Electronic Journal of Qualitative Theory of Differential Equations
2016, No. 5, 1–18; doi: 10.14232/ejqtde.2016.1.5 http://www.math.u-szeged.hu/ejqtde/

On solvability of periodic boundary value problems for second order linear functional differential equations

Eugene Bravyi ✉

State National Research Polytechnical University of Perm, Komsomolsky pr. 29, Perm, 614990, Russia

Received 20 September 2015 appeared 5 February 2016
Communicated by Ivan Kiguradze

Abstract. The periodic boundary value problem for second order linear functional differential equations with pointwise restrictions (instead of integral ones) is considered. Sharp sufficient conditions for the solvability are obtained.

Keywords: periodic problem, functional differential equations.

2010 Mathematics Subject Classification: 34K06, 34K10, 34K13.

1 Introduction

In the last decades, periodic boundary problems for functional differential equations have attracted a lot of attention. First of all because of their meaningful interest for modeling real-life processes (see for instance [1,4,6,7,11–13,16,17,21,26,27,29,30,34,36] and references therein). The problem on the existence of periodic solution for linear functional differential equations is of interest by itself [13,17,21,33,35], but results concerning linear equations are often used to investigate periodic solutions to some kinds of nonlinear functional differential equations (for example, [7–9,23,24,34]). In many publications on functional differential equations [3–7,11–13,16–18,20–22,25,27,30,33–36], there are no specific restrictions on deviating arguments. Therefore, it is important to obtain optimal conditions for the existence of periodic solutions to linear functional differential equations that will be valid for all possible delays (or for more general deviating arguments). Some such conditions are obtained in [2,8–10,12,14,23,24,26,31,32] for various boundary value conditions under integral restrictions on the coefficients of the equations.

Here we research a rather common case when the linear functional operator T of the second-order functional differential equation is the difference of two positive operators: $T = T^+ - T^-$. A new class of sufficient conditions for the existence of periodic solutions is offered. For arbitrary given non-negative functions p^+, p^-, we find sharp sufficient conditions for the existence of periodic solutions to all functional differential equations with linear positive

✉ Email: bravyi@perm.ru

operators T^+, T^- such that $T^+\mathbf{1} = p^+$, $T^-\mathbf{1} = p^-$ ($\mathbf{1}$ is the unit function). To the best of our knowledge, for arbitrary functions p^+, p^-, such conditions are new.

Consider the periodic boundary value problem for a second order functional differential equation

$$\begin{cases} \ddot{x}(t) = \lambda(Tx)(t) + f(t) & \text{for almost all } t \in [0,1], \\ x(0) = x(1), \quad \dot{x}(0) = \dot{x}(1), \end{cases} \tag{1.1}$$

where λ is a real number, $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ is a linear bounded operator, $f \in \mathbf{L}[0,1]$. Here $\mathbf{C}[0,1]$ is the space of continuous functions $x : [0,1] \to \mathbb{R}$ with norm $\|x\| = \max_{t\in[0,1]} |x(t)|$, $\mathbf{L}[0,1]$ is the space of Lebesgue integrable functions $z : [0,1] \to \mathbb{R}$ with norm $\|z\| = \int_0^1 |z(t)|\, dt$.

A linear bounded operator $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ is called *positive* if it maps every non-negative function into an almost everywhere non-negative function. Let $\mathcal{S}$ be the set of all positive linear operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$.

Suppose p^+, $p^- \in \mathbf{L}[0,1]$ are given non-negative functions. Define the family of operators $\mathcal{S}(p^+, p^-)$ by the equality

$$\mathcal{S}(p^+, p^-) = \{T^+ - T^- : \ T^+ \in \mathcal{S},\ T^+\mathbf{1} = p^+,\ T^- \in \mathcal{S},\ T^-\mathbf{1} = p^-\},$$

where $\mathbf{1}$ is the *unit function*: $\mathbf{1}(t) \equiv 1$ for all $t \in [0,1]$.

Suppose a real number λ and a linear operator $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ are given. Boundary value problem (1.1) is called *uniquely solvable* if for all $f \in \mathbf{L}[0,1]$ there exists a unique absolutely continuous function $x : [0,1] \to \mathbb{R}$ with an absolutely continuous derivative $\dot{x}$ satisfying the first equation of (1.1) for almost all $t \in [0,1]$ and satisfying the periodic boundary conditions $x(0) = x(1)$, $\dot{x}(0) = \dot{x}(1)$.

Our main result is that we can find all real numbers λ such that problem (1.1) is uniquely solvable for all operators T from the operator family $\mathcal{S}(p^+, p^-)$. It allows to obtain some new sufficient conditions for the solvability. These conditions are unimprovable in a sense. It means that if our conditions are not fulfilled, then there exists an operator $T \in \mathcal{S}(p^+, p^-)$ such that boundary value problem (1.1) is not uniquely solvable.

Suppose two different non-negative numbers $\mathcal{P}^+$, $\mathcal{P}^-$ are given. Sharp sufficient conditions for the solvability of (1.1) for all operators T from the unify of sets $\mathcal{S}(p^+, p^-)$ with all non-negative p^+, p^- satisfying the equalities

$$\int_0^1 p^+(s)\, ds = \mathcal{P}^+, \qquad \int_0^1 p^-(s)\, ds = \mathcal{P}^-,$$

are obtained in [10] (see condition (2.7)). However, if we consider the boundary value problem (1.1) only for the operator family $\mathcal{S}(p^+, p^-)$ with given non-negative functions p^+, p^-, we can essentially improve the results (see Example 2.6).

Here in Section 2, Theorems 2.1, 2.2 contain conditions for the unique solvability of (1.1) for all operators T from the family $\mathcal{S}(p^+, p^-)$ with arbitrary non-negative functions p^+, p^-. Further, we refine these results when p^- is the zero function, and the function p^+ has one symmetry axis (Theorem 2.7), two symmetry axes (Theorem 2.10), or three symmetry axes (Theorem 2.12). It follows from Theorem 2.12 that the consideration of the cases with more symmetries does not improve the results.

In Section 3, all proofs are given.

2 Main results

Theorem 2.1. *Let p^+, $p^- \in \mathbf{L}[0,1]$ be non-negative functions, $\int_0^1 p^-(t)\,dt \neq \int_0^1 p^+(t)\,dt$.*

Then there exists a number $\lambda^*(p^+, p^-) > 0$ such that boundary value problem (1.1) is uniquely solvable for all $T \in \mathbb{S}(p^+, p^-)$ if

$$\lambda \neq 0, \qquad |\lambda| < \lambda^*(p^+, p^-). \tag{2.1}$$

If $|\lambda| \geqslant \lambda^*(p^+, p^-)$, there exists an operator $T \in \mathbb{S}(p^+, p^-)$ such that problem (1.1) is not uniquely solvable.

It turns out that we can compute $\lambda^*(p^+, p^-)$ (see equality (3.11)).

