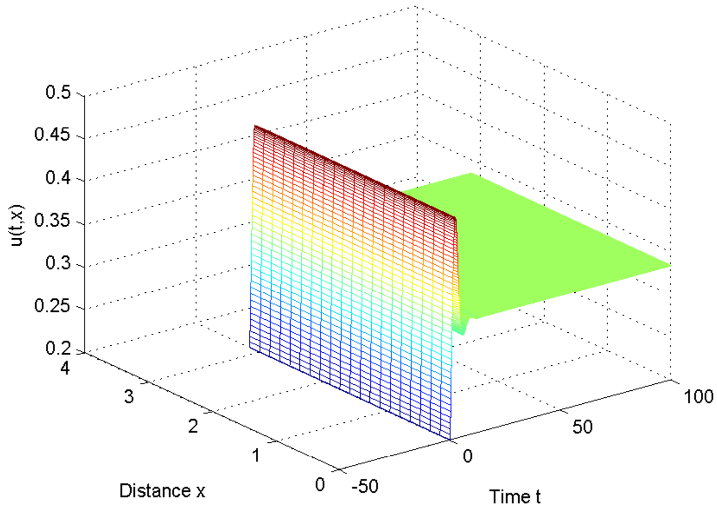

Figure 1: The trajectory graph(u(t,x)) of system (4.1) with $0.3 = \tau < \tau_0 = 1.4796$.

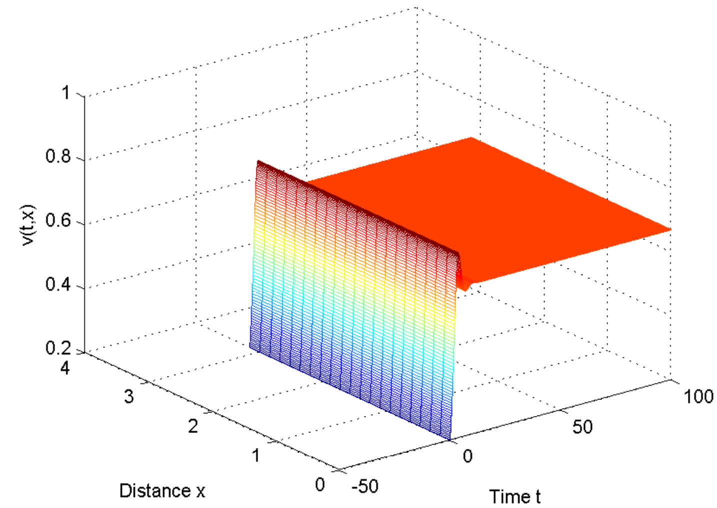

Figure 2: The trajectory graph(v(t,x)) of system (4.1) with $0.3 = \tau < \tau_0 = 1.4796$.

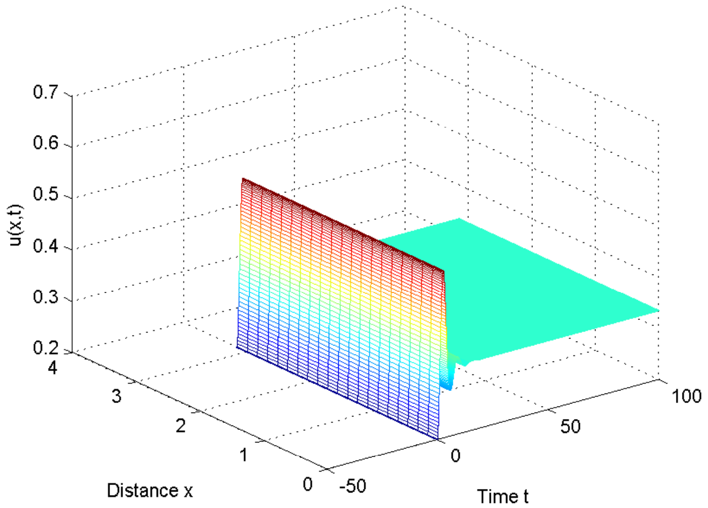

Figure 3: The trajectory graph(u(t,x)) of system (4.1) with $0.6 = \tau < \tau_0 = 1.4796$.

