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On an eigenvalue problem with variable exponents and sign-changing potential

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Abstract. In this paper we study a non-homogeneous eigenvalue problem involving variable growth conditions and a sign-changing potential. We prove that any $\lambda > 0$ sufficiently small is an eigenvalue of the nonhomogeneous eigenvalue problem

$$\begin{cases} -\mathrm{div}(a(|\nabla u|)\nabla u) = \lambda V(x)|u|^{q(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

The proofs of the main results are based on Ekeland's variational principle.

Keywords: nonhomogeneous differential operator, eigenvalue problem, Orlicz–Sobolev space, indefinite potential.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded domain with smooth boundary $\partial\Omega$. We assume that the function $a : (0, \infty) \to \mathbb{R}$ is such that the mapping $\phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(t) = \begin{cases} a(|t|)t, & \text{for } t \neq 0, \\ 0, & \text{for } t = 0, \end{cases}$$

is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. We also suppose throughout this paper that $\lambda > 0$, V is an indefinite sign-changing weight and $q : \overline{\Omega} \to (1, \infty)$ is a continuous function. In this note we study the following nonlinear eigenvalue problem:

$$\begin{cases} -\mathrm{div}(a(|\nabla u|)\nabla u) = \lambda V(x)|u|^{q(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{P}$$

The interest in analyzing this kind of problems is motivated by some recent advances in the study of eigenvalue problems involving non-homogeneous operators in the divergence form. We refer especially to the results in [5, 6, 11, 13–16, 18].

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Mihăilescu and Rădulescu, in [13], studied the same nonhomogeneous eigenvalue problem in the particular case when $V(x) = 1$. The authors proved, under the assumption $1 < \inf_{x\in\Omega} q(x) < p_0$, that there exists $\lambda_0 > 0$ such that any $\lambda \in (0, \lambda_0)$ is an eigenvalue for problem (P).

In order to go further we introduce the functional space setting where problem (P) will be discussed. In this context we notice that the operator in the divergence form is not homogeneous and thus, we introduce an Orlicz–Sobolev space setting for problems of this type. Orlicz–Sobolev spaces have been used in the last decades to model various phenomena. Chen, Levine and Rao [3] proposed a framework for image restoration based on a variable exponent Laplacian. A second application which uses variable exponent type Laplace operators is modelling electrorheological fluids [9]. On the other hand, the presence of the continuous functions s and q as exponents appeals to a suitable variable exponent Lebesgue space setting. In the following, we give a brief description of the Orlicz–Sobolev spaces and of the variable exponent Lebesgue spaces.

We first recall some basic facts about Orlicz spaces. Define

$$\Phi(t) = \int_0^t \phi(s)ds,\ \Phi^*(t) = \int_0^t \phi^{-1}(s)ds, \qquad \forall t \in \mathbb{R}.$$

We observe that Φ is a Young function, that is, $\Phi(0) = 0$, Φ is convex, and $\lim_{t\to\infty} \Phi(t) = +\infty$. Furthermore, since $\Phi(0) = 0$ if and only if $t = 0$, $\lim_{t\to 0} \frac{\Phi(t)}{t} = 0$, and $\lim_{t\to\infty} \frac{\Phi(t)}{t} = +\infty$, then Φ is called an N-function. The function Φ^* is called the complementary function of Φ and it satisfies

$$\Phi^*(t) = \sup\{st - \Phi(s) : s \geq 0\}, \qquad \forall t \geq 0.$$

We also observe that Φ^* is also an N-function and the following Young's inequality holds true:

$$st \leq \Phi(s) + \Phi^*(t), \qquad \forall s, t \geq 0.$$

The Orlicz spaces $L_\Phi(\Omega)$ defined by the N-function Φ (see [1,2,4]) is the space of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L_\Phi} = \sup\left\{\int_\Omega uvdx : \int_\Omega \Phi^*(|v|)dx \leq 1\right\} < +\infty.$$