So, let non-negative integrable functions p^+, p^- be given,

$$p \equiv p^+ - p^-, \qquad \mathcal{P} \equiv \int_0^1 p(s)\,ds. \tag{2.2}$$

For every $0 \leqslant t_1 < t_2 \leqslant 1$, we define the piecewise linear function

$$q_{t_1,t_2}(t) \equiv \begin{cases} t(t_2 - t_1), & t \in [0, t_1), \\ t_2 - t - (1-t)(t_2 - t_1), & t \in [t_1, t_2), \\ -(1-t)(t_2 - t_1), & t \in [t_2, 1]. \end{cases} \tag{2.3}$$

For every $z \in \mathbf{L}[0,1]$, let

$$q_{t_1,t_2,z}(t) \equiv q_{t_1,t_2}(t) - \int_0^1 z(s) q_{t_1,t_2}(s)\,ds, \qquad t \in [0,1]; \tag{2.4}$$

for every $a \in \mathbb{R}$

$$[a]^+ \equiv (|a| + a)/2, \qquad [a]^- \equiv (|a| - a)/2.$$

Theorem 2.2. *Let $\mathcal{P} = 1$. Then*

$$\lambda^*(p^+, p^-) = \frac{1}{\max\limits_{0 \leqslant t_1 < t_2 \leqslant 1} \int_0^1 \left(p^+(t)[q_{t_1,t_2,p}(t)]^+ + p^-(t)[q_{t_1,t_2,p}(t)]^- \right) dt}. \tag{2.5}$$

Remark 2.3. If $\mathcal{P} = 1$, then we have

$$\int_0^1 \left(p^+(t)[q_{t_1,t_2,p}(t)]^+ + p^-(t)[q_{t_1,t_2,p}(t)]^-\right) ds > 0, \qquad 0 < t_1 < t_2 < 1,$$

and

$$\lambda^*(p^+, p^-) = \frac{1}{\max\limits_{0 \leqslant t_1 < t_2 \leqslant 1} \int_0^1 \left(p^+(t)[q_{t_1,t_2,p}(t)]^- + p^-(t)[q_{t_1,t_2,p}(t)]^+ \right) dt}. \tag{2.6}$$

Example 2.4. Suppose p^+, p^- are nonnegative constants and $p \equiv p^+ - p^- = 1$. Then $\int_0^1 p\, q_{t_1,t_2}(s)\,ds = (t_1 + t_2 - 1)(t_2 - t_1)/2$, and one can readily check that

$$\lambda^*(p^+, p^-) = \frac{32}{p^+ + p^-}.$$

Example 2.5. Set

$$p(t) = 4 - 6\,t, \qquad p^+(t) = \begin{cases} p(t), & t \in [0, 2/3], \\ 0, & t \in (2/3, 1], \end{cases} \quad p^-(t) = \begin{cases} 0, & t \in [0, 2/3], \\ -p(t), & t \in (2/3, 1]. \end{cases}$$

To compute $\lambda^*(p^+, p^-)$ we have to consider all possible cases of relative positions of the points t_1, t_2, $2/3$, and the zeros of $q_{t_1,t_2,p}$. After that we can conclude that

$$\frac{1}{\lambda^*(p^+, p^-)} = \max_{1/3 \leqslant t_1 \leqslant 2/3 \leqslant t_2 \leqslant 1} \left((t_2 - t_1)\,(A^2 - 1)(A - 1) + \frac{B^3}{27(1 - (t_2 - t_1)^2)} \right),$$

where

$$A = (1 - t_1)^2 + (1 - t_2)^2 + t_1 t_2, \qquad B = 3t_1^3 - 6t_1^2 + 2t_1 - 5t_2 + 6t_2^2 - 3t_2^3 + 2.$$

After some elementary computations, we get

$$15.4 < \lambda^*(p^+, p^-) < 15.5.$$

Let $\mathcal{P}^+ \equiv \int_0^1 p^+(t)\,dt = 4/3$, $\mathcal{P}^- \equiv \int_0^1 p^-(t)\,dt = 1/3$. The well-known integral sufficient conditions for the solvability of (1.1) from [10]

$$\lambda \neq 0, \qquad \frac{|\lambda|\mathcal{P}^-}{1 - |\lambda|\mathcal{P}^-/4} \leqslant |\lambda|\mathcal{P}^+ \leqslant 8(1 + \sqrt{1 - |\lambda|\mathcal{P}^-/4}) \tag{2.7}$$

gives the following result: problem (1.1) is uniquely solvable for all $T \in \mathbb{S}(p^+, p^-)$ if

$$0 < |\lambda| \leqslant 9.$$

It is obvious that the sufficient condition for the solvability obtained in Theorem 2.2

$$0 < |\lambda| \leqslant 15.4$$

is better. Moreover, if $|\lambda| \geqslant 15.5$, then there exists an operator $T \in \mathbb{S}(p^+, p^-)$ such that problem (1.1) is not uniquely solvable.

Let $\mathbf{0}(t) = 0$, $t \in [0, 1]$, be the zero function.

Example 2.6. If $p^+(t) = 2t$, $t \in [0, 1]$, $p^- \equiv \mathbf{0}$, then $\mathcal{P} = 1$ and

$$\lambda^*(p^+, \mathbf{0}) = \frac{1}{\max\limits_{k \in [0,1/2], s \in [k, 1-k]} g_1(k, s) g_2(k, s)},$$

where

$$g_1(k, s) = \left(\frac{-1 + 3k + k^2 - 3s + 3s^2}{9(1 - 2k)} \right)^2,$$

$$g_2(k, s) = -2k(1 + 2k - 7k^2 + 4k^3 - 6s + 6sk - 3s^2 + 12s^2 k).$$

It is easy to compute that

$$\left(\max_{k \in [0,1/2], s \in [k, 1-k]} g_1(k, s) g_2(k, s) \right)^{-1} \in (29.328, 29.329).$$

Therefore, in this case, periodic boundary value problem (1.1) is uniquely solvable for every operator $T \in \mathbb{S}(p^+, p^-)$ if $|\lambda| \in (0, 29.328]$. If $|\lambda| > 29.329$, then there exists $T \in \mathbb{S}(p^+, p^-)$ such that (1.1) is not uniquely solvable.

Further we consider symmetric functions p^+ and $p^- = \mathbf{0}$. It makes the computation of λ^* much more easier, especially in Theorem 2.12.

Theorem 2.7. *Let* $p^- = \mathbf{0}$,

$$p^+(t) = p^+(1-t) \geqslant 0 \quad \text{for a.a. } t \in [0, 1/2], \qquad \int_0^1 p^+(t)\, dt = 1. \tag{2.8}$$

Then

$$\lambda^*(p^+, \mathbf{0}) = \frac{1}{\max\limits_{0 \leqslant t_1 \leqslant 1/2,\ 1-t_1 \leqslant t_2 \leqslant 1} \int_0^1 p^+(t)[q_{t_1,t_2,p^+}(t)]^+\, dt}.$$

It is not difficult to show that $\lambda^*(p^+, p^-)$ can take any value from the interval $(0, +\infty)$ for different functions p^+, p^- under the conditions of Theorem 2.2. Moreover, under the conditions of Theorems 2.2 and 2.7, we have $\lambda^*(p^+, \mathbf{0}) \in (16, \infty)$. It follows from this that the periodic boundary value problem (1.1) is uniquely solvable for every $T \in \mathbb{S}(p^+, \mathbf{0})$ if the function $p^+ \in \mathbf{L}[0, 1]$ is non-negative and

$$0 < |\lambda| \int_0^1 p^+(s)\, ds \leqslant 16.$$

This result is well known. For the first time, the best constant 16 was obtained in [15] for ordinary differential equations, and in [19] for functional differential equations (non-linear).

Example 2.8. If $p^+(t) = 6t(1-t)$, $t \in [0, 1]$, $p^- = \mathbf{0}$, then $\mathcal{P} = 1$ and

$$\lambda^*(p^+, \mathbf{0}) = \frac{1}{\max\limits_{t \in [0, 1/2]} g_3(t)},$$

where

$$g_3(t) = \frac{t(2t-1)(4t^2 - 6t - 3)}{16}.$$

We have

$$\frac{1}{\max\limits_{t \in [0,1]} g_3(t)} \in (29.737, 29.738).$$

Therefore, in this case, the periodic boundary value problem (1.1) has a unique solution for every operator $T \in \mathbb{S}(p^+, \mathbf{0})$ if $|\lambda| \in (0, 29.737]$. But if $|\lambda| \geqslant 29.738$, then there exists $T \in \mathbb{S}(p^+, \mathbf{0})$ such that (1.1) is not uniquely solvable.

