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Iterative technique for a third-order differential equation with three-point nonlinear boundary value conditions

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Abstract. In this paper, we study the existence of extremal solutions for a nonlinear third-order differential equation with three-point nonlinear boundary value conditions. By means of the method of upper and lower solutions and different monotone iterative techniques, the sufficient conditions which guarantee the existence of extremal solutions are given. An example illustrates the main results.

Keywords: third-order differential equation, nonlinear boundary value conditions, the method of upper and lower solutions, iterative technique.

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1 Introduction

Nonlinear boundary value conditions in differential equations can describe many phenomena in applied mathematics, engineering, physical or biological processes. In this paper, we consider the following third-order differential equation with three-point nonlinear boundary value conditions

$$\begin{cases} -u'''(t) = f(t, u(t)), \quad t \in [0,1], \\ u(0) = u''(0) = 0, \\ p\left(u(1), u(\xi)\right) = 0, \end{cases} \tag{1.1}$$

where $\xi \in (0,1)$, $f : [0,1] \times R \to R$ and $p : R \times R \to R$ are continuous.

In recent years, third-order differential equations with nonlinear boundary value conditions have been discussed in many papers (see [1–15] and the references therein). For example, [4] considered a class of two-point nonlinear boundary value conditions by using a priori estimate, Nagumo condition, upper and lower solutions and Leray–Schauder degree. Papers [7–9, 13] considered some nonlinear nonlocal boundary conditions. However, according to our knowledge, for third-order differential equation, the three-point nonlinear boundary

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value conditions in problem (1.1) are never discussed in literature. Hence the aim of this paper is to discuss this issue.

The main contributions are as follows: (a) we present problems with linear boundary value conditions, and on this basis we obtain the existence of the extremal solutions for problem (1.1) by applying the method of upper and lower solutions and monotone iterative technique; (b) the iterative technique is not unique and an example illustrates the result.

2 Notations and preliminaries

In this section, we present some definitions and lemmas that will be used throughout the paper.

Definition 2.1. Assume $\xi \in (0,1)$, $f : [0,1] \times R \to R$ is continuous. A function $u(t)$ is called an upper solution for problem (1.1) if it satisfies

$$\begin{cases} u'''(t) + f(t,u(t)) \le 0, \quad t \in [0,1], \\ u(0) = u''(0) = 0, \\ p\left(u(1), u(\xi)\right) \ge 0. \end{cases}$$

Similarly, a function $u(t)$ is called a lower solution for problem (1.1) if it satisfies

$$\begin{cases} u'''(t) + f(t,u(t)) \ge 0, \quad t \in [0,1], \\ u(0) = u''(0) = 0, \\ p\left(u(1), u(\xi)\right) \le 0. \end{cases}$$

Lemma 2.2. *Assume that $\alpha\xi \neq 1$, $b \in R$ and $h : [0,1] \to R$ is continuous. Then the boundary value problem*

$$\begin{cases} -u'''(t) = h(t), \quad t \in [0,1], \\ u(0) = u''(0) = 0, \\ u(1) = \alpha u(\xi) + b, \end{cases}$$

has a unique solution

$$u(t) = \frac{b}{1-\alpha\xi} t + \frac{\alpha t}{1-\alpha\xi} \int_0^1 G(\xi,s)h(s)ds + \int_0^1 G(t,s)h(s)ds,$$

where

$$G(t,s) = \frac{1}{2} \begin{cases} (1-t)(t-s^2), & 0 \le s \le t \le 1, \\ t(1-s)^2, & 0 \le t \le s \le 1. \end{cases} \tag{2.1}$$

Proof. Integrating the equation

$$-u'''(t) = h(t), \qquad t \in [0,1],$$

over $[0,t]$ for three times, we have

$$u(t) = -\frac{1}{2}\int_0^t (t-s)^2 h(s)ds + \frac{1}{2}u''(0)t^2 + u'(0)t + u(0).$$

Due to the boundary conditions $u(0) = u''(0) = 0$, $u(1) = \alpha u(\xi) + b$, it follows that

$$-\frac{1}{2}\int_0^1 (1-s)^2 h(s)ds + u'(0) = \alpha u(\xi) + b.$$

And then

$$\begin{aligned} u(t) &= t(\alpha u(\xi)+b) + \frac{1}{2}t\int_0^1 (1-s)^2 h(s)ds - \frac{1}{2}\int_0^t (t-s)^2 h(s)ds \\ &= t(\alpha u(\xi)+b) + \frac{1}{2}\int_0^t t(1-s)^2 h(s)ds - \frac{1}{2}\int_0^t (t-s)^2 h(s)ds + \frac{1}{2}\int_t^1 t(1-s)^2 h(s)ds \\ &= t(\alpha u(\xi)+b) + \frac{1}{2}\int_0^t (1-t)(t-s^2)h(s)ds + \frac{1}{2}\int_t^1 t(1-s)^2 h(s)ds \\ &= t(\alpha u(\xi)+b) + \int_0^1 G(t,s)h(s)ds. \end{aligned} \tag{2.2}$$