Then $(L_\Phi(\Omega), \|\cdot\|_{L_\Phi})$ is a reflexive Banach space whose norm is equivalent to the Luxemburg norm

$$\|u\|_\Phi = \inf\left\{\mu > 0 : \int_\Omega \Phi\left(\frac{u}{\mu}\right) dx \leq 1\right\}.$$

For Orlicz spaces, Hölder's inequality reads as follows (see [17]):

$$\int_\Omega uvdx \leq 2\|u\|_{L_\Phi}\|v\|_{L_{\Phi^*}}, \qquad \forall u \in L_\Phi(\Omega),\ \forall v \in L_{\Phi^*}(\Omega).$$

We denote by $W_0^1 L_\Phi(\Omega)$ the corresponding Orlicz–Sobolev space for problem (P), equipped with the norm

$$\|u\| = \|\nabla u\|_\Phi$$

(see [8]). The space $W_0^1 L_\Phi(\Omega)$ is also a Banach space.

Throughout this paper we assume that

$$1 < \liminf_{t\to\infty} \frac{t\phi(t)}{\Phi(t)} \leq \limsup_{t>0} \frac{t\phi(t)}{\Phi(t)} < \infty \tag{1.1}$$

and the function $[0,+\infty) \ni t \to \Phi(\sqrt{t}\ )$ is convex. Due to assumption (1.1), we may define the numbers

$$p_0 = \inf_{t>0} \frac{t\phi(t)}{\Phi(t)} \quad \text{and} \quad p^0 = \sup_{t>0} \frac{t\phi(t)}{\Phi(t)}.$$

Note that for $a(|t|) = |t|^{p-2}, p > 1$, one has $p_0 = p^0 = p$.

On the other hand, the following relations hold true:

$$\|u\|^{p^0} \le \int_\Omega \Phi(|\nabla u|)dx \le \|u\|^{p_0}, \qquad \forall u \in W_0^1 L_\Phi(\Omega) \text{ with } \|u\| < 1, \tag{1.2}$$

$$\|u\|^{p_0} \le \int_\Omega \Phi(|\nabla u|)dx \le \|u\|^{p^0}, \qquad \forall u \in W_0^1 L_\Phi(\Omega) \text{ with } \|u\| > 1 \tag{1.3}$$

(see [12, Lemma 1]).

Let us now introduce the Orlicz–Sobolev conjugate Φ_* of Φ, which is given by

$$\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}} ds, \tag{1.4}$$

(see [1]), where we suppose that

$$\lim_{t\to 0} \int_t^1 \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}} ds < +\infty \quad \text{and} \quad \lim_{t\to\infty} \int_1^t \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}} ds = \infty. \tag{1.5}$$

In the case $\Phi(t) = \frac{1}{p}|t|^p$, (1.5) holds if and only if $p < N$.

2 The main result and proof of the theorem

We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (P) if there exists $u \in W_0^1 L_\Phi(\Omega) \setminus \{0\}$ such that

$$\int_\Omega a(|\nabla u|)\nabla u \nabla v dx - \lambda \int_\Omega V(x)|u|^{q(x)-2} u v dx = 0,$$

for all $v \in W_0^1 L_\Phi(\Omega)$. We point out that if λ is an eigenvalue of problem (P), then the corresponding eigenfunction $v \in W_0^1 L_\Phi(\Omega) \setminus \{0\}$ is a weak solution of problem (P).

Our main result is given by the following theorem.

Theorem 2.1. *Suppose that* (1.5) *and the following conditions hold:*

$\mathbf{H}(\mathbf{q},\mathbf{s}):$ $1 < q(x) < p_0 \le p^0 < s(x),\ \forall x \in \overline{\Omega}$.

