Existence and stability of mild solutions to parabolic stochastic partial differential equations driven by Lévy space-time noise

Chaoliang Luo[✉ 1, 2] and Shangjiang Guo[2]

[1]College of Science and Technology, Hunan University of Technology,
Zhuzhou, Hunan 412008, P. R. China
[2]College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, P. R. China

Abstract. This paper is concerned with the well-posedness and stability of parabolic stochastic partial differential equations. Firstly, we obtain some sufficient conditions ensuring the existence and uniqueness of mild solutions, and some $\mathcal{H}$-stability criteria for a class of parabolic stochastic partial differential equations driven by Lévy space-time noise under the local/non-Lipschitz condition. Secondly, we establish some existence-uniqueness theorems and present sufficient conditions ensuring the $\mathcal{H}'$-stability of mild solutions for a class of parabolic stochastic partial functional differential equations driven by Lévy space-time noise under the local/non-Lipschitz condition. These theoretical results generalize and improve some existing results. Finally, two examples are given to illustrate the effectiveness of our main results.

Keywords: Lévy space-time noise, parabolic stochastic partial differential equation, non-Lipschitz, well-posedness, stability.

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1   Introduction

It is well known that stochastic partial differential equations (SPDEs) are appropriate mathematical models for many multiscale systems with uncertain and fluctuating influences, which are playing an increasingly important role in accurately describing complex phenomena in physics, geophysics, biology, etc. In recent years, the theoretical research of SPDEs has attracted a large number of research workers, and has already achieved fruitful results. There are many interesting problems, such as well-posed problem, blow-up problem, stability, invariant measures and other properties, which have been extensively investigated for different kinds of SPDEs. We refer the reader to [4, 9, 20, 23–26, 29] for more details and some new developments. There are many results in which the coefficients satisfy the global Lipschitz condition and the linear growth condition [8, 14]. However, the global Lipschitz condition,

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✉ Corresponding author. Email: lcl197511@163.com

even the local Lipschitz condition, is seemed to be considerably strong in discussing variable applications in the real world. Obviously, we need to find some weaker or more general conditions ensuring the existence and uniqueness of solutions of SPDEs.

Xie [27] investigated the following stochastic heat equation driven by space-time white noise

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u(t,x)}{\partial x^2} + b\Big(t,x,u(t,x)\Big) + \sigma\Big(t,x,u(t,x)\Big)\dfrac{\partial^2 W(t,x)}{\partial t \partial x}, \ t \geq 0, \ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $\{W(t,x), t \geq 0, x \in \mathbb{R}\}$ is a two-sided Brownian sheet and the coefficients $b, \sigma : [0, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous nonlinear functions. Such an equation arises in many fields, such as population biology, quantum field, statistical physics, neurophysiology, and so on, see [6, 12, 19]. By using the successive approximation argument, Xie [27] studied the existence of mild solutions to equation (1.1) under some conditions weaker than the Lipschitz condition.

It is worth pointing out that the work of [27] focuses on the SPDE driven by Browian motion whose path is continuous. However, many abrupt changes such as environmental shocks for the population, sudden earthquakes, hurricanes or epidemics may lead to the discontinuity of the sample paths. Therefore, SPDEs driven by Brownian motions are not appropriate to model some real situations where large external and/or internal fluctuations with possible large jumps might exist. But Lévy noise can produce large jumps or exhibit long heavy tails of the distribution which makes the sample paths discontinuous in time, so SPDEs driven by Lévy noise are more suitable for the actual situation (see [1, 18, 21]). In [1], Albeverio, Wu and Zhang established the existence and uniqueness of mild solutions for a class of stochastic heat equations driven by compensated Poisson random measures. In [21], Shi and Wang discussed the mild solutions to SPDEs driven by Lévy space-time white noise under the uniform Lipschitz condition. So far as we know, however, there has been no mathematical treatment about the pathwise uniqueness to parabolic SPDEs driven by Lévy noise under some kinds of conditions weaker than the Lipschitz condition. Inspired by the work of Xu, Pei and Guo [28], in this paper we shall promote the work of Xie [27] and investigate the following parabolic SPDE

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u(t,x)}{\partial x^2} + b\Big(t,x,u(t,x)\Big) + \sigma\Big(t,x,u(t,x)\Big)\dot{L}(t,x), \ t \geq t_0, \ x \in \mathbb{R}, \\ u(t_0,x) = u_0(x), \end{cases} \tag{1.2}$$

where $t_0 \geq 0$, $(t,x) \in [t_0, +\infty) \times \mathbb{R}$, $\dot{L}(t,x)$ is Lévy space-time white noise, and the coefficients $b, \sigma : [t_0, +\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are usually continuous nonlinear functions. Here, our primary task is to investigate the existence and uniqueness of mild solutions to (1.2) under the local/non-Lipschitz condition which includes the Lipschitz condition as a special case. Furthermore, in order to obtain the dynamical properties of solutions to SPDEs driven by Lévy noise, we shall seek for some $\mathcal{H}$-stability conditions under which the mild solutions of (1.2) are $\mathcal{H}$-stable.

We also notice that the above mentioned results are based on the fact that the future of systems is independent of the past states and is determined solely by the present. However, in realistic models many dynamical systems depend on not only the present but also the past states and even the future states of the systems. Stochastic functional differential equations (SFDEs) give a mathematical formulation for such models. Recently, the investigation of stochastic partial functional differential equations (SPFDEs) has attracted the considerable

attentions of researchers and many qualitative theories of SPFDEs have been obtained in literature [3, 7, 8, 10, 11, 15, 16, 22]. There are a lot of substantial results on the existence and uniqueness of solutions to SPFDEs. For example, Taniguchi [22] and Luo [15] employed the Banach fixed point theorem and the successive approximation method to study the existence and uniqueness of mild solutions for SPFDEs under the global Lipschitz condition and the linear growth condition. By using the stochastic convolution, Govindan [8] investigated the existence, uniqueness and almost sure exponential stability of neutral SPFDEs under the global Lipschitz condition and the linear growth condition. Luo and Guo [17] studied the existence and uniqueness of mild solutions for parabolic SPFDEs driven by Winner space-time white noise under the non-Lipschitz condition. To the best of our knowledge, there is few work about the well-posedness and stability for mild solutions to parabolic SPFDEs driven by Lévy space-time white noise.

Motivated by the previous problems, in this paper we further investigate the following parabolic SPFDE driven by Lévy space-time noise

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u(t,x)}{\partial x^2} + b\Big(t,x,u_t(x)\Big) + \sigma\Big(t,x,u_t(x)\Big)\dot{L}(t,x), \ t_0 \le t \le T, x \in \mathbb{R}, \\ u_{t_0}(x) = \Big\{\xi(\theta,x) : -\tau \le \theta \le 0, x \in \mathbb{R}\Big\}, \end{cases} \tag{1.3}$$

where $u_t(x) \triangleq \{u(t+\theta,x), -\tau \le \theta \le 0, x \in \mathbb{R}\}$ is regarded as an $\mathcal{F}_t$-measurable $\mathbb{C}([-\tau,0] \times \mathbb{R};\mathbb{R})$-valued stochastic process, $u_{t_0}(x) = \{\xi(\theta,x) : -\tau \le \theta \le 0, x \in \mathbb{R}\}$ is an $\mathcal{F}_{t_0}$-measurable $\mathbb{C}([-\tau,0] \times \mathbb{R};\mathbb{R})$-valued stochastic variable satisfying $\mathbb{E}[\|\xi\|^2] < \infty$. The coefficients $b, \sigma : [t_0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Borel measurable functions and are perhaps not Lipschitz, and $\dot{L}(t,x)$ is Lévy space-time white noise. Therefore, the other two tasks of this paper are to discuss the well-posedness of mild solutions to (1.3) under the local/non-Lipschitz condition, and to obtain some sufficient conditions ensuring the $\mathcal{H}'$-stability of mild solutions to (1.3).

The rest of the paper is organized as follows. After presenting some preliminaries in the next section, we establish some existence-uniqueness theorems under the local/non-Lipschitz condition and provide some sufficient conditions ensuring the $\mathcal{H}$-stability of mild solutions to SPDE (1.2) in Section 3. Section 4 is devoted to the well-posedness and $\mathcal{H}'$-stability of mild solutions to SPFDE (1.3) under the local/non-Lipschitz condition. Two examples are provided in Section 5 to illustrate our main results.

Throughout this paper, the letters C and C' represent some positive constants which may change occasionally their values from line to line. If C and C' are essential to depending on some parameters, e.g. T etc, which will be written as C_T and C_T', respectively.

2 Preliminaries

In this section, let us recall some basic definitions and introduce some notations and assumptions. Assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \ge t_0}, \mathbb{P})$ is a complete probability space with the filtration $\{\mathcal{F}_t\}_{t \ge t_0}$, which satisfies the usual condition, i.e., $\{\mathcal{F}_t\}$ is a right continuous, increasing family of sub σ-algebras of $\mathcal{F}$ and $\mathcal{F}_{t_0}$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. Let $\mathcal{H}$ be the family of all random fields $\{X(t,x), t \ge t_0, x \in \mathbb{R}\}$ defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \ge t_0}, \mathbb{P})$ such that

$$\|X\|_{\mathcal{H}} \triangleq \sup_{t \in [t_0,T], x \in \mathbb{R}} e^{-r|x|} \mathbb{E}\left[|X(t,x)|^2\right] < \infty,$$

where $r > 0$. Following from Borel-Cantell's lemma [5], $\mathcal{H}$ equipped with the norm $\|\cdot\|_{\mathcal{H}}$ is a Banach space.

2.1 Lévy space-time white noise

Let $(E, \varepsilon, \mu) \triangleq ([t_0, \infty) \times \mathbb{R}, \mathcal{B}([t_0, \infty) \times \mathbb{R}), dt \times dx)$ and $(U, \mathcal{B}(U), \nu)$ denote two σ-finite measurable spaces. We define $N : (E, \varepsilon, \mu) \times (U, \mathcal{B}(U), \nu) \times (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{N} \cup \{0\} \cup \{\infty\}$ as a Poisson noise, if for all $A \in \varepsilon, B \in \mathcal{B}(U)$ and $n \in \mathbb{N} \cup \{0\} \cup \{\infty\}$,

$$\mathbb{P}\Big(N(A, B) = n\Big) = \frac{e^{-\mu(A)\nu(B)} \left[\mu(A)\nu(B)\right]^n}{n!}.$$

Furthermore, for all $(t, \mathbb{A}, \mathbb{B}) \in [t_0, \infty) \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(U)$, we define a compensated Poisson random martingale measure by

$$\tilde{N}(\mathbb{B}, \mathbb{A}, t) = N\Big([t_0, t] \times \mathbb{A}, \mathbb{B}\Big) - \mu\Big([t_0, t] \times \mathbb{A}\Big)\nu(\mathbb{B}),$$

provided that $\mu([t_0, t] \times \mathbb{A})\nu(\mathbb{B}) < \infty$.