Example 2.9. If $p^+(t) = 30t^2(1-t)^2$, $t \in [0, 1]$, $p^- = \mathbf{0}$, then $\mathcal{P} = 1$ and

$$\lambda^*(p^+, \mathbf{0}) = \frac{1}{\max\limits_{t \in [0,1]} g_4(t)},$$

where

$$g_4(t) = \frac{t(1-2t)(16t^4 - 40t^3 + 20t^2 + 10t + 5)}{32}.$$

We have

$$\frac{1}{\max\limits_{t \in [0,1]} g_4(t)} \in (30.117, 30.118).$$

Therefore, in this case, the periodic boundary value problem (1.1) is uniquely solvable for every operator $T \in \mathbb{S}(p^+, \mathbf{0})$ if $|\lambda| \in (0, 30.117]$. But if $|\lambda| \geqslant 30.118$, then there exists $T \in \mathbb{S}(p^+, p^-)$ such that (1.1) is not uniquely solvable.

Theorem 2.10. *Let $p^- = \mathbf{0}$, $p^+ = p$, where*

$$p(t) = p(1/2-t) = p(1/2+t) = p(1-t) \geqslant 0 \quad \text{for a.a. } t \in [0,1/4], \tag{2.9}$$

and $\int_0^1 p(t)\,dt = 1$. Then

$$\lambda^*(p,\mathbf{0}) = \frac{1}{\max\limits_{0\leqslant t\leqslant 1/4}\left(\int_0^{1/4-t}(1/4-t-s)p(s)\,ds - \int_0^t (t-s)p(s)\,ds + t/4\right)}.$$

If, moreover,

$$\int_{1/4-t}^{1/4} p(t)\,dt \geqslant \int_0^t p(t)\,dt, \qquad t\in[0,1/8], \tag{2.10}$$

then

$$\lambda^*(p,\mathbf{0}) = \frac{1}{\int_0^{1/4} s\,p(s)\,ds};$$

if

$$\int_{1/4-t}^{1/4} p(t)\,dt \geqslant \int_0^t p(t)\,dt, \qquad t\in[0,1/8], \tag{2.11}$$

then

$$\lambda^*(p,\mathbf{0}) = \frac{1}{1/16 - \int_0^{1/4} s\,p(s)\,ds}.$$

Remark 2.11. If p is increasing on $[0,1/4]$, then (2.10) is fulfilled. If p is decreasing on $[0,1/4]$, then (2.11) is fulfilled.

It is easy to show that $\lambda^*(p,\mathbf{0})$ can take any value from the interval $(16,32]$ under the conditions of Theorem 2.10.

Theorem 2.12. *Let $p^- = \mathbf{0}$, $p^+ = p$, where*

$$p(t) = p(1/2-t) = p(1/2+t) = p(1-t) = p(1/4-t) \geqslant 0, \qquad t\in[0,1/4],$$

and

$$\int_0^1 p(t)\,dt = 1.$$

Then

$$\lambda^*(p,\mathbf{0}) = 32.$$

Theorems 2.2, 2.7, 2.10 and 2.12 can be reformulated in the form of sharp sufficient conditions for the solvability. In particular, Theorems 2.1 and 2.2 have the following equivalent version.

Theorem 2.13. *Let p^+, $p^- \in \mathbf{L}[0,1]$ be non-negative functions, $T \in \mathbb{S}(p^+,p^-)$, $p \equiv p^+ - p^-$, $\mathcal{P} \equiv \int_0^1 p(s)\,ds \neq 0$, $f \in \mathbf{L}[0,1]$. The periodic boundary value problem*

$$\begin{cases} \ddot{x}(t) = (Tx)(t) + f(t) & \text{for a.a. } t\in[0,1], \\ x(0) = x(1), \quad \dot{x}(0) = \dot{x}(1), \end{cases} \tag{2.12}$$

is uniquely solvable if

$$\max_{0\leqslant t_1<t_2\leqslant 1} \int_0^1 \left(p^+(t)[q_{t_1,t_2,p/\mathcal{P}}(t)]^+ + p^-(t)[q_{t_1,t_2,p/\mathcal{P}}(t)]^-\right)\,dt < 1. \tag{2.13}$$

If $\mathcal{P} = 0$ or inequality (2.13) is not fulfilled, then there exists an operator $T \in \mathbb{S}(p^+,p^-)$ such that problem (2.12) is not uniquely solvable.

3 Proofs

We need two lemmas for proving Theorems 2.1 and 2.2. The proof of the following lemma on the Fredholm property can be found in [28, p. 85] or [2, pp. 1, 7–8, 60–62].

Lemma 3.1 ([2, 28])**.** *Let $T \in \mathbb{S}(p^+, p^-)$. Boundary value problem (2.12) is uniquely solvable if and only if the homogeneous problem*

$$\begin{cases} \ddot{x}(t) = (Tx)(t) & \text{for a.a. } t \in [0,1], \\ x(0) = x(1), \quad \dot{x}(0) = \dot{x}(1), \end{cases} \tag{3.1}$$

has only the trivial solution.

Lemma 3.2. *Let $T \in \mathbb{S}(p^+, p^-)$. Then for every $x \in \mathbf{C}[0,1]$, there exist (depending on x) points t_1, $t_2 \in [0,1]$ and functions p_1, $p_2 \in \mathbf{L}[0,1]$ satisfying the conditions*

$$\begin{gathered} p_1(t) + p_2(t) = p^+(t) - p^-(t), \\ -p^-(t) \leqslant p_i(t) \leqslant p^+(t), \quad \text{for a.a. } t \in [0,1],\ i = 1,2, \end{gathered} \tag{3.2}$$

such that the following equality holds:

$$(Tx)(t) = p_1(t)x(t_1) + p_2(t)x(t_2) \quad \text{for a.a. } t \in [0,1].$$

Proof. Suppose $x \in \mathbf{C}[0,1]$,

$$\max_{t\in[0,1]} x(t) = x(t_2), \qquad \min_{t\in[0,1]} x(t) = x(t_1).$$

Then

$$p^+(t)x(t_1) - p^-(t)x(t_2) \leqslant (Tx)(t) \leqslant p^+(t)x(t_2) - p^-(t)x(t_1), \qquad t \in [0,1].$$

Hence, there exists a measurable function $\xi : [0,1] \to [0,1]$ such that

$$(Tx)(t) = x(t_1)\big((1-\xi(t))p^+(t) - \xi(t)p^-(t)\big) + x(t_2)\big((\xi(t)-1)p^-(t) + \xi(t)p^+(t)\big),\ t \in [0,1].$$

Therefore, the chosen points t_1, t_2 and the functions p_1, p_2 defined by the equalities

$$\begin{aligned} p_1(t) &= (1-\xi(t))p^+(t) - \xi(t)p^-(t), \qquad t \in [0,1], \\ p_2(t) &= (\xi(t)-1)p^-(t) + \xi(t)p^+(t), \end{aligned}$$

satisfy the conditions of the lemma. □

Remark 3.3. It follows from the proof of Lemma 3.2, that some maximum and minimum points of x can be taken as points t_1 and t_2 in Lemma 3.2.