$\mathbf{H}(\boldsymbol{\Phi}):$ $\lim_{t\to\infty} \frac{|t|^{\frac{s^- q^+}{S^- - q^+}}}{\Phi_*(kt)} = 0,\ \forall k > 0.$

$\mathbf{H}(\mathbf{V}):$ *$V \in L^{s(x)}(\Omega)$ and there exists a measurable set $\Omega_0 \subset \Omega$ of positive measure such that $V(x) > 0, \forall x \in \overline{\Omega}_0$.*

Then there exists $\lambda_0 > 0$ such that any $\lambda \in (0, \lambda_0)$ is an eigenvalue of the problem (P).

Proof. In order to formulate the variational problem (P), let us introduce the functionals $F,\ G,\ \varphi_\lambda : W_0^1 L_\Phi(\Omega) \to \mathbb{R}$ defined by

$$F(u) = \int_\Omega \Phi(|\nabla u|)dx,\ G(u) = \int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}dx$$

and

$$\varphi_\lambda(u) = F(u) - \lambda G(u).$$

Denote by $s'(x)$ the conjugate exponent of the function $s(x)$ and put $\alpha(x) := \frac{s(x)q(x)}{s(x)-q(x)}$. From $\mathbf{H}(\mathbf{q},\mathbf{s})$, we have $s'(x)q(x) < \alpha(x)$, $\forall x \in \overline{\Omega}$, $\alpha(x) < \frac{s^-q^+}{s^--q^+}$, $\forall x \in \overline{\Omega}$. Thus, by relation (1.5), condition $\mathbf{H}(\mathbf{\Phi})$ and Theorem 2.2 in [7], we deduce that $W_0^1L_\Phi(\Omega)$ is compactly embedded in $L^{\frac{s^-q^+}{s^--q^+}}(\Omega)$. That fact combined with the continuous embedding of $L^{\frac{s^-q^+}{s^--q^+}}(\Omega)$ in $L^{\alpha(x)}(\Omega)$ ensures that $W_0^1L_\Phi(\Omega)$ is compactly embedded in $L^{\alpha(x)}(\Omega)$. In an analogous way, we can show that the embedding $X \hookrightarrow L^{s'(x)q(x)}(\Omega)$ is compact.

The proof is divided into the following four steps.

Step 1. We will show that $\varphi_\lambda \in C^1(W_0^1L_\Phi(\Omega), \mathbb{R})$.

Firstly, by Lemma 3.4 in [7] we deduce that F is a C^1 convex functional, with Fréchet derivative given by

$$\langle F'(u), v\rangle = \int_\Omega a(|\nabla u|)\nabla u \nabla v dx.$$

Therefore, we only need to prove that $G \in C^1(W_0^1L_\Phi(\Omega), \mathbb{R})$, that is, we show that for all $h \in W_0^1L_\Phi(\Omega)$,

$$\lim_{t\downarrow 0} \frac{G(u+th)-G(u)}{t} = \langle dG(u), h\rangle,$$

and $dG : W_0^1L_\Phi(\Omega) \to (W_0^1L_\Phi(\Omega))^*$ is continuous, where we denote by $(W_0^1L_\Phi(\Omega))^*$ the dual space of $W_0^1L_\Phi(\Omega)$, $\langle\cdot,\cdot\rangle$ is the pairing between $(W_0^1L_\Phi(\Omega))^*$ and $W_0^1L_\Phi(\Omega)$.

For all $h \in W_0^1L_\Phi(\Omega)$, we have

$$\begin{aligned}
\lim_{t\downarrow 0} \frac{G(u+th)-G(u)}{t} &= \frac{d}{dt}G(u+th)\bigg|_{t=0} \\
&= \left(\frac{d}{dt}\int_\Omega \frac{V(x)}{q(x)}|u+th|^{q(x)}dx\right)\bigg|_{t=0} \\
&= \int_\Omega \frac{d}{dt}\left(\frac{V(x)}{q(x)}|u+th|^{q(x)}\right)\bigg|_{t=0} dx \\
&= \int_\Omega V(x)|u+th|^{q(x)-2}(u+th)h\big|_{t=0}dx \\
&= \int_\Omega V(x)|u|^{q(x)-2}uhdx \\
&= \langle dG(u), h\rangle.
\end{aligned}$$