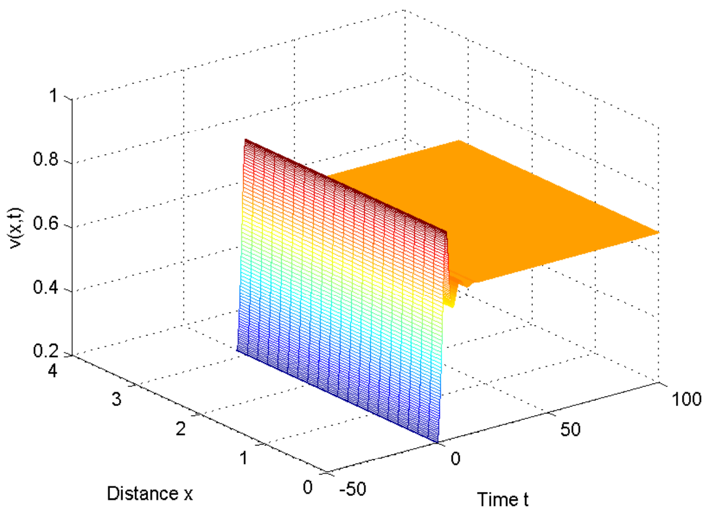

Figure 4: The trajectory graph(v(t,x)) of system (4.1) with $0.6 = \tau < \tau_0 = 1.4796$.

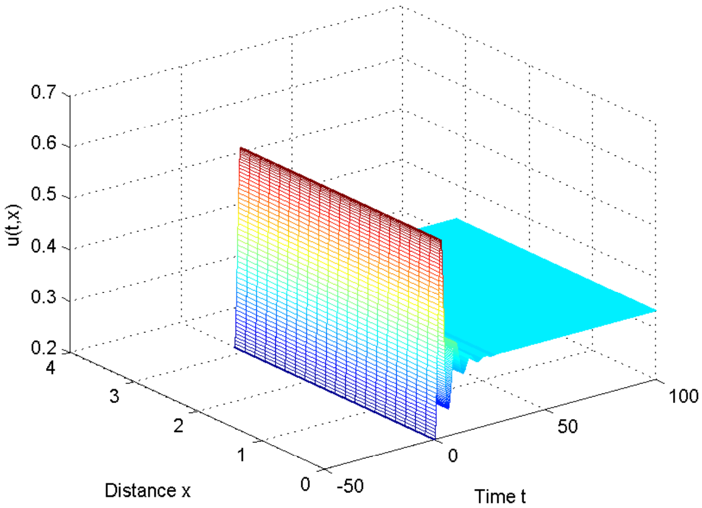

Figure 5: The trajectory graph(u(t,x)) of system (4.1) with $0.8 = \tau < \tau_0 = 1.4796$.

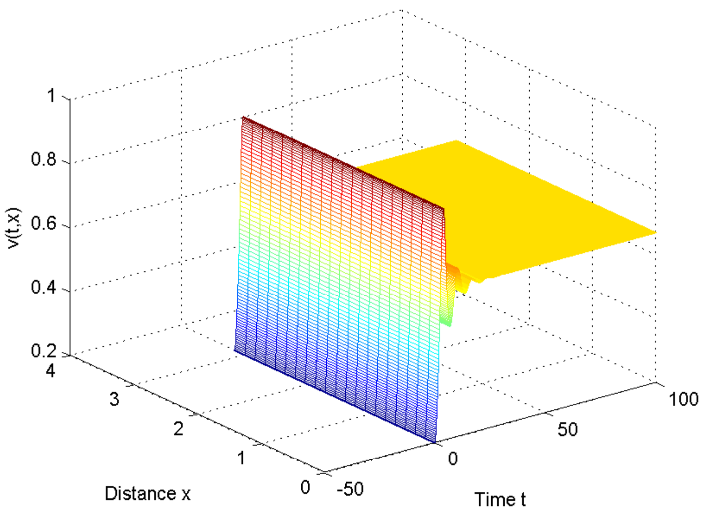

Figure 6: The trajectory graph(v(t,x)) of system (4.1) with $0.8 = \tau < \tau_0 = 1.4796$.