Proof of Remark 2.3. Let $0 < t_1 < t_2 < 1$ and $\mathcal{P} = 1$. Denote here $r = q_{t_1,t_2,p}$. Using equalities (2.2) and (2.4), one can easily get that if $\mathcal{P} = 1$, then

$$\int_0^1 p(s)r(s)\,ds = 0.$$

That is,

$$\int_0^1 \big(p^+(s) - p^-(s)\big)\big([r(s)]^+ - [r(s)]^-\big)\,ds = 0.$$

Therefore,

$$\int_0^1 (p^+(s)[r(s)]^+ + p^-(s)[r(s)]^-)\,ds = \int_0^1 (p^+(s)[r(s)]^- + p^-(s)[r(s)]^+)\,ds. \tag{3.3}$$

Thus, if

$$\int_0^1 \left(p^+(s)[r(s)]^+ + p^-(s)[r(s)]^-\right) ds = 0, \tag{3.4}$$

then

$$\int_0^1 (p^+(s) + p^-(s))([r(s)]^+ + [r(s)]^-)\,ds = \int_0^1 (p^+(s) + p^-(s))|r(s)|\,ds = 0.$$

Since for $|r(s)| > 0$ for almost all $s \in [0,1]$, it means that assumption (3.4) is not fulfilled. Equality (2.6) follows from (3.3). □

Proof of Theorems 2.1 and 2.2. First we will prove Theorem 2.2. Suppose $\mathcal{P} = 1$. By Lemma 3.1, the boundary value problem (1.1) is not uniquely solvable if and only if the homogeneous problem

$$\begin{cases} \ddot{y}(t) = \lambda(Ty)(t) & \text{for a.a. } t \in [0,1], \\ y(0) = y(1), \quad \dot{y}(0) = \dot{y}(1), \end{cases} \tag{3.5}$$

has a non-zero solution. Suppose (3.5) has a non-zero solution y. By Lemma 3.2, there exist points t_1, $t_2 \in [0,1]$, $t_1 < t_2$, and functions p_1, $p_2 \in \mathbf{L}[0,1]$ satisfying (3.2) such that

$$(Ty)(t) = p_1(t)y(t_1) + p_2(t)y(t_2) \quad \text{for a.a. } t \in [0,1].$$

Therefore, y is a solution of the periodic problem

$$\begin{cases} \ddot{y}(t) = \lambda(p_1(t)y(t_1) + p_2(t)y(t_2)) & \text{for a.a. } t \in [0,1], \\ y(0) = y(1), \quad \dot{y}(0) = \dot{y}(1). \end{cases} \tag{3.6}$$

Thus,

$$y(t) = y(0) + \dot{y}(0)t + \lambda \int_0^t (t-s)\,(p_1(s)y(t_1) + p_2(s)y(t_2))\,ds, \qquad t \in [0,1]. \tag{3.7}$$

From the condition $y(0) = y(1)$, we get

$$\dot{y}(0) = -\lambda \int_0^1 (1-s)\,(p_1(s)y(t_1) + p_2(s)y(t_2))\,ds. \tag{3.8}$$

From the condition $\dot{y}(0) = \dot{y}(1)$ it follows that

$$\lambda \int_0^1 (p_1(s)y(t_1) + p_2(s)y(t_2))\,ds = 0. \tag{3.9}$$

Substituting $\dot{y}(0)$ from (3.8) in (3.7) for $t = t_1$ and $t = t_2$, we obtain

$$\begin{aligned} y(t_1) = y(0) - \lambda t_1 \int_0^1 (1-s)\,(p_1(s)y(t_1) + p_2(s)y(t_2))\,ds \\ + \lambda \int_0^{t_1} (t_1-s)\,(p_1(s)y(t_1) + p_2(s)y(t_2))\,ds, \\ y(t_2) = y(0) - \lambda t_2 \int_0^1 (1-s)\,(p_1(s)y(t_1) + p_2(s)y(t_2))\,ds \\ + \lambda \int_0^{t_2} (t_2-s)\,(p_1(s)y(t_1) + p_2(s)y(t_2))\,ds. \end{aligned}$$

Excluding $y(0)$ from these equations, we get

$$y(t_1) - y(t_2) + \lambda \int_0^1 q_{t_1,t_2}(s)\,(p_1(s)y(t_1) + p_2(s)y(t_2))\,ds = 0. \tag{3.10}$$

Problem (3.6) has a non-zero solution if and only if the system of two equations (3.9), (3.10) (with respect to scalar variables $y(t_1)$ and $y(t_2)$) has a non-zero solution. This system has a non-zero solution if and only if

$$\begin{aligned}
\Delta(t_1,t_2,p_1) &\equiv \begin{vmatrix} \lambda \int_0^1 p_1(s)\,ds & \lambda \int_0^1 p_2(s)\,ds \\ 1+\lambda \int_0^1 p_1(s)q_{t_1,t_2}(s)\,ds & -1+\lambda \int_0^1 p_2(s)q_{t_1,t_2}(s)\,ds \end{vmatrix} \\
&= \begin{vmatrix} \lambda \int_0^1 p_1(s)\,ds & \lambda \\ 1+\lambda \int_0^1 p_1(s)q_{t_1,t_2}(s)\,ds & \lambda \int_0^1 p(s)q_{t_1,t_2}(s)\,ds \end{vmatrix} \\
&= \lambda\left(-\lambda \int_0^1 p_1(s)q_{t_1,t_2,p}(s)\,ds - 1\right) = 0.
\end{aligned}$$

Denote by R the set of all $\{t_1,t_2,p_1\}$ such that t_1, $t_2 \in [0,1]$, $0 \leqslant t_1 \leqslant t_2 \leqslant 1$, the functions $p_1 \in \mathbf{L}[0,1]$ and $p_2 = p - p_1$ satisfy condition (3.2). Using Remark 2.3, we get

$$\begin{aligned}
\max_{\{t_1,t_2,p_1\}\in R} \int_0^1 p_1(s)q_{t_1,t_2,p}(s)\,ds &= -\min_{\{t_1,t_2,p_1\}\in R} \int_0^1 p_1(s)q_{t_1,t_2,p}(s)\,ds \\
&= \max_{0\leqslant t_1<t_2\leqslant 1} \int_0^1 \left(p^+(t)[q_{t_1,t_2,p}(t)]^+ + p^-(t)[q_{t_1,t_2,p}(t)]^-\right) dt > 0.
\end{aligned}$$

Moreover,

$$\left\{\int_0^1 p_1(s)q_{t_1,t_2,p}(s)\,ds : \{t_1,t_2,p_1\}\in R\right\} = \left[-\frac{1}{\lambda^*(p^+,p^-)}, \frac{1}{\lambda^*(p^+,p^-)}\right],$$

where $\lambda^*(p^+,p^-)$ is defined by (2.5).

So, if condition (2.1) with $\lambda^*(p^+,p^-)$ from equality (2.5) is fulfilled, then $\Delta(t_1,t_2,p_1) \neq 0$ for all t_1, $t_2 \in [0,1]$ and for all $p_1 \in \mathbf{L}[0,1]$, satisfying (3.2). Thus, neither problem (3.6) and, therefore, problem (3.5) have no non-zero solutions.

This contradiction proves that condition (2.1) with $\lambda^*(p^+,p^-)$ from equality (2.5) implies the unique solvability of problem (1.1).