The differentiation under the integral is allowed for t close to zero. Indeed, for $|t| < 1$, using Hölder's inequality and condition $\mathbf{H}(\mathbf{q},\mathbf{s})$, we have

$$\begin{aligned}
\int_\Omega |V(x)|u+th|^{q(x)-2}(u+th)h|dx &\le \int_\Omega |V(x)||u+th|^{q(x)-1}|h|dx \\
&\le \int_\Omega |V(x)|(|u|+|h|)^{q(x)-1}|h|dx \\
&\le 3|V|_{s(x)}\big||u|+|h|\big|_{q(x)}^{q^i-1}|h|_{\alpha(x)} \\
&< +\infty,
\end{aligned}$$

where $i = +$ if $\big||u|+|h|\big|_{q(x)} > 1$ and $i = -$ if $\big||u|+|h|\big|_{q(x)} \le 1$. Since $W_0^1L_\Phi(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$, $W_0^1L_\Phi(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ and $V \in L^{s(x)}(\Omega)$.

On the other hand, since $W_0^1L_\Phi(\Omega)$ is continuously embedded in $L^{\alpha(x)}(\Omega)$ it follows that there exists positive constants c_1 such that $|h|_{\alpha(x)} \leq c_1\|h\|$. Therefore, by condition $\mathbf{H}(\mathbf{q},\mathbf{s})$, we have

$$\begin{aligned}
|\langle dG(u),h\rangle| &= \left|\int_\Omega |V(x)|u|^{q(x)-2}uhdx\right| \\
&\leq \int_\Omega |V(x)||u|^{q(x)-1}|h|dx \\
&\leq \left(\frac{1}{s^-}+\frac{q^+}{q^+-1}+\frac{1}{\alpha^-}\right)|V|_{s(x)}\left||u|^{q(x)-1}\right|_{\frac{q(x)}{q(x)-1}}|h|_{\alpha(x)} \\
&\leq \left(\frac{1}{s^-}+\frac{q^+}{q^+-1}+\frac{1}{\alpha^-}\right)|V|_{s(x)}|u|_{q(x)}^{q^i-1}|h|_{\alpha(x)} \\
&\leq c_1\left(\frac{1}{s^-}+\frac{q^+}{q^+-1}+\frac{1}{\alpha^-}\right)|V|_{s(x)}|u|_{q(x)}^{q^i-1}\|h\|,
\end{aligned}$$

for any $h \in W_0^1L_\Phi(\Omega)$.

Thus there exists $c_2 = c_1\left(\frac{1}{s^-}+\frac{q^+}{q^+-1}+\frac{1}{\alpha^-}\right)|V|_{s(x)}|u|_{q(x)}^{q^i-1}$ such that

$$|\langle dG(u),h\rangle| \leq c_2\|h\|.$$

Using the linearity of $dG(u)$ and the above inequality we deduce that $dG(u) \in (W_0^1L_\Phi(\Omega))^*$

Note that map $L^{q(x)}(\Omega) \ni u \mapsto |u|^{q(x)-2}u \in L^{\frac{q(x)}{q(x)-1}}(\Omega)$ is continuous. For the Fréchet differentiability, we conclude that G is Fréchet differentiable. Furthermore,

$$\langle G'(u),v\rangle = \int_\Omega V(x)|u|^{q(x)-2}uvdx,$$

for all $u,v \in W_0^1L_\Phi(\Omega)$. The Step 1 is completed.

It is clear that (u,λ) is a solution of (P) if and only if $F'(u) = \lambda G'(u)$ in $(W_0^1L_\Phi(\Omega))^*$.