Let $\phi : E \times U \times \Omega \to \mathbb{R}$ be an $\{\mathcal{F}_t\}_{t \geq t_0}$-predictable function satisfying

$$\mathbb{E}\left[\int_{t_0}^t \int_{\mathbb{A}} \int_{\mathbb{B}} |\phi(s, x, y)|^2 \nu(dy) dx ds\right] < \infty,$$

for all $t > t_0$ and $(\mathbb{A}, \mathbb{B}) \in \mathcal{B}(\mathbb{R}) \times \mathcal{B}(U)$. Then, the stochastic integral process

$$\int_{t_0}^{t^+} \int_{\mathbb{A}} \int_{\mathbb{B}} \phi(s, x, y) \tilde{N}(dy, dx, ds)$$

can be well defined for $t > t_0$ (see [13]), which is a square integrable $(\mathbb{P}, \{\mathcal{F}_t\}_{t \geq t_0})$-martingale. Moreover, the stochastic integral process has the following isometry property

$$\mathbb{E}\left\{\left[\int_{t_0}^{t^+} \int_{\mathbb{A}} \int_{\mathbb{B}} \phi(s, x, y) \tilde{N}(dy, dx, ds)\right]^2\right\} = \mathbb{E}\left[\int_{t_0}^t \int_{\mathbb{A}} \int_{\mathbb{B}} |\phi(s, x, y)|^2 \nu(dy) dx ds\right]. \tag{2.1}$$

Now, we introduce a pure jump Lévy space-time white noise which possesses the following structure (see [2] for details)

$$\dot{L}(t, x) = \dot{W}(t, x) + \int_{U_0} h_1(t, x, y) \dot{\tilde{N}}(dy, x, t) + \int_{U/U_0} h_2(t, x, y) \dot{N}(dy, x, t), \tag{2.2}$$

for some $U_0 \in \mathcal{B}(U)$ satisfying $\nu(U/U_0) < \infty$ and $\int_{U_0} y^2 \nu(dy) < +\infty$. Here, $h_1, h_2 : [t_0, \infty) \times \mathbb{R} \times U \to \mathbb{R}$ are some measurable functions, and $\dot{W}(t, x) = \frac{\partial^2}{\partial t \partial x} W(t, x)$ is a Gaussian space-time white noise on $[t_0, \infty) \times \mathbb{R}$. $\dot{\tilde{N}}$ and $\dot{N}$ are the Radon–Nikodym derivatives, i.e.,

$$\dot{\tilde{N}}(dy, x, t) = \frac{\tilde{N}(dy, dx, dt)}{dt \times dx}, \quad \dot{N}(dy, x, t) = \frac{N(dt \times dx, dy)}{dt \times dx},$$

where $(t, x, y) \in [t_0, \infty) \times \mathbb{R} \times U$.

2.2 Gaussian kennel and its properties

Let the Gaussian kernel $G(t, x)$ denote the fundamental solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t} G(t, x) = \dfrac{1}{2} \dfrac{\partial^2}{\partial x^2} G(t, x), & t > 0, \ x \in \mathbb{R}, \\ G(0, x) = \delta_0(x), & t = 0, \end{cases} \tag{2.3}$$

where δ_0 stands for the Dirac function. By Fourier transform, we obtain

$$G(t,x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}, \qquad t > 0, \ x \in \mathbb{R}.$$

Let $g(t,x,z) = G(t,x-z)$ for all $t > 0$, $x,z \in \mathbb{R}$, then the heat kernel $g(t,x,z)$ have the following properties (see [27] for details).

Lemma 2.1.

  (i) *For each $r \in \mathbb{R}$ and $T > 0$, there exists a constant C depending only on r, t_0 and T such that*

$$\int_{\mathbb{R}} g(t,x,z)e^{r|z|}dz \le Ce^{r|x|}, \qquad \forall(x,t) \in \mathbb{R} \times [t_0, T]. \tag{2.4}$$

  (ii) *If $0 < p < 3$, then there exists a positive constant C such that*

$$\int_{t_0}^{t} \int_{\mathbb{R}} |g(t-s,x,z)|^p dsdy \le Ct^{\frac{3-p}{2}}, \qquad \forall(x,t) \in \mathbb{R} \times [t_0, T]. \tag{2.5}$$

  (iii) *If $\frac{3}{2} < p < 3$, then there exists a positive constant C such that for all $t \in [t_0, T]$,*

$$\int_{t_0}^{t} \int_{\mathbb{R}} |g(t-s,x,z) - g(t-s,x',z)|^p dsdy \le C|x-x'|^{3-p}, \qquad \forall x, x' \in \mathbb{R}. \tag{2.6}$$

  (iv) *If $1 < p < 3$, then there exists a positive constant C such that for all t and t' ($t_0 \le t \le t' \le T$),*

$$\int_{t_0}^{t} \int_{\mathbb{R}} |g(t-s,x,z) - g(t'-s,x,z)|^p dsdy \le C|t-t'|^{\frac{3-p}{2}}, \qquad \forall x \in \mathbb{R}, \tag{2.7}$$

  and

$$\int_{t}^{t'} \int_{\mathbb{R}} |g(t'-s,x,z)|^p dsdy \le C|t-t'|^{\frac{3-p}{2}}, \qquad \forall x \in \mathbb{R}. \tag{2.8}$$

2.3 Bihari's lemma

We give two lemmas without proofs (see [27] for the proofs), which will be used many times in the following analysis.

Lemma 2.2 (Bihari inequality). *Let $\alpha \ge 0$ and $T > 0$, $\chi(t), v(t)$ be two nonnegative continuous functions defined on $[0, T]$. Assume that φ is a positive, continuous and nondecreasing concave function defined on $[0, \infty)$ such that $\varphi(q) > 0$ for $q > 0$. If $v(t)$ is integrable on $[0, T]$ and for each $t \in [0, T]$,*

$$\chi(t) \le \alpha + \int_0^t v(s)\varphi\Big(\chi(s)\Big)ds,$$

then for every $t \in [0, T]$,

$$\chi(t) \le G^{-1}\left(G(\alpha) + \int_0^t v(s)ds\right)$$

and

$$G(\alpha) + \int_0^t v(s)ds \in \mathrm{Dom}(G^{-1}),$$

where G^{-1} is the inverse function of G and $G(q) = \int_1^q \frac{1}{\varphi(s)}ds$. In particular, if $\alpha = 0$, $G(q) = \int_{0^+} \frac{1}{\varphi(s)}ds = \infty$, then $\chi(t) = 0$ for all $t \in [0, T]$.

Lemma 2.3. *Under the conditions of Lemma 2.2, if for every $\epsilon > 0$, there exists $t_1 \geq 0$ such that*

$$\int_{t_1}^T v(t)dt \leq \int_\alpha^\epsilon \frac{1}{\varphi(s)} ds$$

for $0 \leq \alpha < \epsilon$. Then the estimate $\chi(t) \leq \epsilon$ holds for all $t \in [t_1, T]$.

3 Existence and stability of mild solutions to SPDE (1.2)

This section is devoted to the existence and uniqueness of mild solutions to (1.2) under the non-Lipschitz condition. Equation (1.2) can be given by the following integral equation

$$u(t,x) = \int_\mathbb{R} g(t,x,z)u(t_0,z)dz + \int_{t_0}^t \int_\mathbb{R} g(t-s,x,z)b\Big(s,z,u(s,z)\Big)dzds$$

$$+ \int_{t_0}^t \int_\mathbb{R} g(t-s,x,z)\sigma\Big(s,z,u(s,z)\Big)\dot{L}(z,s)dzds$$

for $t \in [t_0, T]$ and $x \in \mathbb{R}$. In view of the definition of Lévy space-time white noise, we obtain

$$
\begin{aligned}
u(t,x) = {} & \int_\mathbb{R} g(t,x,z)u(t_0,z)dz + \int_{t_0}^t \int_\mathbb{R} g(t-s,x,z)b\Big(s,z,u(s,z)\Big)dzds \\
& + \int_{t_0}^t \int_\mathbb{R} g(t-s,x,z)\sigma\Big(s,z,u(s,z)\Big)W(dz,ds) \\
& + \int_{t_0}^t \int_\mathbb{R} g(t-s,x,z)\sigma\Big(s,z,u(s,z)\Big)\psi(s,z)dzds \\
& + \int_{t_0}^{t^+} \int_\mathbb{R} \int_U g(t-s,x,z)\sigma\Big(s,z,u(s,z)\Big)h(s,z,y)\dot{\tilde{N}}(dy,dz,ds)
\end{aligned}
$$

(3.1)

for all $(t,x) \in [t_0, T] \times \mathbb{R}$, with the mappings $\psi(s,z)$ and $h(t,z,y)$ defined by

$$\psi(s,z) = \int_{U/U_0} h_2(t,z,y)v(dy),$$

$$h(t,z,y) = h_1(t,z,y)I_{U_0}(y) + h_2(t,z,y)I_{U/U_0}(y),$$

where we suppose that all integrals on the right-hand side of (3.1) exist, and I_{U_0} denotes the indicator function of the set U_0.

3.1 Well-posedness of mild solutions to (1.2)

Now, we recall Shi and Wang's recent work [21] on the existence of mild solutions of the following SPDE with Lévy space-time white noise, which is induced by the λ-fractional differential operator,

$$
\begin{cases}
\dfrac{\partial u(t,x)}{\partial t} = \Delta_\lambda u(t,x) + b\Big(t,x,u(t,x)\Big) + \sigma\Big(t,x,u(t,x)\Big)\dot{L}(t,x), \\
u(0,x) = u_0(x),
\end{cases}
$$

(3.2)

where $(t,x) \in [0,+\infty) \times \mathbb{R}$, $0 < \lambda \leq 2$, $\dot{L}(t,x)$ is Lévy space-time white noise, Δ_λ is λ-fractional differential operator and is defined via Fourier transform $\Im$ by

$$\Im(\Delta_\lambda u)(\xi) = -|\xi|^\lambda \Im(u)(\xi), \qquad u \in \text{Dom}(\Delta_\lambda), \ \xi \in \mathbb{R}.$$

We introduce the following assumptions:

(P1) b, σ are uniform Lipschitz, i.e. there exists a constant $C > 0$ such that $|b(t, x, u_1) - b(t, x, u_2)| + |\sigma(t, x, u_1) - \sigma(t, x, u_2)| \leq C|u_1 - u_2|$ for all $(t, x) \in [0, T] \times \mathbb{R}$ and $u_1, u_2 \in \mathbb{R}$;

(P2) for every $p \in \left(\frac{2(\lambda+1)}{\lambda-1}, \infty \right)$ with $\lambda \in (1, 2]$,

$$\sup_{0 \leq t \leq T} \|\psi(t, \cdot)\|_p^p < \infty, \quad \sup_{0 \leq t \leq T} \left\| \int_U |h(t, \cdot, y)|^2 \nu(dy) \right\|_{\frac{p}{2}}^{\frac{p}{2}} < \infty,$$

where the functions $\psi(t, \cdot)$ and $h(t, \cdot, y)$ are specified in (3.1), and the norm $\| \cdot \|_p$ is defined as $\|\psi(t, x)\|_p \triangleq \left[\int_{\mathbb{R}} \psi^p(t, x) dx \right]^{\frac{1}{p}}$; and

(P3) the initial condition $u_0(x)$ is $\mathcal{F}_0$-measurable and satisfies $\mathbb{E}\left[\|u_0(\cdot)\|_p^p \right] < \infty$.

Under the assumptions (P1)–(P3), Shi and Wang [21] obtained the following result by the Banach's fixed point theorem.