Putting $t = \xi$, we have

$$u(\xi) = \frac{1}{1-\alpha\xi}\left(\xi b + \int_0^1 G(\xi,s)h(s)ds\right).$$

Substituting it into (2.2), we get

$$u(t) = \frac{b}{1-\alpha\xi}t + \frac{\alpha t}{1-\alpha\xi}\int_0^1 G(\xi,s)h(s)ds + \int_0^1 G(t,s)h(s)ds.$$

□

Remark 2.3. It is easy to see that $G(t,s) > 0$ for all $(t,s) \in (0,1)\times(0,1)$ and

$$u'(t) = \frac{b}{1-\alpha\xi} + \frac{\alpha}{1-\alpha\xi}\int_0^1 G(\xi,s)h(s)ds + \int_0^1 G'(t,s)h(s)ds, \tag{2.3}$$

where

$$G'(t,s) = \frac{1}{2}\begin{cases} 1-2t+s^2, & 0 \le s \le t \le 1, \\ (1-s)^2, & 0 \le t \le s \le 1. \end{cases} \tag{2.4}$$

Remark 2.4. For $(t,s) \in [0,1]\times[0,1]$,

$$0 \le t(1-t)\varphi(s) \le G(t,s) \le \varphi(s), \tag{2.5}$$

where $\varphi(s) = \frac{1}{8}(1+s)^2(1-s)^2$.

In fact, for $t \in [s,1]$, $G(\cdot,s) = \frac{1}{2}(1-t)(t-s^2)$ attains its maximum at $t = \frac{1}{2}(1+s^2) \in [s,1]$, so that

$$G(\cdot,s) \le \varphi(s) = \frac{1}{8}(1+s)^2(1-s)^2.$$

For $t \in [0,s]$, clearly $G(\cdot,s) \le \frac{1}{2}s(1-s)^2 \le \varphi(s) = \frac{1}{8}(1+s)^2(1-s)^2$. That is, $G(t,s) \le \varphi(s)$ for $(t,s) \in [0,1]\times[0,1]$.

Also, for $s \in [0,t]$,

$$\frac{G(t,s)}{\varphi(s)} = \frac{\frac{1}{2}(1-t)(t-s^2)}{\frac{1}{8}(1+s)^2(1-s)^2} \ge \frac{4(1-t)(t-st)}{(1+s)^2(1-s)^2} = \frac{4t(1-t)}{(1+s)^2(1-s)} \ge t(1-t).$$

If for $s \in [t,1]$,

$$\frac{G(t,s)}{\varphi(s)} = \frac{\frac{1}{2}t(1-s)^2}{\frac{1}{8}(1+s)^2(1-s)^2} = \frac{4t}{(1+s)^2} \geq t.$$

Since $\min\{t, t(1-t)\} = t(1-t)$, we have

$$0 \leq t(1-t)\varphi(s) \leq G(t,s) \leq \varphi(s).$$

Lemma 2.5. *Assume that* $0 < \alpha\xi < 1$. *If* $u(t) \in C[0,1]$, *satisfying* $u'''(t) \in C[0,1]$ *and*

$$\begin{cases} u'''(t) \leq 0, \quad t \in [0,1], \\ u(0) = u''(0) = 0, \\ u(1) \geq \alpha u(\xi), \end{cases}$$

then $u(t) \geq 0, \ \ t \in [0,1]$.

Proof. Let $-u'''(t) = h(t) \geq 0, \ t \in [0,1]$ and $u(1) = \alpha u(\xi) + b, \ b \geq 0$. Then by Lemma 2.2, we get

$$\begin{aligned} u(t) &= \frac{b}{1-\alpha\xi}t + \frac{\alpha t}{1-\alpha\xi}\int_0^1 G(\xi,s)h(s)ds + \int_0^1 G(t,s)h(s)ds \\ &\geq 0. \end{aligned}$$

□

Lemma 2.6. *Assume that* $\kappa(t), \mu(t) \in C[0,1]$ *and*

$$\int_0^1 (1+s)^2(1-s)^2|\kappa(s)|ds < \frac{8(1-\alpha\xi)}{\alpha+1-\alpha\xi}. \tag{2.6}$$

Then the following linear boundary value problem

$$\begin{cases} -u'''(t) = \kappa(t)u(t) + \mu(t), \quad t \in [0,1], \\ u(0) = u''(0) = 0, \\ u(1) = \alpha u(\xi) + b, \end{cases} \tag{2.7}$$

has a unique solution $u(t) \in C[0,1]$, *where* $0 < \alpha\xi < 1, b \geq 0$.