Step 2. There exists $\lambda_0 > 0$ such that for any $\lambda \in (0,\lambda_0)$ there exist $\tau, a > 0$ such that $\varphi_\lambda(u) \geq a > 0$ for any $u \in W_0^1L_\Phi(\Omega)$ with $\|u\| = \tau$.

Since the embedding $W_0^1L_\Phi(\Omega) \hookrightarrow L^{s'(x)q(x)}(\Omega)$ is continuous, we can find a constant $c_3 > 0$ such that

$$|u|_{s'(x)q(x)} \leq c_3\|u\|, \qquad \forall u \in W_0^1L_\Phi(\Omega). \tag{2.1}$$

Let us fix $\tau \in (0,1)$ such that $\tau < \frac{1}{c_3}$. Then relation (2.1) implies $|u|_{s'(x)q(x)} < 1$, for all $u \in W_0^1L_\Phi(\Omega)$ with $\|u\| = \tau$. Thus,

$$\int_\Omega V(x)|u|^{q(x)}dx \leq |V|_{s(x)}\left||u|^{q(x)}\right|_{s'(x)} \leq |V|_{s(x)}|u|_{q(x)s'(x)}^{q^-}, \tag{2.2}$$

for all $u \in W_0^1L_\Phi(\Omega)$ with $\|u\| = \tau$.

Combining (2.1) and (2.2), we obtain

$$\int_\Omega V(x)|u|^{q(x)}dx \leq c_3^{q^-}|V|_{s(x)}\|u\|^{q^-}, \tag{2.3}$$

for all $u \in W_0^1L_\Phi(\Omega)$ with $\|u\| = \rho$.

Taking into account relations (1.2) and (2.3) we deduce that for any $u \in W_0^1 L_\Phi(\Omega)$ with $\|u\| = \tau < 1$, we have

$$\begin{aligned}
\varphi_\lambda(u) &= \int_\Omega \Phi(|\nabla u|)dx - \lambda \int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}dx \\
&\geq \|u\|^{p^0} - \frac{\lambda c_3^{q^-}}{q^-}|V|_{s(x)}\|u\|^{q^-} \\
&= \tau^{q^-}\left(\tau^{p^0-q^-} - \frac{\lambda c_3^{q^-}}{q^-}|V|_{s(x)}\right).
\end{aligned}$$

Putting

$$\lambda_0 = \frac{\tau^{p^0-q^-}}{2}\frac{q^-}{c_3^{q^-}|V|_{s(x)}},$$

then for any $\lambda \in (0, \lambda_0)$ and $u \in X$ with $\|u\| = \tau$, there exists $a = \frac{\tau^{p^0}}{2}$, such that

$$\varphi_\lambda(u) \geq a > 0.$$

Step 3. There exists $\xi \in W_0^1 L_\Phi(\Omega)$ such that $\xi \geq 0$, $\xi \neq 0$ and $\varphi_\lambda(t\xi) < 0$, for $t > 0$ small enough.

In fact, assumption $\mathbf{H}(\mathbf{q}, \mathbf{s})$ implies $q(x) < p_0$, $\forall x \in \overline{\Omega}_0$. In the sequel, we use the notation $q_0^- = \inf_{\Omega_0} q(x)$ and $q_0^+ = \sup_{\Omega_0} q(x)$. Thus, there exists $\varepsilon_0 > 0$ such that $q_0^- + \varepsilon_0 < p_0$.

Since $q \in C(\overline{\Omega}_0)$, there exists an open set $\Omega_1 \subset \Omega_0$ such that

$$|q(x) - q_0^-| < \varepsilon_0, \qquad \forall x \in \Omega_1.$$

Thus, we deduce

$$q(x) \leq q_0^- + \varepsilon_0, \qquad \forall x \in \Omega_1. \tag{2.4}$$

Take $\xi \in C_0^\infty(\Omega_0)$ such that $\overline{\Omega}_1 \subset \text{supp}(\xi)$, $\xi(x) = 1$ for $x \in \overline{\Omega}_1$ and $0 < \xi < 1$ in Ω_0.