If condition (2.1) (with $\lambda^*(p^+,p^-)$ from equality (2.5)) does not hold, then there exist t_1, $t_2 \in [0,1]$ and $p_1 \in \mathbf{L}[0,1]$, $p_2 = p - p_1$ such that $\Delta(t_1,t_2,p_1) = 0$, therefore, problem (3.6) has a non-zero solution. Thus, periodic problem (1.1) is not uniquely solvable for the operator T

$$(Tx)(t) = p_1(t)x(t_1) + p_2(t)x(t_2), \qquad t \in [0,1].$$

It is clear, that $T \in \mathbb{S}(p^+,p^-)$. Therefore, Theorem 2.2 is proved. For arbitrary $\mathcal{P} \neq 0$, using equalities (2.5) (for $\mathcal{P} > 0$) and (2.6) (for $\mathcal{P} < 0$), we can obtain that Theorem 2.1 is valid with $\lambda^*(p^+,p^-)$ defined by the equality

$$\lambda^*(p^+,p^-) = \frac{1}{\max\limits_{0\leqslant t_1<t_2\leqslant 1} \int_0^1 \left(p^+(t)[q_{t_1,t_2,p/\mathcal{P}}(t)]^+ + p^-(t)[q_{t_1,t_2,p/\mathcal{P}}(t)]^-\right) dt}. \tag{3.11}$$

□

For proving Theorems 2.7 and 2.10, we need Lemmas 3.4, 3.7 and 3.8.

Lemma 3.4. *Suppose $p^+ \in \mathbf{L}[0,1]$ is non-negative, $T \in \mathbb{S}(p^+, \mathbf{0})$, and the boundary value problem*

$$\begin{cases} \ddot{x}(t) = \lambda (Tx)(t) \quad \text{for a.a. } t \in [0,1], \\ x(0) = x(1), \quad \dot{x}(0) = \dot{x}(1), \end{cases} \tag{3.12}$$

has a non-trivial solution y such that

$$\min_{t\in[0,1]} y(t) = y(\tau_1), \qquad \max_{t\in[0,1]} y(t) = y(\tau_2), \qquad \tau_1 < \tau_2. \tag{3.13}$$

Then there exists a measurable function

$$g : [0,1] \to [\tau_1, \tau_2] \tag{3.14}$$

such that

$$\begin{cases} \ddot{y}(t) = \lambda p^+(t) y(g(t)) \quad \text{for a.a. } t \in [0,1], \\ y(0) = y(1), \quad \dot{y}(0) = \dot{y}(1). \end{cases} \tag{3.15}$$

Proof. By Lemma 3.2, the solution y satisfies the equality

$$\ddot{y}(t) = \lambda \left(p_1(t) y(\tau_1) + (p^+(t) - p_1(t)) y(\tau_2) \right) \quad \text{for a.a. } t \in [0,1],$$

where $p_1 \in \mathbf{L}[0,1]$, $0 \leqslant p_1(t) \leqslant p(t)$, $t \in [0,1]$. Therefore,

$$\ddot{y}(t) = \lambda p^+(t) \widetilde{y}(t) \quad \text{for a.a. } t \in [0,1],$$

where $\widetilde{y}$ is measurable and $\widetilde{y}(t) \in [y(\tau_1), y(\tau_2)]$ for a.a. $t \in [0,1]$. From this, it follows that there exists a measurable g satisfying the conditions of the lemma. □

Remark 3.5. It is obvious that if y is a solution of (3.12), then $-y$ is also a solution. Therefore, if (3.12) has a non-trivial solution, then this problem has a solution satisfying (3.13).

Remark 3.6. It is clear that we can replace condition (3.14) in Lemma 3.4 by the condition

$$g : [0,1] \to [0, \tau_1] \cup [\tau_2, 1].$$

Define the sets

$$R_1 \equiv \{(t_1, t_2) : 0 \leqslant t_1 \leqslant 1/2 \leqslant t_2 \leqslant 1,\ t_1 + t_2 \geqslant 1\},$$
$$R_2 \equiv \{(t_1, t_2) : 1/4 \leqslant t_1 \leqslant 1/2,\ 3/4 \leqslant t_2 \leqslant 1,\ t_2 - t_1 \leqslant 1/2\}.$$

Lemma 3.7. *Suppose $p^+ \in \mathbf{L}[0,1]$ satisfies (2.8), $T \in \mathbb{S}(p^+, \mathbf{0})$, and homogeneous boundary value problem (3.12) has a non-trivial solution. Then there exists a measurable function $h : [0,1] \to [0,1]$ such that problem*

$$\begin{cases} \ddot{x}(t) = \lambda p^+(t) x(h(t)) \quad \text{for a.a. } t \in [0,1], \\ x(0) = x(1), \quad \dot{x}(0) = \dot{x}(1). \end{cases} \tag{3.16}$$

has a non-zero solution with some maximum and minimum points $(t_1, t_2) \in R_1$.

Proof. By Lemma 3.4 and Remark 3.5, there exists a measurable $g : [0,1] \to [\tau_1, \tau_2]$ such that boundary value problem (3.15) has a non-trivial solution y satisfying (3.13). Note, that under the conditions of the lemma, a solution of (3.15) has a zero at some point t_0.

Define the intervals $J_1 \equiv [0,1/2]$, $J_2 \equiv [1/2,1]$. If both points τ_1, τ_2 belong to the same interval, set

$$h(t) = \begin{cases} g(t), & t \in [\tau_1, \tau_2], \\ g(1-t), & t \in [1-\tau_2, 1-\tau_1], \\ t_0, & \text{otherwise.} \end{cases}$$

Then, using the equality $p^+(t) = p^+(1-t)$, $t \in [0,1]$, it is easy to prove that the boundary value problem (3.16) has the solution

$$x(t) = \begin{cases} -y(t), & t \in [\tau_1, \tau_2], \\ -y(1-t), & t \in [1-\tau_2, 1-\tau_1], \\ -y(\tau_1), & t \in [0,\tau_1] \cup [1-\tau_1, 1], \\ -y(\tau_2), & t \in [\tau_2, 1-\tau_2], \end{cases}$$

with a minimum at the point $1/2$ and a maximum at the point 1.

If $\tau_1 \in J_1$, $\tau_2 \in J_2$, set $h(t) = 1 - g(1-t)$. Using the equality $p^+(t) = p^+(1-t)$, $t \in [0,1]$, we obtain that $x(t) = -y(1-t)$, $t \in [0,1]$, is a solution of (3.16) with the minimum point $\theta_1 = 1 - \tau_2 \in J_1$ and the maximum point $\theta_2 = 1 - \tau_1 \in J_2$. Since either $\tau_2 + \tau_1 = 2 - (\theta_1 + \theta_2) \geqslant 1$ or $\theta_2 + \theta_1 \geqslant 1$, the lemma is proved. □

Lemma 3.8. *Let $p^+ \in \mathbf{L}[0,1]$ satisfy (2.9), $T \in \mathbb{S}(p^+, \mathbf{0})$, and the homogeneous boundary value problem (3.12) have a non-trivial solution. Then there exists a measurable function $h : [0,1] \to [0,1]$ such that problem (3.16) has a non-zero solution with some maximum and minimum points $(t_1, t_2) \in R_2$.*

Proof. By Lemmas 3.4 and Remark 3.5, there exists a measurable $g : [0,1] \to [\tau_1, \tau_2]$ such that the boundary value problem (3.15) has a non-trivial solution y with a minimum point τ_1 and a maximum point $\tau_2 > \tau_1$. Under the conditions of Lemma 3.8 a solution of (3.15) has a zero at some point t_0.

Define the intervals $I_1 \equiv [0,1/4]$, $I_2 \equiv [1/4,1/2]$, $I_3 \equiv [1/2,3/4]$, $I_4 \equiv [3/4,1]$.

By Lemma 3.7, we have to consider only three cases.