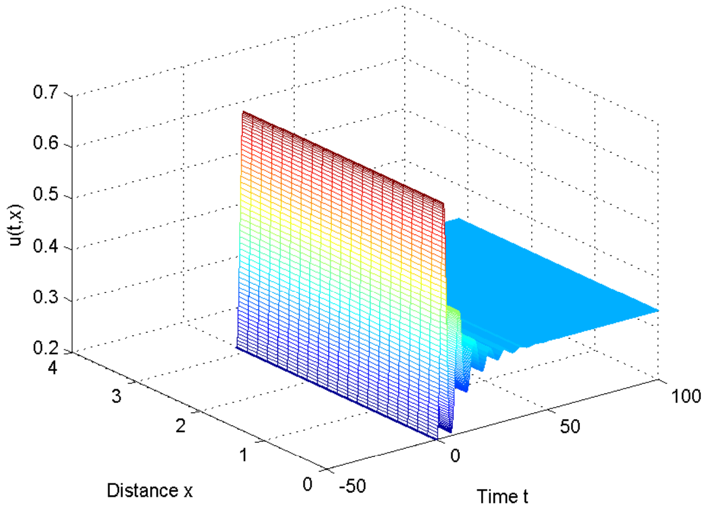

Figure 7: The trajectory graph(u(t,x)) of system (4.1) with $1.0 = \tau < \tau_0 = 1.4796$.

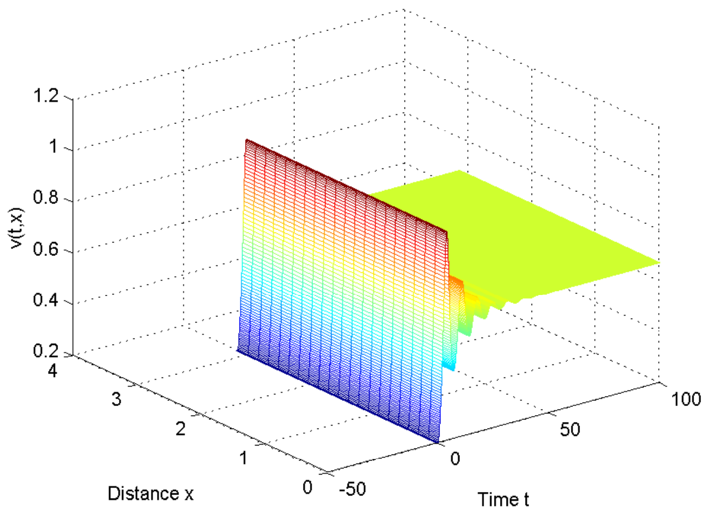

Figure 8: The trajectory graph(v(t,x)) of system (4.1) with $1.0 = \tau < \tau_0 = 1.4796$.

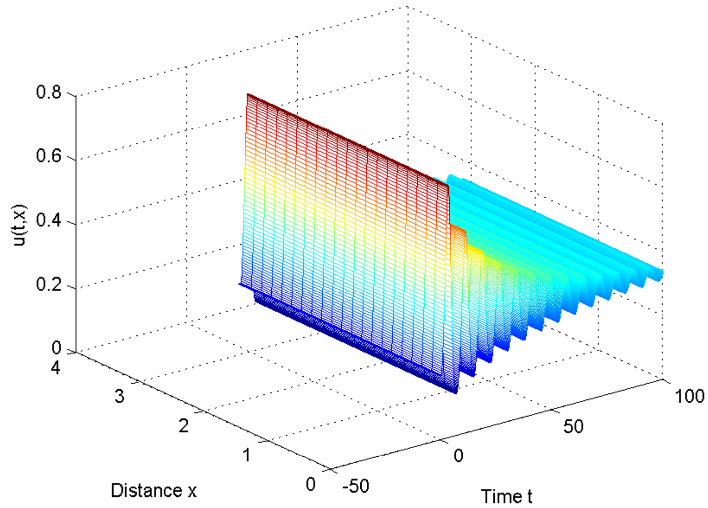

Figure 9: The trajectory graph(u(t,x)) of system (4.1) with $1.3 = \tau < \tau_0 = 1.4796$.