We also point out that there exists $t_0 \in (0,1)$ such that for any $t \in (0, t_0)$ we have

$$\|t|\nabla \xi|\| = t\|\xi\| < 1. \tag{2.5}$$

Using (1.3), (2.4) and (2.5), for all $t \in (0,1)$, we get the estimate

$$\begin{aligned}
\varphi_\lambda(t\xi) &= \int_\Omega \Phi(t|\nabla \xi|)dx - \lambda \int_\Omega \frac{t^{q(x)}}{q(x)}V(x)|\xi|^{q(x)}dx \\
&\leq t^{p_0}\|\xi\|^{p_0} - \lambda \int_{\Omega_0} \frac{t^{q(x)}}{q(x)}V(x)|\xi|^{q(x)}dx \\
&\leq t^{p_0}\|\xi\|^{p_0} - \frac{\lambda}{q_0^+}\int_{\Omega_1} t^{q(x)}V(x)|\xi|^{q(x)}dx \\
&\leq t^{p_0}\|\xi\|^{p_0} - \frac{\lambda t^{q_0^-+\varepsilon_0}}{q_0^+}\int_{\Omega_1} V(x)|\xi|^{q(x)}dx.
\end{aligned}$$

Then, for any $t < \tau^{\frac{1}{p_0 - q_0^- - \varepsilon_0}}$ with $0 < \tau < \min\left\{1, \frac{\lambda \int_{\Omega_1} V(x)|\phi|^{q(x)}dx}{q_0^+\|\xi\|^{p_0}}\right\}$, we conclude that

$$\varphi_\lambda(t\xi) < 0.$$

By Step 2, we have

$$\inf_{v\in\partial B_\rho(0)} \varphi_\lambda(v) > 0. \tag{2.6}$$

On the other hand, by Step 3, there exists $\xi \in W_0^1 L_\Phi(\Omega)$ such that $\varphi_\lambda(t\xi) < 0$ for $t > 0$ small enough. Using (2.3), it follows that

$$\varphi_\lambda(u) \geq \|u\|^{p^0} - \lambda c_3^{q^-} |V|_{s(x)} \|u\|^{q^-}, \qquad \forall u \in B_\rho(0).$$

Thus,

$$-\infty < c_\lambda := \inf_{v\in\overline{B_\rho(0)}} \varphi_\lambda(v) < 0.$$

Now let ε be such that $0 < \varepsilon < \inf_{v\in\partial B_\rho(0)} \varphi_\lambda(v) - \inf_{v\in B_\rho(0)} \varphi_\lambda(v)$. Then, by applying Ekeland's variational principle to the functional

$$\varphi_\lambda : \overline{B_\rho(0)} \to \mathbb{R},$$

there exists $u_\varepsilon \in \overline{B_\rho(0)}$ such that

$$\varphi_\lambda(u_\varepsilon) \leq \inf_{v\in\overline{B_\rho(0)}} \varphi_\lambda(v) + \varepsilon,$$

$$\varphi_\lambda(u_\varepsilon) < \varphi_\lambda(u) + \varepsilon\|u - u_\varepsilon\|, \qquad u \neq u_\varepsilon.$$

Since

$$\varphi_\lambda(u_\varepsilon) \leq \inf_{v\in\overline{B_\rho(0)}} \varphi_\lambda(v) + \varepsilon \leq \inf_{v\in B_\rho(0)} \varphi_\lambda(v) + \varepsilon < \inf_{v\in\partial B_\rho(0)} \varphi_\lambda(v),$$

we deduce that $u_\varepsilon \in B_\rho(0)$.