If $\tau_1 \in I_2$, $\tau_2 \in I_3$, we set

$$h(t) = \begin{cases} g(t), & t \in [\tau_1, \tau_2], \\ g(3/2-t), & t \in [3/2-\tau_2, 1], \\ g(1/2-t), & t \in [0, 1/2-\tau_1], \\ t_0, & \text{otherwise.} \end{cases}$$

Using condition (2.9), it is easy to prove that the boundary value problem (3.16) has the solution

$$x(t) = \begin{cases} y(t), & t \in [\tau_1, \tau_2], \\ y(3/2-t), & t \in [3/2-\tau_2, 1], \\ y(1/2-t), & t \in [0, 1/2-\tau_1], \\ y(\tau_1), & t \in [1/2-\tau_1, \tau_1], \\ y(\tau_2), & t \in [\tau_2, 3/2-\tau_2], \end{cases}$$

with a minimum at the point $1/4$ and a maximum at the point $3/4$.

If $\tau_1 \in I_2$, $\tau_2 \in I_4$, set

$$h(t) = \begin{cases} g(t+1/2), & t \in [0,1/2), \\ g(t-1/2), & t \in [1/2,1]. \end{cases}$$

Then, using the equality $p^+(t) = p^+(t+1/2)$, $t \in [0,1/2]$, it is easy to prove that

$$x(t) = \begin{cases} -y(t+1/2), & t \in [0,1/2), \\ -y(t-1/2), & t \in [1/2,1], \end{cases}$$

is a solution of problem (3.16) with the minimum point at $\theta_1 = \tau_2 - 1/2$ and the maximum point at $\theta_2 = 1/2 + \tau_1$. Since, either $\tau_2 - \tau_1 \leqslant 1/2$ or $\theta_2 - \theta_1 = 1 - (\tau_2 - \tau_1) \leqslant 1/2$, we obtain that at least one of the pairs (τ_1, τ_2) or (θ_1, θ_2) belongs to the set R_2.

If $\tau_1 \in I_1$, $\tau_2 \in I_4$, we use Remark 3.6. Set

$$h(t) = \begin{cases} g(t), & t \in [0,\tau_1] \cup [\tau_2, 1], \\ g(3/2-t), & t \in [1/2, 3/2-\tau_2], \\ g(1/2-t), & t \in [1/2-\tau_1, 1/2], \\ t_0, & \text{otherwise}. \end{cases}$$

In this case, using condition (2.9), we can also show that the boundary value problem (3.16) has the solution

$$x(t) = \begin{cases} y(t), & t \in [0,\tau_1] \cup [\tau_2, 1], \\ y(3/2-t), & t \in [1/2, 3/2-\tau_2], \\ y(1/2-t), & t \in [1/2-\tau_1, 1/2], \\ y(\tau_1), & t \in [\tau_1, 1/2-\tau_1], \\ y(\tau_2), & t \in [3/2-\tau_2, \tau_2], \end{cases}$$

with a minimum at the point $1/4$ and a maximum at the point $3/4$.

So, in all cases, there exists a measurable function h with the required properties. □

Proof of Theorems 2.7 and 2.10. Define $\lambda_1^*(p^+, \mathbf{0})$ and $\lambda_2^*(p^+, \mathbf{0})$ by the equalities

$$\lambda_i^*(p^+, \mathbf{0}) \equiv \frac{1}{\max\limits_{(t_1,t_2)\in R_i} \int_0^1 p^+(t) q^+_{t_1,t_2,p}(t)\, dt}, \quad i = 1,2.$$

We will show that if p^+ satisfies conditions (2.8), then

$$\lambda^*(p^+, \mathbf{0}) = \lambda_i^*(p^+, \mathbf{0}) \tag{3.17}$$

for $i = 1$, and if p^+ satisfies conditions (2.9), then equality (3.17) holds for $i = 2$.

For every $i = 1,2$, repeating the proof of Theorems 2.1 and 2.2 for the set of pairs (t_1, t_2) R_i instead of the set of all pairs $\{(t_1, t_2) : 0 \leqslant t_1 \leqslant t_2 \leqslant 1\}$, we obtain that

$$|\lambda| \geqslant \lambda_i^*(p^+, \mathbf{0})$$

if and only if there exists an operator $T \in \mathbb{S}(p^+, \mathbf{0})$ such that problem (3.5) has a non-zero solution y with $(\tau_1, \tau_2) \in R_i$, where τ_1, τ_2 are defined by condition (3.13). From this, the definition of $\lambda^*(p^+, \mathbf{0})$, and Lemma 3.7, it follows that if p^+ satisfies (2.8), then

$$\lambda_1^*(p^+, \mathbf{0}) = \lambda^*(p^+, \mathbf{0}),$$

and from Lemma 3.8 it follows that

$$\lambda_2^*(p^+,\mathbf{0}) = \lambda^*(p^+,\mathbf{0})$$

if p^+ satisfies (2.9).

It proves Theorem 2.7, but for proving Theorem 2.10 we have to compute $\lambda_2^*(p^+,\mathbf{0})$.

Let p^+ satisfy (2.9) and $\int_0^1 p^+(t)\,dt = 1$. For $0 \leqslant t_1 \leqslant t_2 \leqslant 1$, we have

$$I_{t_1,t_2} \equiv \int_0^1 p^+(s)\, q_{t_1,t_2}(s)\, ds = g(\theta_2) - g(\theta_1) + 1/8 - \theta_2/2,$$

where $q_{t_1,t_2}(s)$ is defined by (2.3),

$$g(t) \equiv \int_0^t (t-s)p^+(s)\, ds, \qquad t \in [0,1/4], \tag{3.18}$$

$$\theta_1 \equiv 1/2 - t_1, \qquad \theta_2 \equiv 1 - t_2.$$

We introduce some notation: the points $t_3 \in [0,t_1]$ and $t_4 \in [t_1,t_2]$ satisfy the equalities

$$q_{t_1,t_2}(t_3) = q_{t_1,t_2}(t_4) = I_{t_1,t_2},$$

and

$$\theta_3 = t_3, \qquad \theta_4 = t_4 - 1/2.$$

Let

$$M(\theta_1,\theta_2) \equiv \int_0^1 p^+(s)\, [q_{t_1,t_2}(s) - I_{t_1,t_2}]^+\, ds,$$

where $t_1 = 1/2 - \theta_1$, $t_2 = 1 - \theta_2$. Then

$$M(\theta_1,\theta_2) = \frac{\theta_1+\theta_2}{8} + g(\theta_3)\left(\frac{1}{2} - (\theta_2-\theta_1)\right) + g(\theta_4)\left(\frac{1}{2} + (\theta_2-\theta_1)\right) - \frac{g(\theta_1)+g(\theta_2)}{2}, \tag{3.19}$$

where $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 1$ and

$$\theta_3 = \theta_3(\theta_1,\theta_2) = \frac{1/8 - \theta_2/2 + g(\theta_2) - g(\theta_1)}{\frac{1}{2} - (\theta_2-\theta_1)}, \tag{3.20}$$

$$\theta_4 = \theta_4(\theta_1,\theta_2) = \frac{1/8 - \theta_1/2 - (g(\theta_2) - g(\theta_2))}{\frac{1}{2} + \theta_2 - \theta_1}. \tag{3.21}$$

Thus we have

$$\lambda^*(p^+,\mathbf{0}) = \frac{1}{\max\limits_{(t_1,t_2)\in R_2} \int_0^1 p^+(s)[q_{t_1,t_2,p^+}(s)]^+\, ds} = \frac{1}{\max\limits_{0\leqslant\theta_1\leqslant\theta_2\leqslant 1/4} M(\theta_1,\theta_2)}. \tag{3.22}$$

We will show that

$$\max_{0\leqslant\theta_1\leqslant\theta_2\leqslant 1/4} M(\theta_1,\theta_2) = \max_{0\leqslant\theta\leqslant 1/4} M(\theta,\theta). \tag{3.23}$$

It will prove Theorem 2.10, because

$$M(\theta,\theta) = \frac{\theta}{4} + g(1/4-\theta) - g(\theta), \qquad \theta \in [0,1/4].$$

To prove (3.23), we will check the inequality

$$(M(\theta_1,\theta_1)+M(\theta_2,\theta_2))/2 \geqslant M(\theta_1,\theta_2) \quad \text{if } 0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 1/4. \tag{3.24}$$

Indeed, from (3.24), it follows that

$$M(\theta_1,\theta_2) \leqslant \max\{M(\theta_1,\theta_1), M(\theta_2,\theta_2)\} \leqslant \max_{\theta\in[0,1/4]} M(\theta,\theta).$$

Therefore, (3.23) holds.