Proposition 3.1 ([21])**.** *Under the assumptions (P1)–(P3), there exists a unique mild solution to SPDE* (3.2). *Moreover, for every $p \in \left(\frac{2(\lambda+1)}{\lambda-1}, \infty \right)$, this mild solution satisfies*

$$\sup_{0 \leq t \leq T} \|u(t, \cdot)\|_p^p < \infty.$$

Remark 3.2. It is worthwhile to point out that there exists a unique mild solution $u(t, x)$ to SPDE (1.2) under the conditions (P1)–(P3), because we can employ the same method as that of Proposition 3.1 to prove the existence and uniqueness of mild solutions.

Therefore, we can obtain the following existence result for SPDE (1.2).

Corollary 3.3. *Suppose that the conditions (P1)–(P3) hold. Then there exists a unique mild solution $u(t, x)$ to SPDE* (1.2) *with initial condition $u_0(x)$ satisfying $\mathbb{E}[\|u_0(\cdot)\|_2^2] < \infty$. Moreover, the solution $u(t, x)$ satisfies $\sup_{t_0 \leq t \leq T} \|u(t, \cdot)\|_2^2 < \infty$.*

However, it is easy to find that the condition (P1) is very stringent in Proposition 3.1 and Corollary 3.3. A natural question arises: Whether or not can the condition (P1) be relaxed to the local Lipschitz case or the non-Lipschitz case? In what follows, we shall investigate the existence and uniqueness of SPDE (1.2) under the following assumptions (i.e., the local Lipschitz condition and the non-Lipschitz condition).

(S1) b, σ are local Lipschitz, i.e. there exists a positive constant K_n such that for all $(t, x) \in [t_0, T] \times \mathbb{R}$ and $u_1, u_2 \in \mathbb{R}$ with $|u_1| \vee |u_2| \leq n$,

$$|b(t, x, u_1) - b(t, x, u_2)|^2 + |\sigma(t, x, u_1) - \sigma(t, x, u_2)|^2 \leq K_n|u_1 - u_2|^2.$$

(S'1) If there exist a strictly positive, nondecreasing function $\lambda(t)$ defined on $[t_0, T]$ and a nondecreasing, continuous function $\phi(u)$ defined on $\mathbb{R}_+$ such that for all $(t, x) \in [t_0, T] \times \mathbb{R}$ and $u_1, u_2 \in \mathbb{R}$,

$$|b(t, x, u_1) - b(t, x, u_2)|^2 + |\sigma(t, x, u_1) - \sigma(t, x, u_2)|^2 \leq \lambda(t)\phi(|u_1 - u_2|^2), \qquad (3.3)$$

where $\lambda(t)$ is a locally integrable function, $\phi(u)$ or $\phi^2(u)/u$ is a concave function with $\phi(0) = 0$ satisfying $\int_{0^+} \frac{1}{\phi(u)} du = \infty$.

(S2) The functions $\psi(t,z)$ and $h(t,z,y)$ satisfy the following conditions

$$\sup_{t_0 \leq t \leq T} \int_{\mathbb{R}} \psi^2(t,z)dz < \infty,$$

$$\sup_{t_0 \leq t \leq T} \int_{\mathbb{R}} \int_U |h(t,z,y)|^2 \nu(dy)dz < \infty.$$

(S3) $b(t,x,0)$ and $\sigma(t,x,0)$ are locally integrable functions with respect to t and x.

(S4) The initial condition $u_0(x)$ is $\mathcal{F}_{t_0}$-measurable and satisfies $\sup_{x \in \mathbb{R}} \mathbb{E}\left[u_0^2(x)\right] < \infty$.

Remark 3.4. The condition (S'1) is so-called non-Lipschitz condition. In particular, when $\lambda(t) = k$ is a positive constant and $\phi(u) = u$, then the condition (S'1) can be reduced to the Lipschitz condition.

Utilizing the method of [21], we obtain the following conclusion.

Corollary 3.5. *Assume that the conditions (P1) and (S2)–(S4) hold. Then there exists a unique $\mathcal{H}$-valued solution $u(t,x)$ to SPDE (1.2) with the initial value $u_0(x)$.*

Before stating our main results, we give the following auxiliary conclusion.

Lemma 3.6. *Suppose that the conditions (S1) and (S3) or (S'1) and (S3) hold, then there exists a positive constant K such that for all $(t,x,u) \in [t_0, T] \times \mathbb{R} \times \mathbb{R}$,*

$$|b(t,x,u)|^2 + |\sigma(t,x,u)|^2 \leq K(1 + |u|^2). \tag{3.4}$$

Proof. Here, we shall only prove the conclusion under the conditions (S'1) and (S3). For the conditions (S1) and (S3), it can be shown by the same techniques. Since $\phi(u)$ or $\phi^2(u)/u$ is a concave and non-negative function satisfying $\phi(0) = 0$, we can choose two appropriate positive constants k_1 and k_2 such that

$$\phi(u) \leq k_1 + k_2 u, \qquad u \geq 0.$$

Therefore, by utilizing (3.3), we have

$$\begin{aligned}
|b(t,x,u)|^2 &+ |\sigma(t,x,u)|^2 \\
&\leq 2|b(t,x,u) - b(t,x,0)|^2 + 2|b(t,x,0)|^2 + 2|\sigma(t,x,u) - \sigma(t,x,0)|^2 + 2|\sigma(t,x,0)|^2 \\
&\leq 2\Big(|b(t,x,u) - b(t,x,0)|^2 + |\sigma(t,x,u) - \sigma(t,x,0)|^2\Big) + 2\Big(|b(t,x,0)|^2 + |\sigma(t,x,0)|^2\Big) \\
&\leq 2\lambda(t)\phi(|u|^2) + \sup_{t_0 \leq t \leq T, x \in \mathbb{R}} 2\Big(|b(t,x,0)|^2 + |\sigma(t,x,0)|^2\Big) \\
&\leq K(1 + |u|^2),
\end{aligned}$$

where $K = \max_{t_0 \leq t \leq T}\{\sup_{t_0 \leq t \leq T, x \in \mathbb{R}} 2(|b(t,x,0)|^2 + |\sigma(t,x,0)|^2 + k_1\lambda(t)), 2k_2\lambda(t)\}$. Thus the proof of Lemma 3.6 is completed. $\square$

Remark 3.7. Lemma 3.6 tells us that the linear growth condition can be obtained by utilizing the local Lipschitz condition or the non-Lipschitz condition. This result plays an important role in proving the following important conclusion.

Now, we shall present a well-posed result for the mild solutions of (1.2) under the local Lipschitz condition.

Theorem 3.8. *Under the conditions (S1)–(S4), there exists an $\mathcal{H}$-valued solution $u(t,x)$ to SPDE (1.2) with the initial value $u_0(x)$.*

Proof. Here we only outline the proof by utilizing a truncation procedure. For every integer $n \geq 1$, we define truncation functions $b_n(t,x,u(t,x))$ and $\sigma_n(t,x,u(t,x))$ by

$$b_n\Big(t,x,u(t,x)\Big) = b\left(t,x,\frac{n \wedge |u(t,x)|}{|u(t,x)|}u(t,x)\right),$$

$$\sigma_n\Big(t,x,u(t,x)\Big) = b\left(t,x,\frac{n \wedge |u(t,x)|}{|u(t,x)|}u(t,x)\right),$$

where we set $\frac{|u(t,x)|}{|u(t,x)|} = 1$ if $u(t,x) \equiv 0$. Then b_n and σ_n satisfy the uniform Lipschitz condition (P1). Therefore, by Corollary 3.5, there exists a unique $\mathcal{H}$-valued solution $u_n(t,x)$ to the following equation

$$
\begin{aligned}
u_n(t,x) = {} & \int_{\mathbb{R}} g(t,x,z)u(t_0,z)dz + \int_{t_0}^{t}\int_{\mathbb{R}} g(t-s,x,z)b_n\Big(s,z,u_n(s,z)\Big)dzds \\
& + \int_{t_0}^{t}\int_{\mathbb{R}} g(t-s,x,z)\sigma_n\Big(s,z,u_n(s,z)\Big)W(dz,ds) \\
& + \int_{t_0}^{t}\int_{\mathbb{R}} g(t-s,x,z)\sigma_n\Big(s,z,u_n(s,z)\Big)\psi(s,z)dzds \\
& + \int_{t_0}^{t^+}\int_{\mathbb{R}}\int_{U} g(t-s,x,z)\sigma_n\Big(s,z,u_n(s,z)\Big)h(s,z,y)\mathring{\tilde{N}}(dy,dz,ds).
\end{aligned}
\tag{3.5}
$$

Introduce the stopping time

$$\eta_n = T \wedge \inf\Big\{t \in [t_0,T] : |u_n(t,x)| \geq n\Big\},$$

where we set $\inf \phi = \infty$ if possible. It is easy to see that

$$u_n(t,x) = u_{n+1}(t,x) \quad \text{if } t_0 \leq t \leq \eta_n,$$

and $\eta_n \uparrow T$ as $n \to \infty$. Therefore, there exists an integer n_0 such that $\eta_n = T$ when $n \geq n_0$. Let

$$u(t,x) = u_{n_0}(t,x) \quad \text{for } (t,x) \in [t_0,T] \times \mathbb{R}.$$

Thus $u(t \wedge \eta_n, x) = u_n(t \wedge \eta_n, x)$, which combining with (3.5), yields that

$$
\begin{aligned}
u(t \wedge \eta_n, x) = {} & \int_{\mathbb{R}} g(t,x,z)u(t_0,z)dz + \int_{t_0}^{t \wedge \eta_n}\int_{\mathbb{R}} g(t-s,x,z)b_n\Big(s,z,u(s,z)\Big)dzds \\
& + \int_{t_0}^{t \wedge \eta_n}\int_{\mathbb{R}} g(t-s,x,z)\sigma_n\Big(s,z,u(s,z)\Big)W(dz,ds) \\
& + \int_{t_0}^{t \wedge \eta_n}\int_{\mathbb{R}} g(t-s,x,z)\sigma_n\Big(s,z,u(s,z)\Big)\psi(s,z)dzds \\
& + \int_{t_0}^{t \wedge \eta_n^+}\int_{\mathbb{R}}\int_{U} g(t-s,x,z)\sigma_n\Big(s,z,u(s,z)\Big)h(s,z,y)\mathring{\tilde{N}}(dy,dz,ds).
\end{aligned}
$$

From the definition of truncation functions $b_n(t, x, u(t, x))$ and $\sigma_n(t, x, u(t, x))$, it follows that

$$
\begin{aligned}
u(t \wedge \eta_n, x) = & \int_{\mathbb{R}} g(t, x, z) u(t_0, z) dz + \int_{t_0}^{t \wedge \eta_n} \int_{\mathbb{R}} g(t - s, x, z) b\Big(s, z, u(s, z)\Big) dz ds \\
& + \int_{t_0}^{t \wedge \eta_n} \int_{\mathbb{R}} g(t - s, x, z) \sigma\Big(s, z, u(s, z)\Big) W(dz, ds) \\
& + \int_{t_0}^{t \wedge \eta_n} \int_{\mathbb{R}} g(t - s, x, z) \sigma\Big(s, z, u(s, z)\Big) \psi(s, z) dz ds \\
& + \int_{t_0}^{t \wedge \eta_n^+} \int_{\mathbb{R}} \int_U g(t - s, x, z) \sigma\Big(s, z, u(s, z)\Big) h(s, z, y) \dot{\tilde{N}}(dy, dz, ds).
\end{aligned}
$$

Let $n \to \infty$ we observe that $u(t, x)$ satisfies equation (3.1), which implies that $u(t, x)$ is the solution of equation (1.2). Therefore, we complete the proof of the theorem. □

The following Lemma is a corollary of Bihari's lemma which can be found in [27] and will provide some help in the forthcoming proof.