Now, we define $T_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by

$$T_\lambda(u) = \varphi_\lambda(u) + \varepsilon\|u - u_\varepsilon\|.$$

It is clear that u_ε is a minimum of T_λ. Therefore, for small $t > 0$ and $v \in B_1(0)$, we have

$$\frac{T_\lambda(u_\varepsilon + tv) - T_\lambda(u_\varepsilon)}{t} \geq 0,$$

which implies that

$$\frac{\varphi_\lambda(u_\varepsilon + tv) - \varphi_\lambda(u_\varepsilon)}{t} + \varepsilon\|v\| \geq 0.$$

As $t \to 0$, we have

$$\langle d\varphi_\lambda(u_\varepsilon), v\rangle + \varepsilon\|v\| \geq 0, \qquad \forall v \in B_1(0).$$

Hence, $\|\varphi'_\lambda(u_\varepsilon)\|_{X^*} \leq \varepsilon$. We deduce that there exists a sequence $\{u_n\}_{n=1}^\infty \subset B_\rho(0)$ such that

$$\varphi_\lambda(u_n) \to c_\lambda \quad \text{and} \quad \varphi'_\lambda(u_n) \to 0. \tag{2.7}$$

It is clear that $\{u_n\}_{n=1}^\infty$ is bounded in $W_0^1 L_\Phi(\Omega)$. Since $W_0^1 L_\Phi(\Omega)$ is reflexive, there exists a subsequence, still denoted by $\{u_n\}_{n=1}^\infty$, and $u \in W_0^1 L_\Phi(\Omega)$ such that $\{u_n\}_{n=1}^\infty$ converges weakly to u in $W_0^1 L_\Phi(\Omega)$.

Step 4. We will show that $u_n \to u$ in $W_0^1 L_\Phi(\Omega)$.

Claim:

$$\lim_{n\to\infty} \int_\Omega V(x)|u_n|^{q(x)-2} u_n(u_n - u)dx = 0.$$

In fact, from the Hölder type inequality, we have

$$\begin{aligned}
&\int_\Omega V(x)|u_n|^{q(x)-2}u_n(u_n-u)dx\\
&\quad\le |V|_{s(x)}\left||u_n|^{q(x)-2}u_n(u_n-u)\right|_{s'(x)}\\
&\quad\le |V|_{s(x)}\left||u_n|^{q(x)-2}u_n\right|_{\frac{q(x)}{q(x)-1}}|u_n-u|_{\alpha(x)}\\
&\quad\le |V|_{s(x)}\left(1+|u_n|_{q(x)}^{q^+-1}\right)|u_n-u|_{\alpha(x)}.
\end{aligned}$$

Since $W_0^1L_\Phi(\Omega)$ is continuously embedded in $L^{q(x)}(\Omega)$ and $\{u_n\}_n^\infty$ is bounded in $W_0^1L_\Phi(\Omega)$, so $\{u_n\}_n^\infty$ is bounded in $L^{q(x)}(\Omega)$. On the other hand, since the embedding $W_0^1L_\Phi(\Omega)\hookrightarrow L^{\alpha(x)}(\Omega)$ is compact, we deduce that $|u_n-u|_{\alpha(x)}\to 0$ as $n\to+\infty$. Hence, the proof of the claim is complete.

Moreover, since $d\varphi_\lambda(u_n)\to 0$ and $\{u_n\}_n^\infty$ is bounded in $W_0^1L_\Phi(\Omega)$, we have

$$\begin{aligned}
&|\langle d\varphi_\lambda(u_n),u_n-u\rangle|\\
&\quad\le |\langle d\varphi_\lambda(u_n),u_n\rangle|+|\langle d\varphi_\lambda(u_n),u\rangle|\\
&\quad\le \|d\varphi_\lambda(u_n)\|_{(W_0^1L_\Phi(\Omega))^*}\|u_n\|+\|d\varphi_\lambda(u_n)\|_{(W_0^1L_\Phi(\Omega))^*}\|u\|,
\end{aligned}$$

that is,

$$\lim_{n\to+\infty}\langle d\varphi_\lambda(u_n),u_n-u\rangle=0.$$

Using the previous claim and the last relation we deduce that

$$\lim_{n\to+\infty}\int_\Omega a(|\nabla u_n|)\nabla u_n\nabla(u_n-u)dx=0. \tag{2.8}$$