Let $\theta_0 \equiv 1/4-(\theta_1+\theta_2)/2$. We have

$$(\theta_3-\theta_0)\left(\frac{1}{2}-(\theta_2-\theta_1)\right) = g(\theta_2)-g(\theta_1)-\frac{\theta_2^2-\theta_1^2}{2} = (\theta_0-\theta_4)\left(\frac{1}{2}+(\theta_2-\theta_1)\right). \tag{3.25}$$

Therefore, the points θ_3, θ_4 are on opposite sides of the point θ_0.

Using (3.25), one can prove that

$$\begin{aligned} M(\theta_1,\theta_2) = g(\theta_0)+\frac{\theta_1+\theta_2}{8}-\frac{g(\theta_1)+g(\theta_2)}{2}+\int_{\theta_0}^{\theta_3}(\theta_3-s)p^+(s)\,ds\left(\frac{1}{2}-(\theta_2-\theta_1)\right) \\ +\int_{\theta_0}^{\theta_4}(\theta_4-s)p^+(s)\,ds\left(\frac{1}{2}+(\theta_2-\theta_1)\right). \end{aligned}$$

It is clear that inequality (3.24) is equivalent to inequality

$$\begin{aligned} \frac{g(1/4-\theta_1)+g(1/4-\theta_2)}{2}-g(\theta_0) \geqslant \int_{\theta_0}^{\theta_3}(\theta_3-s)p^+(s)\,ds\left(\frac{1}{2}-(\theta_2-\theta_1)\right) \\ +\int_{\theta_0}^{\theta_4}(\theta_4-s)p^+(s)\,ds\left(\frac{1}{2}+(\theta_2-\theta_1)\right). \end{aligned} \tag{3.26}$$

Using the integral representation (3.18) for the function g, we can rewrite the latter inequality in the form

$$\int_{1/4-\theta_2}^{1/4-\theta_1} A(s)p^+(s)\,ds \geqslant \int_{\tau_3}^{\tau_4} B(s)p^+(s)\,ds, \tag{3.27}$$

where

$$\begin{aligned} \int_{1/4-\theta_2}^{1/4-\theta_1} A(s)p^+(s)\,ds &= \frac{g(1/4-\theta_1)+g(1/4-\theta_2)}{2}-g(\theta_0), \\ \int_{\tau_3}^{\tau_4} B(s)p^+(s)\,ds &= \int_{\theta_0}^{\theta_3}(\theta_3-s)p^+(s)\,ds\left(\frac{1}{2}-(\theta_2-\theta_1)\right) \\ &\quad +\int_{\theta_0}^{\theta_4}(\theta_4-s)p^+(s)\,ds\left(\frac{1}{2}+(\theta_2-\theta_1)\right), \end{aligned}$$

$\tau_3=\min\{\theta_3,\theta_4\}$, $\tau_4=\max\{\theta_3,\theta_4\}$, and the continuous function $A(s)$ is linear on the intervals $[1/4-\theta_2,\theta_0]$, $[\theta_0,1/4-\theta_1]$ and is equal to zero at the ends of the interval $[1/4-\theta_2,1/4-\theta_1]$: the equalities $A(1/4-\theta_2)=A(1/4-\theta_1)=0$ hold; the continuous function $B(s)$ is linear on the intervals $[\tau_3,\theta_0]$, $[\theta_0,\tau_4]$ and is equal to zero at the ends of the interval $[\tau_3,\tau_4]$: $B(\tau_3)=B(\tau_4)=0$. Moreover, we have

$$A(\theta_0)=(\theta_2-\theta_1)/4, \qquad B(\theta_0)=\left|g(\theta_2)-g(\theta_1)-\frac{\theta_2^2-\theta_1^2}{2}\right|.$$

If we prove that $A(\theta_0) \geqslant B(\theta_0)$ and

$$\theta_3 \geqslant 1/4 - \theta_2, \qquad \theta_4 \geqslant 1/4 - \theta_1, \tag{3.28}$$

then inequalities (3.26) and (3.27) are fulfilled and the theorem is proved.

We have

$$\begin{aligned} 0 \leqslant g(\theta_2) - g(\theta_1) &= \int_0^{\theta_2} (\theta_2 - s) p^+(s)\, ds - \int_0^{\theta_1} (\theta_1 - s) p^+(s)\, ds \\ &= (\theta_2 - \theta_1) \int_0^{\theta_1} p^+(s)\, ds + (\theta_2 - \theta_1) \int_{\theta_1}^{\theta_2} (\theta_2 - s) p^+(s)\, ds \\ &\leqslant (\theta_2 - \theta_1) \int_0^{\theta_2} p^+(s)\, ds \leqslant \frac{\theta_2 - \theta_1}{4}. \end{aligned} \tag{3.29}$$

Hence,

$$-\frac{\theta_2 - \theta_1}{4} \leqslant -\frac{\theta_2^2 - \theta_1^2}{2} \leqslant g(\theta_2) - g(\theta_1) - \frac{\theta_2^2 - \theta_1^2}{2} \leqslant \frac{\theta_2 - \theta_1}{4}$$

for all $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 1/4$, therefore, $A(\theta_0) \geqslant B(\theta_0)$.

Using equalities (3.20), (3.21) and inequality (3.29), it is easy to check that conditions (3.28) are also fulfilled. □

Proof of Theorem 2.12. If p^+ satisfies condition (2.9), then $p^+(t) = p^+(1/4 - t)$, $t \in [0, 1/4]$, and $\int_0^{1/8} p^+(s)\, ds = 1/8$. Therefore, for $M(\theta, \theta)$ defined by (3.19), we get

$$\begin{aligned} M(\theta, \theta) &= \theta/4 - \int_0^{1/4} (1/4 - s) p^+(s)\, ds = \int_0^{1/4} s p^+(s)\, ds \\ &= \int_0^{1/8} s\, p^+(s)\, ds + \int_0^{1/8} (1/4 - s)\, p^+(s)\, ds = \frac{1}{4} \int_0^{1/8} p^+(s)\, ds = \frac{1}{32}. \end{aligned}$$

From equalities (3.22) and (3.23) it follows the theorem. □

Proof of Theorem 2.13. If $\mathcal{P} \neq 0$, the assertion follows from Theorems 2.1 and 2.2. One can use the condition

$$1 < \lambda^*(p^+, p^-)$$

for the unique solvability of boundary value problem (2.12), where $\lambda^*(p^+, p^-)$ is defined by (3.11).

If $\mathcal{P} = 0$, define the operator $T \in \mathbb{S}(p^+, p^-)$ by the equality

$$(Tx)(t) = p^+(t) x(0) - p^-(t) x(0) \quad \text{for a.a. } t \in [0, 1].$$

It follows from Lemma 3.1 that boundary value problem (2.12) is not uniquely solvable, since the homogeneous problem (3.1) has a non-trivial solution

$$x(t) = 1 + \int_0^1 s p(s)\, ds\, t + \int_0^t (t - s) p(s)\, ds, \qquad t \in [0, 1].$$

□

Acknowledgements

This research was supported by Grants 14-01-00338 of The Russian Foundation for Basic Research.