Lemma 3.9. *Assume that $\lambda(t)$ and $\phi(u)$ satisfy the condition (S'1). If for some $\alpha \in (0, \frac{1}{2}]$, there exists a nonnegative measurable function $z(t)$ satisfying $z(0) = 0$ and*

$$
z(t) \le \int_0^t \frac{\lambda(s) \phi\Big(z(s)\Big)}{(t - s)^\alpha} ds, \qquad \forall t \in [t_0, T],
$$

then $z(t) \equiv 0$ on $[t_0, T]$.

In what follows, we study the existence of mild solutions for SPDE (1.2) with initial value $u_0(x)$ under the non-Lipschitz condition.

Theorem 3.10. *Under the conditions (S'1), (S2)–(S4), there exists a unique $\mathcal{H}$-valued solution $u(t, x)$ to SPDE (1.2) with the initial value $u_0(x)$.*

We prepare a lemma in order to prove this theorem.

Lemma 3.11. *Under the conditions (S'1) and (S3), the solution $u(t, x)$ of SPDE (1.2) with the initial value $u_0(x)$ satisfies*

$$
\|u\|_{\mathcal{H}} \le \Big(1 + C_0 \sup_{x \in \mathbb{R}} \mathbb{E}\left[|u_0(x)|^2\right]\Big) e^{(C_1 + C_2 + C_3 + C_4)\sqrt{T - t_0}}, \tag{3.6}
$$

where C_i ($i = 0, 1, \ldots, 4$) are positive constants specified in the following proof. In particular, $u(t, x)$ is an element of the Banach space $\mathcal{H}$.

Proof. For any integer $n \ge 1$, we define the stopping time

$$
\tau_n = T \wedge \inf\Big\{t \in [t_0, T] : |u(t, x)| \ge n\Big\}.
$$

Clearly, $\tau_n \uparrow T$ as $n \to \infty$ a.s. Let $u_n(t, x) = u(t \wedge \tau_n, x)$ for $t \in [t_0, T]$. Then $u_n(t, x)$ satisfies the following equation

$$
\begin{aligned}
u_n(t, x) = & \int_{\mathbb{R}} g(t, x, z) u(t_0, z) dz + \int_{t_0}^t \int_{\mathbb{R}} g(t - s, x, z) b\Big(s, z, u_n(s, z)\Big) I_{[t_0, \tau_n]} dz ds \\
& + \int_{t_0}^t \int_{\mathbb{R}} g(t - s, x, z) \sigma\Big(s, z, u_n(s, z)\Big) I_{[t_0, \tau_n]} W(dz, ds) \\
& + \int_{t_0}^t \int_{\mathbb{R}} g(t - s, x, z) \sigma\Big(s, z, u_n(s, z)\Big) \psi(s, z) I_{[t_0, \tau_n]} dz ds \\
& + \int_{t_0}^{t^+} \int_{\mathbb{R}} \int_U g(t - s, x, z) \sigma\Big(s, z, u_n(s, z)\Big) h(s, z, y) I_{[t_0, \tau_n]} \dot{\tilde{N}}(dy, dz, ds).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\mathbb{E}\left[|u_n(t,x)|^2\right] \leq{}& 5\mathbb{E}\left[\left|\int_{\mathbb{R}} g(t,x,z)u(t_0,z)dz\right|^2\right] \\
&+ 5\mathbb{E}\left[\left|\int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)b\Big(s,z,u_n(s,z)\Big)dzds\right|^2\right] \\
&+ 5\mathbb{E}\left[\left|\int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)\sigma\Big(s,z,u_n(s,z)\Big)W(dz,ds)\right|^2\right] \\
&+ 5\mathbb{E}\left[\left|\int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)\sigma\Big(s,z,u_n(s,z)\Big)\psi(s,z)dzds\right|^2\right] \\
&+ 5\mathbb{E}\left[\left|\int_{t_0}^{t^+} \int_{\mathbb{R}} \int_U g(t-s,x,z)\sigma\Big(s,z,u_n(s,z)\Big)h(s,z,y)\dot{N}(dy,dz,ds)\right|^2\right] \\
={}& 5\Phi_0(t,x) + 5\Phi_1(t,x) + 5\Phi_2(t,x) + 5\Phi_3(t,x) + 5\Phi_4(t,x).
\end{aligned}
\tag{3.7}
$$

By the assumption (S4) and (2.4), we have

$$
\begin{aligned}
\Phi_0(t,x) &\leq C\int_{\mathbb{R}} g(t,x,z)\mathbb{E}\left[|u(t_0,z)|^2\right]dz \\
&\leq Ce^{r|x|}\mathbb{E}\left[\sup_{x\in\mathbb{R}} e^{-r|x|}|u(t_0,x)|^2\right] \\
&\leq \frac{1}{5}C_0 e^{r|x|}\sup_{x\in\mathbb{R}}\mathbb{E}\left[|u_0(x)|^2\right].
\end{aligned}
\tag{3.8}
$$

We note that (3.4) holds under the conditions (S'1) and (S3). From (2.4), (2.5) and Burkhölder's and Hölder's inequality, we deduce

$$
\begin{aligned}
\Phi_1(t,x) &\leq C\int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)dzds\,\mathbb{E}\left[\int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)b^2\Big(s,z,u_n(s,z)\Big)dzds\right] \\
&\leq C\int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)\Big(1+\mathbb{E}|u_n(s,z)|^2\Big)dzds \\
&\leq \frac{1}{10}C_1 e^{r|x|}\int_{t_0}^t \frac{1}{\sqrt{t-s}}(1+\|u_n\|_{\mathcal{H}})ds,
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
\Phi_2(t,x) &\leq C\mathbb{E}\left[\int_{t_0}^t \int_{\mathbb{R}} g^2(t-s,x,z)b^2\Big(s,z,u_n(s,z)\Big)dzds\right] \\
&\leq C\int_{t_0}^t \int_{\mathbb{R}} \frac{1}{\sqrt{t-s}}g(t-s,x,z)\Big(1+\mathbb{E}|u_n(s,z)|^2\Big)dzds \\
&\leq \frac{1}{10}C_2 e^{r|x|}\int_{t_0}^t \frac{1}{\sqrt{t-s}}(1+\|u_n\|_{\mathcal{H}})ds.
\end{aligned}
\tag{3.10}
$$

Under the condition (S2), utilizing (2.4), (2.5) and (3.4), we get

$$
\begin{aligned}
\Phi_3(t,x) &\leq C\mathbb{E}\left[\int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)b^2\Big(s,z,u_n(s,z)\Big)\psi^2(s,z)dzds\right] \\
&\leq C\sup_{t_0\leq t\leq T}\int_{\mathbb{R}} \psi^2(t,z)dz \cdot \int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)\Big(1+\mathbb{E}|u_n(s,z)|^2\Big)dzds \\
&\leq \frac{1}{10}C_3 e^{r|x|}\int_{t_0}^t \frac{1}{\sqrt{t-s}}(1+\|u_n\|_{\mathcal{H}})ds,
\end{aligned}
\tag{3.11}
$$

and

$$\Phi_4(t,x) \leq C\mathbb{E}\left[\int_{t_0}^t \int_{\mathbb{R}} \int_U g^2(t-s,x,z)\sigma^2\Big(s,z,u_n(s,z)\Big)h^2(s,z,y)\nu(dy)dzds\right]$$

$$\leq C \sup_{t_0 \leq t \leq T} \int_{\mathbb{R}} \int_U |h(t,z,y)|^2\nu(dy)dz \cdot \int_{t_0}^t \int_{\mathbb{R}} \frac{1}{\sqrt{t-s}}g(t-s,x,z)\Big(1+\mathbb{E}|u_n(s,z)|^2\Big)dzds$$

$$\leq \frac{1}{10}C_4 e^{r|x|} \int_{t_0}^t \frac{1}{\sqrt{t-s}}(1+\|u_n\|_{\mathcal{H}})ds.$$

$$(3.12)$$

From (3.7)–(3.12), it follows that

$$\mathbb{E}\left[|u_n(t,x)|^2\right] \leq C_0 e^{r|x|} \sup_{x \in \mathbb{R}} \mathbb{E}\left[|u_0(x)|^2\right] + \frac{1}{2}(C_1+C_2+C_3+C_4)e^{r|x|}\int_{t_0}^t \frac{1}{\sqrt{t-s}}(1+\|u_n\|_{\mathcal{H}})ds,$$

and hence that

$$\|u_n\|_{\mathcal{H}} \leq C_0 \sup_{x \in \mathbb{R}} \mathbb{E}\left[|u_0(x)|^2\right] + \frac{1}{2}(C_1+C_2+C_3+C_4)\int_{t_0}^t \frac{1}{\sqrt{t-s}}(1+\|u_n\|_{\mathcal{H}})ds.$$

Now applying the Gronwall's inequality yields that

$$1+\|u_n\|_{\mathcal{H}} \leq \left(1+C_0 \sup_{x \in \mathbb{R}} \mathbb{E}\left[|u_0(x)|^2\right]\right)e^{(C_1+C_2+C_3+C_4)\sqrt{t-t_0}}.$$

Consequently,

$$\|u_n\|_{\mathcal{H}} \leq \left(1+C_0 \sup_{x \in \mathbb{R}} \mathbb{E}\left[|u_0(x)|^2\right]\right)e^{(C_1+C_2+C_3+C_4)\sqrt{T-t_0}}.$$

By letting $n \to \infty$, we obtain the required inequality (3.6). □

Proof of Theorem 3.10. The proof will be divided into two steps.

Step 1. We firstly show the existence of mild solutions to SPDE (1.2) by the successive approximation scheme. Define $u_0(t,x) = \int_{\mathbb{R}} g(t,x,z)u(t_0,z)dz$, then for $n = 1,2,\ldots$, we set

$$u_n(t,x) = \int_{\mathbb{R}} g(t,x,z)u(t_0,z)dz + \int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)b\Big(s,z,u_{n-1}(s,z)\Big)dzds$$

$$+ \int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)\sigma\Big(s,z,u_{n-1}(s,z)\Big)W(dz,ds)$$

$$+ \int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)\sigma\Big(s,z,u_{n-1}(s,z)\Big)\psi(s,z)dzds$$

$$+ \int_{t_0}^{t^+} \int_{\mathbb{R}} \int_U g(t-s,x,z)\sigma\Big(s,z,u_{n-1}(s,z)\Big)h(s,z,y)\dot{\check{N}}(dy,dz,ds).$$

$$(3.13)$$

Here, we can show that $\{u_n(t,x)\}_{n=0}^\infty$ is a uniformly bounded sequence in $\mathcal{H}$ by induction. From (S4), it is easy to see $u(t_0,x) \in \mathcal{H}$. Assume that $u_{n-1}(t,x) \in \mathcal{H}$, we will prove that $u_n(t,x) \in \mathcal{H}$. Using the similar arguments as above, we have

$$\|u_n\|_{\mathcal{H}} \leq C_0 \sup_{x \in \mathbb{R}} \mathbb{E}\left[|u_0(x)|^2\right] + \frac{1}{2}(C_1+C_2+C_3+C_4)\int_{t_0}^t \frac{1}{\sqrt{t-s}}(1+\|u_{n-1}\|_{\mathcal{H}})ds.$$

By Lemma 3.11, we get

$$\|u_n\|_{\mathcal{H}} \le \left(1 + C_0 \sup_{x \in \mathbb{R}} \mathbb{E}\left[|u_0(x)|^2\right]\right) e^{C\sqrt{t-t_0}} \quad \text{for all } t \in [t_0, T],$$

where $C = (C_1 + C_2 + C_3 + C_4)\left[1 + (1 + C_0 \sup_{x \in \mathbb{R}} \mathbb{E}\left[|u_0(x)|^2\right]) e^{(C_1+C_2+C_3+C_4)\sqrt{T-t_0}}\right]$. Therefore, $\{u_n(t,x)\}_{n=0}^{\infty}$ is a uniformly bounded sequence in $\mathcal{H}$.