From (2.8) and the fact that $u_n\rightharpoonup u$ in $W_0^1L_\Phi(\Omega)$ it follows that

$$\lim_{n\to\to+\infty}\langle F'(u_n),u_n-u\rangle=0. \tag{2.9}$$

Next, we show that $u_n\to u$ in $W_0^1L_\Phi(\Omega)$. Since $\{u_n\}$ converges weakly to u in $W_0^1L_\Phi(\Omega)$ it follows that $\{\|u_n\|\}$ is a bounded sequence of real numbers. That fact and relations (1.2) and (1.3) yield that the sequence $\{F(u_n)\}$ is bounded. Then, up to a subsequence, we deduce that $F(u_n)\to c$. The function F being convex, from Mazur's lemma, it is also weakly lower semi-continuous. Hence

$$F(u)\le\liminf_{n\to\infty}F(u_n)=c. \tag{2.10}$$

On the other hand, since F is convex, we have

$$F(u)\ge F(u_n)+\langle F'(u_n),u-u_n\rangle. \tag{2.11}$$

Furthermore, relations (2.9), (2.10) and (2.11) imply

$$F(u)=c.$$

Taking into account that $\{\frac{u_n+u}{2}\}$ converges weakly to u in $W_0^1L_\Phi(\Omega)$ and using the above method we find

$$c=F(u)\le F\Big(\frac{u_n+u}{2}\Big). \tag{2.12}$$

We assume by contradiction that $\{u_n\}$ does not converge to u in $W_0^1L_\Phi(\Omega)$. Furthermore, we deduce that $\{\frac{u_n-u}{2}\}$ does not converge to u in $W_0^1L_\Phi(\Omega)$. It follows that there exist $\varepsilon > 0$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$\left\|\frac{u_n - u}{2}\right\| \geq \varepsilon, \qquad \forall k \geq 1. \tag{2.13}$$

Thus, relations (1.2), (1.3) and (2.13) imply that there exists $\varepsilon_1 > 0$

$$F\left(\frac{u_n - u}{2}\right) \geq \varepsilon_1, \qquad \forall k \geq 1. \tag{2.14}$$

Moreover, from hypotheses (1.1) we deduce that we can apply Lemma 2.1 in [10] in order to obtain

$$\frac{1}{2}\Big[\Phi(|t|) + \Phi(|s|)\Big] \geq \Phi\left(\frac{|t+s|}{2}\right) + \Phi\left(\frac{|t-s|}{2}\right), \qquad \forall t, s \in \mathbb{R}.$$

The above inequality yields

$$\frac{1}{2}\Big[F(u) + F(v)\Big] \geq F\left(\frac{u+v}{2}\right) + F\left(\frac{u-v}{2}\right), \qquad \forall u, v \in W_0^1L_\Phi(\Omega). \tag{2.15}$$

Hence, from (2.14) and (2.16), we have

$$\frac{1}{2}\Big[F(u) + F(u_{n_k})\Big] - F\left(\frac{u_{n_k} + u}{2}\right) \geq F\left(\frac{u_{n_k} - u}{2}\right) \geq \varepsilon_1, \qquad \forall k \geq 1. \tag{2.16}$$

Letting $k \to \infty$ in the above inequality we have

$$c - \varepsilon_1 \geq \limsup_{k\to\infty} F\left(\frac{u_{n_k} + u}{2}\right),$$

and that is a contradiction with (2.12). We conclude that $u_n \to u$ in $W_0^1L_\Phi(\Omega)$. Thus, in view of (2.7), we obtain

$$\varphi_\lambda(u) = c_\lambda < 0 \quad \text{and} \quad \varphi'_\lambda(u) = 0. \tag{2.17}$$

The proof is complete. □

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