The author thanks Prof. S. Mukhigulashvili and the anonymous referee for their very helpful comments and suggestions.

References

[1] R. P. Agarwal, L. Berezansky, E. Braverman, A. Domoshnitsky, *Nonoscillation theory of functional differential equations with applications*, Springer, New York, 2012. MR2908263; url

[2] N. V. Azbelev, V. P. Maksimov, L. F. Rakhmatullina, *Introduction to the theory of functional differential equations: Methods and applications*, Contemporary Mathematics and its Applications, Vol. 3, Hindawi Publishing Corporation, Cairo, 2007. MR2319815; url

[3] E. I. Bravyi, Solvability of the periodic problem for higher-order linear functional differential equations, *Differ. Equ.* **51**(2015), No. 5, 571–585. MR3374833; url

[4] W. Cheung, J. Ren, Z. Cheng, Existence and Lyapunov stability of periodic solutions for generalized higher-order neutral differential equations, *Bound. Value Probl.* **2011**, Art. ID 635767, 21 pp. MR2679683

[5] A. Domoshnitsky, R. Hakl, J. Šremr, Component-wise positivity of solutions to periodic boundary problem for linear functional differential system, *J. Inequal. Appl.* **2012**, No. 112, 1–23. MR2954534; url

[6] X. Fu, W. Wang, Periodic boundary value problems for second-order functional differential equations, *J. Inequal. Appl.* **2010**, Art. ID 598405, 11 pp. MR2603080; url

[7] D. Jiang, J. J. Nieto, W. Zuo, On monotone method for first and second order periodic boundary value problems and periodic solutions of functional differential equations. *Math. Anal. Appl.* **289**(2004), No. 2, 691–699. MR2026934; url

[8] R. Hakl, A. Lomtatidze, B. Půža, On a boundary value problem for first-order scalar functional differential equations, *Nonlinear Anal.* **53**(2003), No. 3–4, 391–405. MR1964333; url

[9] R. Hakl, A. Lomtatidze, J. Šremr, On a periodic-type boundary value problem for first-order nonlinear functional differential equations, *Nonlinear Anal.* **51**(2002), No. 3, 425–447. MR1942755; url

[10] R. Hakl, S. Mukhigulashvili, A periodic boundary value problem for functional differential equations of higher order, *Georgian Math. J.* **16**(2009), No. 4, 651–665. MR2640795

[11] X. Hou, Z. Wu, Existence and uniqueness of periodic solutions for a kind of Liénard equation with multiple deviating arguments. *J. Appl. Math. Comput.* **38**(2012), No. 1–2, 181–193. MR2886675; url.

[12] I. Kiguradze, N. Partsvania, B. Půža, On periodic solutions of higher-order functional differential equations, *Bound. Value Probl.* **2008**, Art. ID 389028, 18 pp. MR2392912; url

[13] I. Kiguradze, Z. Sokhadze, Positive solutions of periodic type boundary value problems for first order singular functional differential equations, *Georgian Math. J.* **21**(2014), No. 3, 303–311. MR3259074; url

[14] O. Kuzmych, S. Mukhigulashvili, B. Půža, An optimal condition for the uniqueness of a periodic solution for systems of higher order linear functional differential equations, *Miskolc Math. Notes* **11**(2010), No. 1, 63–77. MR2743862

[15] A. Lasota, Z. Opial, Sur les solutions périodiques des équations différentielles ordinaires (in French), *Ann. Polon. Math.* **16**(1964), 69–94. MR0170072

[16] J. Li, J. Luo, Y. Cai, Periodic solutions for prescribed mean curvature Rayleigh equation with a deviating argument, *Adv. Difference Equ.* **2013**, No. 88, 11 pp. MR3047861; url

[17] Q. Li, Y. Li, Existence and multiplicity of positive periodic solutions for second-order functional differential equations with infinite delay, *Electron. J. Differential Equations* **2014**, No. 93, 1–14. MR3193999

[18] A. G. Lomtatidze, R. Hakl, B. Půža, On the periodic boundary value problem for first-order functional-differential equations, *Differ. Equ.* **39**(2003), No. 3, 344–352. url

[19] A. Lomtatidze, S. Mukhigulashvili, On periodic solutions of second order functional differential equations, *Mem. Differ. Equ. Math. Phys.* **5**(1995), 125–126. url

[20] S. Lu, W. Ge, Sufficient conditions for the existence of periodic solutions to some second order differential equations with a deviating argument, *J. Math. Anal. Appl.* **308**(2005), No. 2, 393–419. MR2150099; url

[21] R. Ma, Y. Lu, Existence of positive periodic solutions for second-order functional differential equations, *Monatsh. Math.* **173**(2014), No. 1, 67–81. MR3148661; url

[22] S. V. Mukhigulashvili, On the unique solvability of the Dirichlet problem for a second-order linear functional-differential equation, *Differ. Equ.* **40**(2004), No. 4, 515–523. MR2153646; url

[23] S. V. Mukhigulashvili, J. Šremr, On the solvability of the Dirichlet problem for nonlinear second-order functional-differential equations, *Differ. Equ.* **41**(2005), No. 10, 1425–1435. url

[24] S. V. Mukhigulashvili, On the solvability of the periodic problem for nonlinear second-order function-differential equations, *Differ. Equ.* **42**(2006), No. 3, 380–390. url

[25] S. Mukhigulashvili, On a periodic boundary value problem for third order linear functional differential equations, *Nonlinear Anal.* **66**(2007), No. 2, 527–535. MR2279544; url

[26] S. Mukhigulashvili, N. Partsvania, B. Půža, On a periodic problem for higher-order differential equations with a deviating argument, *Nonlinear Anal.* **74**(2011), No. 10, 3232–3241. MR2793558; url

[27] S. H. Saker, S. Agarwal, Oscillation and global attractivity in a nonlinear delay periodic model of respiratory dynamics, *Comput. Math. Appl.* **44**(2002), No. 5–6, 623–632. MR1925807; url

[28] Š. Schwabik, M. Tvrdý, O. Vejvoda, *Differential and integral equations. Boundary value problems and adjoints*, Academia, Praha, 1979. MR542283

[29] L. Shaikhet, *Lyapunov functionals and stability of stochastic functional differential equations*, Springer, Cham, 2013. MR3076210; url

[30] B. Song, L. Pan, J. Cao, Periodic solutions for a class of n-th order functional differential equations, *Int. J. Differ. Equ.* **2011**, Art. ID 916279, 21 pp. MR2843510

[31] J. Šremr, On the Cauchy type problem for systems of functional differential equations, *Nonlinear Anal.* **67**(2007), No. 12, 3240–3260. MR2350882; url

[32] J. Šremr, R. Hakl, On the Cauchy problem for two-dimensional systems of linear functional differential equations with monotone operators, *Nonlinear Oscil. (N. Y.)* **10**(2007), No. 4, 569–582. MR2394937; url

[33] H. Wang, Positive periodic solutions of functional differential equations, *J. Differential Equations* **202**(2004), No. 2, 354–366. MR2069005; url

[34] W. Wang, J. Shen, J. J. Nieto, Periodic boundary value problems for second order functional differential equations, *J. Appl. Math. Comput.* **36**(2011), No. 1–2, 173–186. MR2794139; url

[35] Y. Wu, Existence of positive periodic solutions for a functional differential equation with a parameter, *Nonlinear Anal.* **68**(2008), No. 7, 1954–1962. MR2388755; url

[36] Y. Wu, Existence nonexistence and multiplicity of periodic solutions for a kind of functional differential equation with parameter, *Nonlinear Anal.* **70**(2009), No. 1, 433–443. MR2468249; url