We shall prove that $\{u_n(t,x)\}_{n=0}^{\infty}$ is a Cauchy sequence of the Banach space $\mathcal{H}$. Suppose that m, n are any two integers, we have

$$\mathbb{E}\left[|u_n(t,x) - u_m(t,x)|^2\right]$$
$$\le 4\mathbb{E}\left[\left|\int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)\left(b\left(s,z,u_{n-1}(s,z)\right) - b\left(s,z,u_{m-1}(s,z)\right)\right) dzds\right|^2\right]$$
$$+ 4\mathbb{E}\left[\left|\int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)\left(\sigma\left(s,z,u_{n-1}(s,z)\right) - \sigma\left(s,z,u_{m-1}(s,z)\right)\right) W(dz,ds)\right|^2\right]$$
$$+ 4\mathbb{E}\left[\left|\int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)\left(\sigma\left(s,z,u_{n-1}(s,z)\right) - \sigma\left(s,z,u_{m-1}(s,z)\right)\right) \psi(s,z)dzds\right|^2\right]$$
$$+ 4\mathbb{E}\left[\left|\int_{t_0}^{t^+} \int_{\mathbb{R}} \int_{U} g(t-s,x,z)\left(\sigma\left(s,z,u_{n-1}(s,z)\right)\right.\right.$$
$$\left.\left. - \sigma\left(s,z,u_{m-1}(s,z)\right)\right) h(s,z,y)\tilde{\tilde{N}}(dy,dz,ds)\right|^2\right].$$

Using the similar arguments as above, we have

$$\mathbb{E}\left[|u_n(t,x) - u_m(t,x)|^2\right] \le C\int_{t_0}^t \int_{\mathbb{R}} \frac{\lambda(s)}{\sqrt{t-s}} g(t-s,x,z)\mathbb{E}\left[\phi\left(|u_{n-1}(s,z) - u_{m-1}(s,z)|^2\right)\right] dzds,$$

which implies that

$$\|u_n - u_m\|_{\mathcal{H}} \le C\int_{t_0}^t \frac{\lambda(s)}{\sqrt{t-s}}\phi(\|u_{n-1} - u_{m-1}\|_{\mathcal{H}})ds.$$

Since $\{u_n(t,x)\}_{n=0}^{\infty}$ is a uniformly bounded sequence in $\mathcal{H}$, we get

$$\sup_{m,n} \|u_n - u_m\|_{\mathcal{H}} < \infty.$$

By Fatou's lemma, for every $t \in [t_0, T]$ we have

$$\lim_{m,n\to\infty} \|u_n - u_m\|_{\mathcal{H}} \le C\int_{t_0}^t \frac{\lambda(s)}{\sqrt{t-s}}\phi\left(\lim_{m,n\to\infty} \|u_{n-1} - u_{m-1}\|_{\mathcal{H}}\right)ds.$$

Therefore, by Lemma 3.9, we deduce that

$$\lim_{m,n\to\infty} \|u_n - u_m\|_{\mathcal{H}} = 0,$$

which implies that $\{u_n(t,x)\}_{n=0}^{\infty}$ is a Cauchy sequence of the Banach space $\mathcal{H}$. Let $u(t,x)$ denote its limit. We pass to limits both sides of the equation (3.13) to prove that $u(t,x), t \in [t_0, T], x \in \mathbb{R}$ satisfies (3.1), $\mathbb{P}$ *a.s.*, which means that $u(t,x)$ is a solution of (1.2).

Step 2. In what follows, let us show the uniqueness of mild solutions to SPDE (1.2). We suppose that $u^{(1)}(t,x)$ and $u^{(2)}(t,x)$ are two solutions of equation (1.2). From Lemma 3.11, it follows that both of them belong to the Banach space $\mathcal{H}$. Moreover, we have

$$
\begin{aligned}
\mathbb{E}&\left[|u^{(1)}(t,x)-u^{(2)}(t,x)|^2\right]\\
&\leq 4\mathbb{E}\left[\left|\int_{t_0}^{t}\int_{\mathbb{R}}g(t-s,x,z)\Big(b\Big(s,z,u^{(1)}(s,z)\Big)-b\Big(s,z,u^{(2)}(s,z)\Big)\Big)dzds\right|^2\right]\\
&\quad+4\mathbb{E}\left[\left|\int_{t_0}^{t}\int_{\mathbb{R}}g(t-s,x,z)\Big(\sigma\Big(s,z,u^{(1)}(s,z)\Big)-\sigma\Big(s,z,u^{(2)}(s,z)\Big)\Big)W(dz,ds)\right|^2\right]\\
&\quad+4\mathbb{E}\left[\left|\int_{t_0}^{t}\int_{\mathbb{R}}g(t-s,x,z)\Big(\sigma\Big(s,z,u^{(1)}(s,z)\Big)-\sigma\Big(s,z,u^{(2)}(s,z)\Big)\Big)\psi(s,z)dzds\right|^2\right]\\
&\quad+4\mathbb{E}\left[\left|\int_{t_0}^{t^+}\int_{\mathbb{R}}\int_{U}g(t-s,x,z)\Big(\sigma\Big(s,z,u^{(1)}(s,z)\Big)\right.\right.\\
&\qquad\qquad\left.\left.-\sigma\Big(s,z,u^{(2)}(s,z)\Big)\Big)h(s,z,y)\dot{\tilde{N}}(dy,dz,ds)\right|^2\right].
\end{aligned}
$$

Using the similar arguments as above, we have

$$
\mathbb{E}\left[|u^{(1)}(t,x)-u^{(2)}(t,x)|^2\right]\leq C\int_{t_0}^{t}\int_{\mathbb{R}}\frac{\lambda(s)}{\sqrt{t-s}}g(t-s,x,z)\mathbb{E}\left[\phi\Big(|u^{(1)}(s,z)-u^{(2)}(s,z)|^2\Big)\right]dzds.
$$

Using Jensen's inequality and (2.4), we obtain

$$
\|u^{(1)}-u^{(2)}\|_{\mathcal{H}}\leq C\int_{t_0}^{t}\frac{\lambda(s)}{\sqrt{t-s}}\phi(\|u^{(1)}-u^{(2)}\|_{\mathcal{H}})ds.
$$

From Lemma 3.9, it follows that

$$
\|u^{(1)}-u^{(2)}\|_{\mathcal{H}}=0 \quad\text{for all } t\in[t_0,T]\text{ and } x\in\mathbb{R},
$$

which means $u^{(1)}(t,x)=u^{(2)}(t,x)$ for all $t\in[t_0,T], x\in\mathbb{R}$, $\mathbb{P}$ *a.s.* Thus the proof of Theorem 3.10 is completed. □

Remark 3.12. If $h_1(t,z,y)=h_2(t,z,y)=0$, then the Lévy space-time white noise will be reduced to Wiener space-time white noise, and equation (1.2) will be converted to the form of (1.1). Therefore, Theorem 2.1 in [27] can be regarded as a special case of Theorem 3.10.

Remark 3.13. Here, we utilize the successive approximation argument to prove Theorem 3.10. Of course, we can use the same method to deduce Theorem 3.1 of [21]. Under the conditions (S'1), (S2)–(S4), we can also employ the same approach to obtain some relevant results about (3.2), which will promote the work of [21]. Indeed, our results under the non-Lipschitz condition can be considered as a generalization of those of Theorem 3.1 of [21] and Theorem 2.2 of [1].

3.2 Stability of mild solutions to (1.2)

In this subsection, we mainly investigate the stability of mild solutions to (1.2). Now let us give the definition of $\mathcal{H}$-stability.

Definition 3.14. A solution $u(t,x)$ of SPDE (1.2) with initial value $u(t_0, x)$ is said to be $\mathcal{H}$-stable if for all $\epsilon > 0$ there exist $\delta > 0$ such that

$$\|u - \bar{u}\|_{\mathcal{H}} < \epsilon \quad \text{for all } t \in [t_0, T] \text{ and } x \in \mathbb{R}, \tag{3.14}$$

whenever $\sup_{x \in \mathbb{R}} \mathbb{E}\left[|u(t_0, x) - \bar{u}(t_0, x)|^2\right] < \delta$, where $\bar{u}(t, x)$ is another solution of SPDE (1.2) with initial value $\bar{u}(t_0, x)$.

Remark 3.15. The $\mathcal{H}$-stability is defined by utilizing the norm of space $\mathcal{H}$, which can be regarded as a promotion of ordinary stochastic stability, and can be applied to describe the dynamic behaviors of non-trivial solutions of SPDEs in the Banach space $\mathcal{H}$.

Theorem 3.16. *Under the conditions (S'1) and (S2)–(S4), there exists a unique $\mathcal{H}$-valued solution $u(t,x)$ to SPDE (1.2). Moreover, the mild solution is $\mathcal{H}$-stable.*

Proof. The existence has been shown in Theorem 3.10, we now only need to prove the $\mathcal{H}$-stability of the mild solution. Let $u(t,x)$ and $\bar{u}(t,x)$ be two solutions of equation (1.2) with initial value $u(t_0, x)$ and $\bar{u}(t_0, x)$, respectively. Then, we have

$$
\begin{aligned}
\mathbb{E}&\left[|u(t,x) - \bar{u}(t,x)|^2\right] \\
&\leq 5\mathbb{E}\left[\left|\int_{\mathbb{R}} g(t,x,z)\left(u(t_0,z) - \bar{u}(t_0,z)\right)dz\right|^2\right] \\
&\quad + 5\mathbb{E}\left[\left|\int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)\left(b\left(s,z,u(s,z)\right) - b\left(s,z,\bar{u}(s,z)\right)\right)dzds\right|^2\right] \\
&\quad + 5\mathbb{E}\left[\left|\int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)\left(\sigma\left(s,z,u(s,z)\right) - \sigma\left(s,z,\bar{u}(s,z)\right)\right)W(dz,ds)\right|^2\right] \\
&\quad + 5\mathbb{E}\left[\left|\int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)\left(\sigma\left(s,z,u(s,z)\right) - \sigma\left(s,z,\bar{u}(s,z)\right)\right)\psi(s,z)dzds\right|^2\right] \\
&\quad + 5\mathbb{E}\left[\left|\int_{t_0}^{t^+} \int_{\mathbb{R}} \int_U g(t-s,x,z)\left(\sigma\left(s,z,u(s,z)\right)\right.\right.\right. \\
&\qquad\qquad\left.\left.\left. - \sigma\left(s,z,\bar{u}(s,z)\right)\right)h(s,z,y)\dot{\tilde{N}}(dy,dz,ds)\right|^2\right].
\end{aligned}
$$