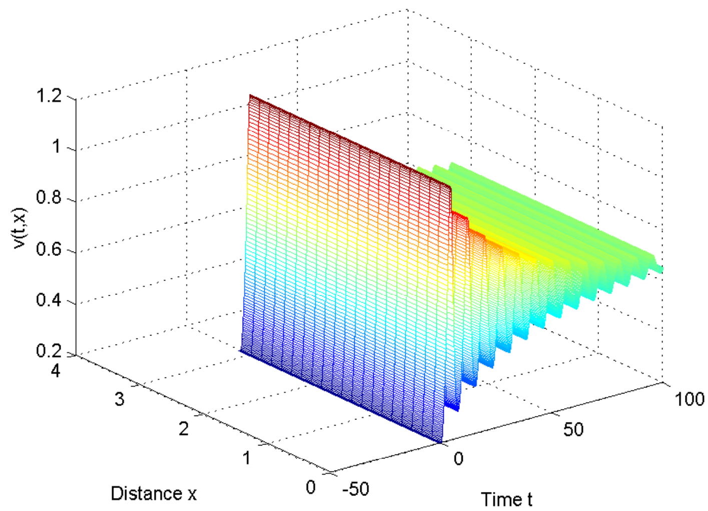

Figure 10: The trajectory graph(v(t,x)) of system (4.1) with $1.3 = \tau < \tau_0 = 1.4796$.

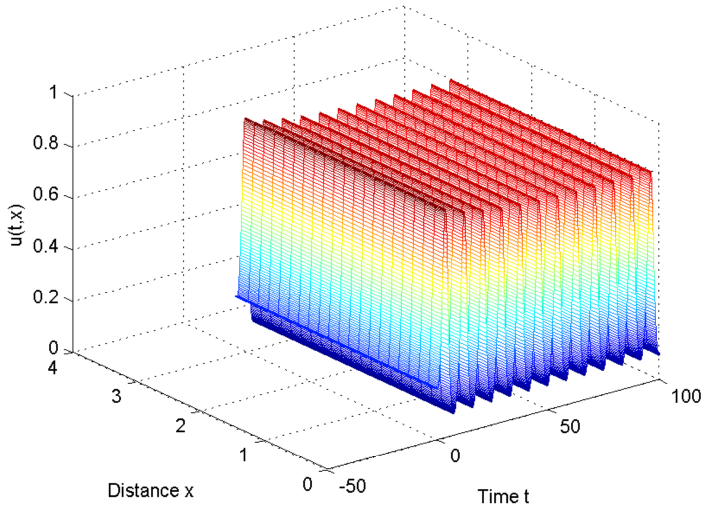

Figure 11: The trajectory graph(u(t,x)) of system (4.1) with $1.5 = \tau > \tau_0 = 1.4796$.

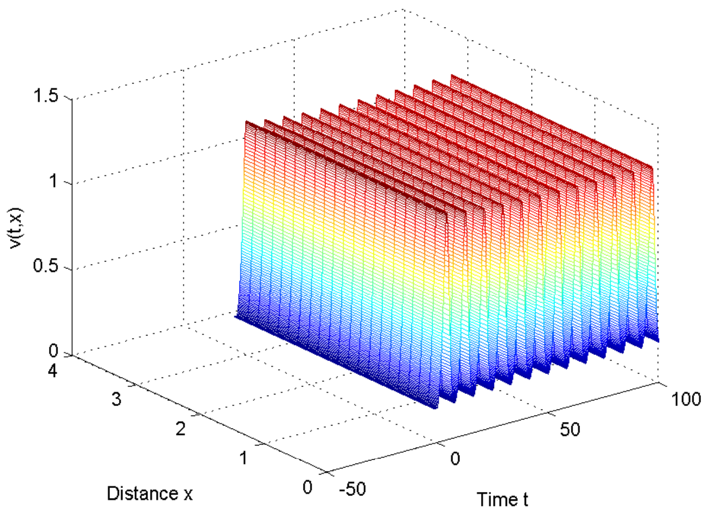

Figure 12: The trajectory graph(v(t,x)) of system (4.1) with $1.5 = \tau > \tau_0 = 1.4796$.