Using the similar arguments as above, it is not difficult to get

$$\mathbb{E}\left[|u(t,x) - \bar{u}(t,x)|^2\right] \leq Ce^{r|x|} \sup_{x \in \mathbb{R}} \mathbb{E}\left[|u(t_0,x) - \bar{u}(t_0,x)|^2\right] + Ce^{r|x|} \int_{t_0}^t \frac{\lambda(s)}{\sqrt{t-s}}\phi(\|u - \bar{u}\|_{\mathcal{H}})ds,$$

which implies

$$\|u - \bar{u}\|_{\mathcal{H}} \leq C \sup_{x \in \mathbb{R}} \mathbb{E}\left[|u(t_0,x) - \bar{u}(t_0,x)|^2\right] + \bar{C} \int_{t_0}^t \frac{\lambda(s)}{\sqrt{t-s}}\phi(\|u - \bar{u}\|_{\mathcal{H}})ds,$$

where C and $\bar{C}$ are two positive constants. We now set $\varphi(m) = \bar{C}\frac{\lambda(s)}{\sqrt{T-s}}\phi(m)$, $s \in [t_0, T]$. From assumption (S'1), it follows that $\varphi(m)$ is obviously a positive, continuous and nondecreasing concave function for any fixed s, which satisfies $\varphi(0) = 0$ and $\int_{0^+} \frac{1}{\varphi(m)}dm = \infty$. Therefore, for any $\epsilon > 0$, set $\epsilon_1 \triangleq \frac{\epsilon}{2}$, we have $\lim_{\gamma \to 0} \int_{\gamma}^{\epsilon_1} \frac{1}{\varphi(m)}dm = \infty$. Thus there exists a positive constant $\delta < \epsilon_1$ satisfying $\int_{\delta}^{\epsilon_1} \frac{1}{\varphi(m)}dm \geq T$.

Let $\alpha = C\sup_{x\in\mathbb{R}}\mathbb{E}\left[|u(t_0,x) - \overline{u}(t_0,x)|^2\right]$, $v(t) = 1$, and $\chi(t) = \|u - \overline{u}\|_{\mathcal{H}}$. When $\alpha \leq \delta \leq \epsilon_1$, we have

$$\int_\alpha^{\epsilon_1} \frac{1}{\varphi(m)}dm \geq \int_\delta^{\epsilon_1} \frac{1}{\varphi(m)}dm \geq T \geq \int_{t_0}^T v(t)dt = T - t_0.$$

Therefore, by utilizing Lemma 2.2 and Lemma 2.3, we can obtain for any $t \in [t_0, T]$, the estimate $\chi(t) = \|u - \overline{u}\|_{\mathcal{H}} < \epsilon$ holds, which implies the stability of the mild solution. Thus, we complete the proof of Theorem 3.16. $\qquad\square$

Remark 3.17. Certainly, under the conditions (P1) and (S2)–(S4), we can obtain that the mild solution of SPDE (1.2) is $\mathcal{H}$-stable as well.

4 Existence and stability of mild solutions to SPFDE (1.3)

In this section, we mainly study the well-posedness and stability of mild solutions to SPFDE (1.3). As before, we still work on the given complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P})$ with the filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ satisfying the usual conditions, and $\dot{L}(t,x)$ is Lévy space-time white noise.

Let $\tau > 0$, $\mathbb{C}([-\tau, 0] \times \mathbb{R}, \mathbb{R})$ be the space of all continuous functions $\varphi(\theta, x)$ defined on $[-\tau, 0] \times \mathbb{R}$ with the norm $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0, x\in\mathbb{R}} |\varphi(\theta, x)|$. We also assume that $u_t(x) \triangleq \{u(t+\theta, x), -\tau \leq \theta \leq 0, x \in \mathbb{R}\}$ is an $\mathcal{F}_t$-measurable $\mathbb{C}([-\tau, 0] \times \mathbb{R}; \mathbb{R})$-valued stochastic process. Let $u_{t_0}(x) = \{\xi(\theta, x) : -\tau \leq \theta \leq 0, x \in \mathbb{R}\}$ be an $\mathcal{F}_{t_0}$-measurable $\mathbb{C}([-\tau, 0] \times \mathbb{R}; \mathbb{R})$-valued stochastic variable satisfying $\mathbb{E}\left[\|\xi\|^2\right] < \infty$. Let $\mathcal{H}'$ be the family of all random fields $\{X(t,x), t \geq t_0 - \tau, x \in \mathbb{R}\}$ defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P})$ such that

$$\|X\|_{\mathcal{H}'} \triangleq \sup_{t\in[t_0-\tau, T], x\in\mathbb{R}} e^{-r|x|}\mathbb{E}\left[|X(t,x)|^2\right] < \infty.$$

where $r > 0$. Then the space $\mathcal{H}'$ with the norm $\|\cdot\|_{\mathcal{H}'}$ will be a Banach space. Therefore, for all $(t,x) \in [t_0, T] \times \mathbb{R}$, the mild solution of (1.3) can be written as follow

$$\begin{aligned}
u(t,x) = &\int_{\mathbb{R}} g(t,x,z)\xi(0,z)dz + \int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)b\Big(s,z,u_s(z)\Big)dzds \\
&+ \int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)\sigma\Big(s,z,u_s(z)\Big)W(dz,ds) \\
&+ \int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)\sigma\Big(s,z,u_s(z)\Big)\psi(s,z)dzds \\
&+ \int_{t_0}^{t^+} \int_{\mathbb{R}} \int_U g(t-s,x,z)\sigma\Big(s,z,u_s(z)\Big)h(s,z,y)\dot{\tilde{N}}(dy,dz,ds),
\end{aligned} \tag{4.1}$$

with the mappings defined by

$$\psi(s,z) = \int_{U/U_0} h_2(t,z,y)v(dy),$$

$$h(t,z,y) = h_1(t,z,y)I_{U_0}(y) + h_2(t,z,y)I_{U/U_0}(y),$$

where we assume that all integrals on the right-hand side of (4.1) exist, and I stands for the indicator function.

4.1 Well-posedness of mild solutions to (1.3)

In this subsection, our main task is to study the existence of mild solutions to (1.3) under the non-Lipschitz condition by the successive approximation method. Before stating our main results, we impose the following assumptions on the coefficients b and σ.

(N1) b, σ are local Lipschitz, i.e. there exists a positive constant K_n such that for all $(t, x) \in [t_0, T] \times \mathbb{R}$ and $u_1, u_2 \in \mathbb{C}([-\tau, 0] \times \mathbb{R}, \mathbb{R})$ with $\|u_1\| \vee \|u_2\| \leq n$,

$$|b(t, x, u_1) - b(t, x, u_2)|^2 + |\sigma(t, x, u_1) - \sigma(t, x, u_2)|^2 \leq K_n \|u_1 - u_2\|^2.$$

(N'1) If there exist a strictly positive, nondecreasing function $\lambda(t)$ defined on $[t_0, T]$ and a nondecreasing, continuous function $\phi(u)$ defined on $\mathbb{R}_+$ such that for all $(t, x) \in [t_0, T] \times \mathbb{R}$ and $u_1, u_2 \in \mathbb{C}([-\tau, 0] \times \mathbb{R}, \mathbb{R})$,

$$|b(t, x, u_1) - b(t, x, u_2)|^2 + |\sigma(t, x, u_1) - \sigma(t, x, u_2)|^2 \leq \lambda(t)\phi(\|u_1 - u_2\|^2),$$

where $\lambda(t)$ is a locally integrable function, $\phi(u)$ or $\phi^2(u)/u$ is a concave function with $\phi(0) = 0$ satisfying $\int_{0+} \frac{1}{\phi(u)} du = \infty$.

Remark 4.1. Here, we can obtain the similar conclusion as Lemma 3.6 by the same argument. Suppose that the conditions (N1) and (S3) or (N'1) and (S3) hold, then there exists a positive constant K such that for all $(t, x, u) \in [t_0, T] \times \mathbb{R} \times \mathbb{C}([-\tau, 0] \times \mathbb{R}, \mathbb{R})$,

$$|b(t, x, u)|^2 + |\sigma(t, x, u)|^2 \leq K(1 + \|u\|^2). \tag{4.2}$$

Now we provide a result on the existence and uniqueness of (1.3) under the non-Lipschitz condition (N'1).

Theorem 4.2. *Under the conditions (N'1), (S2) and (S3), there exists a unique $\mathcal{H}'$-valued solution to (1.3) with the initial value $\xi(\theta, x)$ satisfying $\mathbb{E}[\|\xi\|^2] < \infty$.*

For the reader's convenience, we give a lemma in order to prove this theorem.

Lemma 4.3. *Under the conditions (N'1) and (S3), the solution $u(t, x)$ of SPFDE (1.3) with the initial value $\xi(\theta, x)$ satisfies*

$$\|u\|_{\mathcal{H}'} \leq \left(1 + (1 + C_0')\mathbb{E}\left[\|\xi\|^2\right]\right)e^{C_1'\sqrt{T-t_0}}, \tag{4.3}$$

where the constants C_0' and C_1' will be specified later. In particular, $u(t, x)$ is an element of the Banach space $\mathcal{H}'$.

Proof. Here the proof is very similar to Lemma 3.11, thus we only sketch the proof. For any integer $n \geq 1$, we define the stopping time

$$\delta_n = T \wedge \inf \left\{t \in [t_0, T] : \|u_t\| \geq n \text{ for all } x \in \mathbb{R}\right\}.$$

Clearly, $\delta_n \uparrow T$ as $n \to \infty$ a.s. Let $u^n(t, x) = u(t \wedge \delta_n, x)$. By (4.1), then for $t \in [t_0, T]$, $u^n(t, x)$ satisfies the following equation

$$
\begin{aligned}
u^n(t, x) = {} & \int_{\mathbb{R}} g(t, x, z)\xi(0, z)dz + \int_{t_0}^t \int_{\mathbb{R}} g(t - s, x, z)b\left(s, z, u_s^n(z)\right)I_{[t_0, \delta_n]}dzds \\
& + \int_{t_0}^t \int_{\mathbb{R}} g(t - s, x, z)\sigma\left(s, z, u_s^n(z)\right)I_{[t_0, \delta_n]}W(dz, ds) \\
& + \int_{t_0}^t \int_{\mathbb{R}} g(t - s, x, z)\sigma\left(s, z, u_s^n(z)\right)\psi(s, z)I_{[t_0, \delta_n]}dzds \\
& + \int_{t_0}^{t^+} \int_{\mathbb{R}} \int_U g(t - s, x, z)\sigma\left(s, z, u_s^n(z)\right)h(s, z, y)I_{[t_0, \delta_n]}\dot{N}(dy, dz, ds).
\end{aligned}
\tag{4.4}
$$

Under the condition (S2), utilizing Höld's inequality and (4.2), we can derive that for $t \in [t_0, T]$,

$$\mathbb{E}\left[|u^n(t,x)|^2\right] \leq C_0' e^{r|x|} \mathbb{E}\left[\|\xi\|^2\right] + \frac{1}{2} C_1' e^{r|x|} \int_{t_0}^t \frac{1}{\sqrt{t-s}}\left(1 + e^{-r|x|}\mathbb{E}\left[\|u_s^n\|^2\right]\right)ds.$$

Noting that $\sup_{t \in [t_0-\tau, T], x \in \mathbb{R}} |u^n(t,x)|^2 \leq \|\xi\|^2 + \sup_{t \in [t_0,T], x \in \mathbb{R}} |u^n(t,x)|^2$, we obtain

$$1 + \|u^n\|_{\mathcal{H}'} \leq 1 + (1+C_0')\mathbb{E}\left[\|\xi\|^2\right] + \frac{1}{2}C_1'\int_{t_0}^t \frac{1}{\sqrt{t-s}}(1 + \|u^n\|_{\mathcal{H}'})ds.$$

Therefore the Gronwall inequality yields that

$$1 + \|u^n\|_{\mathcal{H}'} \leq \left(1 + (1+C_0')\mathbb{E}\left[\|\xi\|^2\right]\right)e^{C_1'\sqrt{T-t_0}}.$$

Consequently

$$\|u^n\|_{\mathcal{H}'} \leq \left(1 + (1+C_0')\mathbb{E}\left[\|\xi\|^2\right]\right)e^{C_1'\sqrt{T-t_0}}.$$

Finally, the required inequality follows by letting $n \to \infty$.   □

Proof of Theorem 4.2. The proof will be developed in two steps.

Step 1. We shall show the existence of mild solutions to (1.3) by using the successive approximation method in the Banach space $\mathcal{H}'$. Set $u^0(t,x) = \int_{\mathbb{R}} g(t,x,y)\xi(0,y)dy$, $u_{t_0}^0(t,x) = \int_{\mathbb{R}} g(t,x,y)\xi(\theta,y)dy$. For $n = 1, 2, \cdots$, let $u_{t_0}^n(t,x) = \int_{\mathbb{R}} g(t,x,y)\xi(\theta,y)dy$ and define

$$
\begin{aligned}
u^n(t,x) = {} & \int_{\mathbb{R}} g(t,x,z)\xi(0,z)dz + \int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)b\left(s,z,u_s^{n-1}(z)\right)dzds \\
& + \int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)\sigma\left(s,z,u_s^{n-1}(z)\right)W(dz,ds) \\
& + \int_{t_0}^t \int_{\mathbb{R}} g(t-s,x,z)\sigma\left(s,z,u_s^{n-1}(z)\right)\psi(s,z)dzds \\
& + \int_{t_0}^{t^+} \int_{\mathbb{R}} \int_U g(t-s,x,z)\sigma\left(s,z,u_s^{n-1}(z)\right)h(s,z,y)\dot{N}(dy,dz,ds).
\end{aligned}
$$
(4.5)

Similarly to the proof of Theorem 3.10, we can show that $\{u^n(t,x)\}_{n=0}^\infty$ is a uniformly bounded sequence of the Banach space $\mathcal{H}'$ by induction. Meanwhile, for any two positive integer m and n, we can obtain

$$
\begin{aligned}
& \mathbb{E}\left[\sup_{t \in [t_0,T], x \in \mathbb{R}} |u^n(t,x) - u^m(t,x)|^2\right] \\
& \leq C \int_{t_0}^t \int_{\mathbb{R}} \frac{\lambda(s)}{\sqrt{t-s}} g(t-s,x,z)\phi\left(\mathbb{E}\left[\sup_{l \in [t_0,s], x \in \mathbb{R}} |u^{n-1}(l,z) - u^{m-1}(l,z)|^2\right]\right)dzds,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& \lim_{m,n\to\infty} \mathbb{E}\left[\sup_{t \in [t_0,T], x \in \mathbb{R}} |u^n(t,x) - u^m(t,x)|^2\right] \\
& \leq C \int_{t_0}^t \int_{\mathbb{R}} \frac{\lambda(s)}{\sqrt{t-s}} g(t-s,x,z)\phi\left(\lim_{m,n\to\infty}\mathbb{E}\left[\sup_{l \in [t_0,s], x \in \mathbb{R}} |u^{n-1}(l,z) - u^{m-1}(l,z)|^2\right]\right)dzds.
\end{aligned}
$$

Utilizing Fatou's lemma and Lemma 3.9, we derive that

$$\lim_{m,n\to\infty} \mathbb{E}\left[\sup_{t\in[t_0,T],x\in\mathbb{R}} |u^n(t,x)-u^m(t,x)|^2\right] = 0,$$

which implies that $\{u^n(t,x)\}_{n=0}^{\infty}$ is a Cauchy sequence of the Banach space $\mathcal{H}'$. We assume that $u(t,x) \in \mathcal{H}'$ is its limit. It is easy to see that $u(t,x), t \in [t_0,T], x \in \mathbb{R}$ satisfies equation (4.1) by taking limits both sides of equation (4.4), which means that $u(t,x)$ is the solution of system (1.3).

Step 2. In what follows, we continue to show the uniqueness of mild solutions to system (1.3). Let $u^{(1)}(t,x)$ and $u^{(2)}(t,x)$ be its two solutions. By Lemma 4.3, both of them belong to Banach space $\mathcal{H}'$. then we get

$$
\begin{aligned}
&|u^{(1)}(t,x)-u^{(2)}(t,x)|^2 \\
&\leq 4\left|\int_{t_0}^t\int_{\mathbb{R}} g(t-s,x,z)\left(b\left(s,z,u_s^{(1)}(z)\right)-b\left(s,z,u_s^{(2)}(z)\right)\right)dzds\right|^2 \\
&\quad +4\left|\int_{t_0}^t\int_{\mathbb{R}} g(t-s,x,z)\left(\sigma\left(s,z,u_s^{(1)}(z)\right)-\sigma\left(s,z,u_s^{(2)}(z)\right)\right)W(dz,ds)\right|^2 \\
&\quad +4\left|\int_{t_0}^t\int_{\mathbb{R}} g(t-s,x,z)\left(\sigma\left(s,z,u_s^{(1)}(z)\right)-\sigma\left(s,z,u_s^{(2)}(z)\right)\right)\psi(s,z)dzds\right|^2 \\
&\quad +4\left|\int_{t_0}^{t^+}\int_{\mathbb{R}}\int_U g(t-s,x,z)\left(\sigma\left(s,z,u_s^{(1)}(z)\right)-\sigma\left(s,z,u_s^{(2)}(z)\right)\right)h(s,z,y)\dot{\tilde{N}}(dy,dz,ds)\right|^2.
\end{aligned}
$$

By similar arguments as above, we have

$$\mathbb{E}\left[\sup_{t\in[t_0,T],x\in\mathbb{R}}|u^{(1)}(t,x)-u^{(2)}(t,x)|^2\right] \leq C\int_{t_0}^t\int_{\mathbb{R}}\frac{\lambda(s)}{\sqrt{t-s}}g(t-s,x,z)\mathbb{E}\left[\phi(\|u_s^{(1)}-u_s^{(2)}\|^2)\right]dzds.$$

Using Jensen's inequality and (2.4), we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[t_0,T],x\in\mathbb{R}}|u^{(1)}(t,x)-u^{(2)}(t,x)|^2\right] \\
&\quad \leq C\int_{t_0}^t\frac{\lambda(s)}{\sqrt{t-s}}\phi\left(\mathbb{E}\left[\sup_{l\in[t_0,s],x\in\mathbb{R}}|u^{(1)}(l,z)-u^{(2)}(l,z)|^2\right]\right)ds.
\end{aligned}
$$

From Lemma 2.2, it follows that

$$\mathbb{E}\left[\sup_{t\in[t_0,T],x\in\mathbb{R}}|u^{(1)}(t,x)-u^{(2)}(t,x)|^2\right] = 0, \quad \text{for all } t \in [t_0,T] \text{ and } x \in \mathbb{R},$$

which means $u^{(1)}(t,x) = u^{(2)}(t,x)$ for all $t \in [t_0,T], x \in \mathbb{R}, \mathbb{P}$ *a.s.* The proof is complete. $\quad\square$

Remark 4.4. It is worth pointing out that when $h_1(t,z,y) = h_2(t,z,y) = 0$, the Lévy space-time noise will be reduced to Wiener space-time white noise, and equation (1.3) will be converted to the form of equation (1.3) in [17]. Thus, Theorem 4.2 can be seen as a generalization of Theorem 3.4 of [17].

Remark 4.5. The condition (N'1) is so-called non-Lipschitz condition. In particular, when $\lambda(t) = K$ is a positive constant and $\phi(u) = u$, then the condition (N'1) can be reduced to the uniform Lipschitz condition (N''1), i.e.,

(N"1) b, σ are uniform Lipschitz, i.e. there exists a positive constant K such that for all $(t, x) \in [t_0, T] \times \mathbb{R}$ and $u_1, u_2 \in \mathbb{C}([-\tau, 0] \times \mathbb{R}, \mathbb{R})$,

$$|b(t, x, u_1) - b(t, x, u_2)|^2 + |\sigma(t, x, u_1) - \sigma(t, x, u_2)|^2 \le K\|u_1 - u_2\|^2.$$

Therefore, by Theorem 4.2 we have the following corollary.

Corollary 4.6. *Under the conditions (N"1), (S2) and (S3), there exists a unique $\mathcal{H}'$-valued solution to* (1.3) *with the initial value $\xi(\theta, x)$ satisfying $\mathbb{E}\left[\|\xi\|^2\right] < \infty$.*

Meanwhile, under the local Lipschitz condition (N1) we can also obtain the following result, which is similar to Theorem 3.8 by utilizing the Corollary 4.6.

Theorem 4.7. *Under the conditions (N1), (S2) and (S3), there exists a unique $\mathcal{H}'$-valued solution to* (1.3) *with the initial value $\xi(\theta, x)$ satisfying $\mathbb{E}\left[\|\xi\|^2\right] < \infty$.*

Remark 4.8. We note that Theorem 4.7 can be proved by the truncation procedure as outlined in the proof of Theorem 3.8, and hence we omit the details.

4.2 Stability of mild solutions to (1.3)

In what follows, we shall discuss the stability of mild solutions to SPFDE (1.3). We firstly give the definition of $\mathcal{H}'$-stability of mild solutions to (1.3).

Definition 4.9. A solution $u(t, x)$ of SPFDE (1.3) with initial value $\xi(\theta, x)$ is said to be $\mathcal{H}'$-stable if for all $\epsilon > 0$ there exist $\delta > 0$ such that

$$\|u - \overline{u}\|_{\mathcal{H}'} < \epsilon \quad \text{for all } t \in [t_0, T] \text{ and } x \in \mathbb{R}, \tag{4.6}$$

whenever $\mathbb{E}\left[\|\xi - \overline{\xi}\|^2\right] < \delta$, where $\overline{u}(t, x)$ is another solution of equation (1.3) with initial value $\overline{\xi}(\theta, x)$.

Now, we give some sufficient conditions on the $\mathcal{H}'$-stability of mild solutions to SPFDE (1.3).

Theorem 4.10. *Under the conditions (N'1), (S2) and (S3), there exists a unique $\mathcal{H}'$-valued solution $u(t, x)$ to SPFDE* (1.3), *which is $\mathcal{H}'$-stable.*

Proof. Note that the existence of mild solutions to (1.3) has been shown in Theorem 4.2, then we only need to show the $\mathcal{H}'$-stability. Let $u(t, x)$ and $\overline{u}(t, x)$ denote two solutions of system (1.3) with initial value $\xi(\theta, x)$ and $\overline{\xi}(\theta, x)$, respectively. Then, we have

$$\mathbb{E}\left[|u(t, x) - \overline{u}(t, x)|^2\right]$$

$$\le 5\mathbb{E}\left[\left|\int_{\mathbb{R}} g(t, x, z)\Big(\xi(\theta, x) - \overline{\xi}(\theta, x)\Big)dz\right|^2\right]$$

$$+ 5\mathbb{E}\left[\left|\int_{t_0}^{t}\int_{\mathbb{R}} g(t - s, x, z)\Big(b\Big(s, z, u_s(z)\Big) - b\Big(s, z, \overline{u}_s(z)\Big)\Big)dzds\right|^2\right]$$

$$+ 5\mathbb{E}\left[\left|\int_{t_0}^{t}\int_{\mathbb{R}} g(t - s, x, z)\Big(\sigma\Big(s, z, u_s(z)\Big) - \sigma\Big(s, z, \overline{u}_s(z)\Big)\Big)W(dz, ds)\right|^2\right]$$

$$+ 5\mathbb{E}\left[\left|\int_{t_0}^{t}\int_{\mathbb{R}} g(t - s, x, z)\Big(\sigma\Big(s, z, u_s(z)\Big) - \sigma\Big(s, z, \overline{u}_s(z)\Big)\Big)\psi(s, z)dzds\right|^2\right]$$

$$+ 5\mathbb{E}\left[\left|\int_{t_0}^{t^+}\int_{\mathbb{R}}\int_{U} g(t - s, x, z)\Big(\sigma\Big(s, z, u_s\Big) - \sigma\Big(s, z, \overline{u}_s\Big)\Big)h(s, z, y)\dot{N}(dy, dz, ds)\right|^2\right].$$

By the similar methods as the proof of Theorem 3.16, we have

$$\mathbb{E}\left[|u(t,x) - \bar{u}(t,x)|^2\right] \leq Ce^{r|x|}\mathbb{E}\left[\|\xi - \bar{\xi}\|^2\right] + C\int_{t_0}^t \frac{\lambda(s)}{\sqrt{t-s}} g(t-s,x,z)\phi(\|u_s - \bar{u}_s\|^2)ds.$$

Consequently,

$$\|u - \bar{u}\|_{\mathcal{H}'} \leq (1+C)\mathbb{E}\left[\|\xi - \bar{\xi}\|^2\right] + \bar{C}\int_{t_0}^t \frac{\lambda(s)}{\sqrt{t-s}}\phi(\|u - \bar{u}\|_{\mathcal{H}'})ds,$$

where C and $\bar{C}$ are two positive constants. Set $\varphi(m) = \bar{C}\frac{\lambda(s)}{\sqrt{T-s}}\phi(m)$, $s \in [t_0, T]$. For every fixed s, $\varphi(m)$ is obviously a positive, continuous and nondecreasing concave function, which also satisfies $\varphi(0) = 0$ and $\int_{0^+}\frac{1}{\varphi(m)}dm = \infty$. Let $\epsilon_1 \triangleq \frac{\epsilon}{2}$ for any $\epsilon > 0$. Then we have $\lim_{\gamma \to 0}\int_\gamma^{\epsilon_1}\frac{1}{\varphi(m)}dm = \infty$. Thus, there exists a positive constant $\delta < \epsilon_1$ such that $\int_\delta^{\epsilon_1}\frac{1}{\varphi(m)}dm \geq T$.

Let $\alpha = (1+C)\mathbb{E}\left[\|\xi - \bar{\xi}\|^2\right]$, $v(t) = 1$, and $\chi(t) = \|u - \bar{u}\|_{\mathcal{H}'}$. If $\alpha \leq \delta \leq \epsilon_1$, then

$$\int_\alpha^{\epsilon_1}\frac{1}{\varphi(m)}dm \geq \int_\delta^{\epsilon_1}\frac{1}{\varphi(m)}dm \geq T \geq \int_{t_0}^T v(t)dt = T - t_0.$$

By Lemmas 2.2 and 2.3, we have the estimate $\chi(t) = \|u - \bar{u}\|_{\mathcal{H}'} < \epsilon$ for all $t \in [t_0, T]$, which implies that the mild solution $u(t,x)$ is $\mathcal{H}'$-stable. Therefore, the proof of Theorem 4.10 is completed. $\square$

Remark 4.11. Using the similar arguments, we see that the mild solution of (1.3) is $\mathcal{H}'$-stable under the conditions (N''1), (S2) and (S3).

5 Examples

In this section, we shall give two examples to illustrate our main results.

Example 5.1. We consider the following parabolic SPDE driven by Lévy space-time noise:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u(t,x)}{\partial x^2} + \wp\Big(u(t,x)\Big) + \wp\Big(u(t,x)\Big)\dot{W}(t,x) \\ \qquad + \displaystyle\int_{U_0}\dfrac{y\wp\Big(u(t,x)\Big)}{\sqrt{1+x^2}}\dot{\tilde{N}}(dy,x,t) + \int_{U/U_0}\dfrac{\wp\Big(u(t,x)\Big)}{\sqrt{1+x^2}}\dot{N}(dy,x,t), \\ u(t_0,x) = u_0(x), \end{cases} \quad (5.1)$$

where $u_0(x)$ satisfies $\sup_{x\in\mathbb{R}}\mathbb{E}\left[u_0^2(x)\right] < \infty$, and

$$\wp(u) = \begin{cases} 0, & u = 0, \\ u\sqrt{\ln(u^{-1})}, & 0 < u < \kappa, \\ \kappa\sqrt{\ln(\kappa^{-1})} + \wp'_-(\kappa)(u - \kappa), & u \geq \kappa, \end{cases}$$

$\kappa \in (0,1)$ is sufficiently small, and $\wp'_-(\kappa)$ denotes the left derivative of $\wp(u)$ at the point κ.

Firstly, it is easy to see that (S4) holds. Since $\lim_{u\to 0^+}\frac{\wp(u)}{u} = \infty$, the function $\wp(u)$ is non-Lipschitz continuous. Meanwhile, we easily obtain $\wp(u)$ is a nondecreasing, positive concave

function on $[0, \infty)$ satisfying $\varpi(0) = 0$ and $\int_{0+} \frac{1}{\varpi(u)} du = \infty$. Furthermore, using the concavity of $\varpi(u)$, it is easy to show that for all u and $u' \geq 0$,

$$|\varpi(u) - \varpi(u')| \leq \varpi(|u - u'|).$$

Here, let $b(t, x, u) = \sigma(t, x, u) = \varpi(u)$, $h_1(t, x, y) = \frac{y}{\sqrt{1+x^2}}$ and $h_2(t, x, y) = \frac{1}{\sqrt{1+x^2}}$, $\lambda(t) = 1$ and $\phi(u) = \sqrt{\varpi(u^2)}$. Then $b(t, x, u)$ and $\sigma(t, x, u)$ satisfy the non-Lipschitz condition, $\phi(u)$ is a nondecreasing, positive concave function on $[0, \infty)$ satisfying $\phi(0) = 0$ and $\int_{0+} \frac{1}{\phi(u)} du = \infty$. This implies that the assumption (S'1) holds. Since $\psi(t, x)$ and $h(t, x, y)$ satisfy

$$\int_{\mathbb{R}} \psi^2(t, x) dx = \int_{\mathbb{R}} \left(\int_{U/U_0} h_2(t, x, y) v(dy) \right)^2 dx$$
$$\leq C\pi v(U/U_0)$$
$$< +\infty,$$

and

$$\int_{\mathbb{R}} \int_U |h(t, x, y)|^2 v(dy) dx = \int_{\mathbb{R}} \int_U \left(h_1(t, x, y) I_{U_0}(y) + h_2(t, x, y) I_{U/U_0}(y) \right)^2 v(dy) dx$$
$$\leq C \int_{\mathbb{R}} \left(\int_{U_0} h_1^2(t, x, y) v(dy) + \int_{U/U_0} h_2^2(t, x, y) v(dy) \right) dx$$
$$\leq C\pi \left(v(U/U_0) + \int_{U_0} x^2 v(dy) \right)$$
$$< +\infty,$$

then the assumption (S2) holds as well. In addition, it follows from $b(t, x, 0) = \sigma(t, x, 0) = 0$ that (S3) is satisfied. Therefore, Theorem 3.10 yields that SPDE (5.1) with initial value $u_0(x)$ has a unique mild solution $u(t, x)$. Furthermore, the mild solution $u(t, x)$ is $\mathcal{H}$-stable by Theorem 3.16.

Example 5.2. Consider the following parabolic delayed SPDE driven by Lévy space-time noise:

$$\begin{cases} \dfrac{\partial u(t, x)}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 u(t, x)}{\partial x^2} + \tilde{\omega}\left(u(t - \tau, x) \right) + \tilde{\omega}\left(u(t - \tau, x) \right) \dot{W}(t, x) \\[2mm] \qquad + \displaystyle\int_{U_0} \dfrac{\tilde{\omega}\left(u(t - \tau, x) \right)}{\sqrt{1 + x^2}} e^{-\frac{y^2}{2}} \check{N}(dy, x, t) \\[4mm] \qquad + \displaystyle\int_{U/U_0} \dfrac{\tilde{\omega}\left(u(t - \tau, x) \right)}{\sqrt{1 + x^2}} \dot{N}(dy, x, t), \\[4mm] u_{t_0}(x) = \xi(\theta, x), \end{cases} \tag{5.2}$$

where $\tau > 0$ is a time delay, the initial value $\xi(\theta, x)$ satisfies $\mathbb{E}\left[\|\xi\|^2 \right] < \infty$, and $\tilde{\omega}(u(t - \tau, x))$ is defined as

$$\tilde{\omega}\left(u(t - \tau, x) \right) = \begin{cases} 0, & u = 0, \\[2mm] |u(t - \tau, x)| \sqrt{\ln\left(1 + |u(t - \tau, x)|^{-1} \right)}, & 0 < |u(t - \tau, x)| < 1, \\[2mm] \sqrt{\ln\left(2|u(t - \tau, x)| \right)}, & |u(t - \tau, x)| \geq 1, \end{cases}$$

Here, let $b(t, x, u_t(x)) = \sigma(t, x, u_t(x)) = \tilde{\omega}(u(t - \tau, x))$, $h_1(t, x, y) = \frac{1}{\sqrt{1+x^2}} e^{-\frac{y^2}{2}}$ and $h_2(t, x, y) = \frac{1}{\sqrt{1+x^2}}$, $\lambda(t) = 1$ and $\phi(u) = \sqrt{\tilde{\omega}(u^2)}$. Obviously, $b(t, x, u_t(x))$ and $\sigma(t, x, u_t(x))$ satisfy the non-Lipschitz condition, $\phi(u)$ is a nondecreasing, positive concave function on $[0, \infty)$ with $\phi(0) = 0$ satisfying $\int_{0^+} \frac{1}{\phi(u)} du = \infty$; $\int_{\mathbb{R}} \psi^2(t, x) dx < +\infty$ and $\int_{\mathbb{R}} \int_U |h(t, x, y)|^2 \nu(dy) dx < +\infty$; $b(t, x, 0) = \sigma(t, x, 0) = 0$. Therefore, the assumptions (N'1), (S2), and (S3) hold. From Theorem 4.2, it follows that there exists a unique mild solution $u(t, x)$ to SPDE (5.2) with initial value $\xi(\theta, x)$ satisfying $\mathbb{E}\left[\|\xi\|^2\right] < \infty$. Meanwhile, it follows from Theorem 4.10 that the mild solution $u(t, x)$ is $\mathcal{H}'$-stable.

Acknowledgements

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