Proof. By Lemma 2.2, problem (2.7) is equivalent to the following integral equation

$$\begin{aligned} u(t) = {} & \frac{b}{1-\alpha\xi}t + \frac{\alpha t}{1-\alpha\xi}\int_0^1 G(\xi,s)\left(\kappa(s)u(s) + \mu(s)\right)ds \\ & + \int_0^1 G(t,s)\left(\kappa(s)u(s) + \mu(s)\right)ds =: Tu(t). \end{aligned}$$

Obviously, $T : C[0,1] \longrightarrow C[0,1]$. Note that by (2.5), we have for any $u, v \in C[0,1]$,

$$\begin{aligned} |Tu(t) - Tv(t)| &\leq \frac{\alpha t}{1-\alpha\xi}\int_0^1 G(\xi,s)|\kappa(s)||u(s)-v(s)|ds + \int_0^1 G(t,s)|\kappa(s)||u(s)-v(s)|ds \\ &\leq \frac{1}{8}\left(1 + \frac{\alpha t}{1-\alpha\xi}\right)\|u-v\|\int_0^1 (1+s)^2(1-s)^2|\kappa(s)|ds \\ &\leq \frac{1}{8}\left(\frac{\alpha+1-\alpha\xi}{1-\alpha\xi}\right)\|u-v\|\int_0^1 (1+s)^2(1-s)^2|\kappa(s)|ds = L\|u-v\|, \end{aligned}$$

where $L := \frac{1}{8}\left(\frac{\alpha+1-\alpha\xi}{1-\alpha\xi}\right)\int_0^1(1+s)^2(1-s)^2|\kappa(s)|ds < 1$, which is easy to see from (2.6). Therefore, the operator T is a contraction map in the space $C[0,1]$ and T has a unique fixed point in $C[0,1]$. □

Lemma 2.7. *Assume that* $0 < \alpha\xi < 1$, $\kappa(t)(\in C[0,1]) > 0$ *and satisfies* (2.6). *If* $u(t) \in C[0,1]$, *satisfying* $u'''(t) \in C[0,1]$ *and*

$$\begin{cases} u'''(t) + \kappa(t)u(t) \leq 0, & t \in [0,1], \\ u(0) = u''(0) = 0, \\ u(1) \geq \alpha u(\xi) \end{cases}$$

then $u(t) \geq 0,\ t \in [0,1]$.

Proof. Let $-u'''(t) = \kappa(t)u(t) + \mu(t)$, $t \in [0,1]$, $\kappa(t) > 0$, $\mu(t) \geq 0$, $u(1) = \alpha u(\xi) + b$, $b \geq 0$.

Suppose that the inequality $u(t) \geq 0,\ t \in [0,1]$ is not true. It means that there exists at least a $t^* \in [0,1]$ such that $u(t^*) < 0$. Without loss of generality, we assume $u(t^*) = \min\{u(t) : t \in [0,1]\} = \rho, \rho < 0$. Then by Lemma 2.2 and (2.5), we have

$$\begin{aligned} u(t) &= \frac{b}{1-\alpha\xi}t + \frac{\alpha t}{1-\alpha\xi}\int_0^1 G(\xi,s)(\kappa(s)u(s)+\mu(s))ds \\ &\quad + \int_0^1 G(t,s)(\kappa(s)u(s)+\mu(s))ds \\ &\geq \frac{\alpha t}{1-\alpha\xi}\int_0^1 G(\xi,s)\kappa(s)u(s)ds + \int_0^1 G(t,s)\kappa(s)u(s)ds \\ &\geq u(t^*)\left(\frac{\alpha t}{1-\alpha\xi}\int_0^1 \varphi(s)\kappa(s)ds + \int_0^1 \varphi(s)\kappa(s)ds\right). \end{aligned}$$

Let $t = t^*$, and note that $\rho < 0$, $0 < \alpha\xi < 1$, $0 < \xi < 1$, it follows that

$$\begin{aligned} \rho &\geq \rho\left(\frac{\alpha t^*}{1-\alpha\xi}\int_0^1 \varphi(s)\kappa(s)ds + \int_0^1 \varphi(s)\kappa(s)ds\right) \\ &\geq \rho\left(\frac{\alpha}{1-\alpha\xi}\int_0^1 \varphi(s)\kappa(s)ds + \int_0^1 \varphi(s)\kappa(s)ds\right). \end{aligned}$$

And then

$$1 \leq \frac{\alpha+1-\alpha\xi}{8(1-\alpha\xi)}\int_0^1 (1+s)^2(1-s)^2\kappa(s)ds.$$

That is

$$\int_0^1 (1+s)^2(1-s)^2\kappa(s)ds \geq \frac{8(1-\alpha\xi)}{\alpha+1-\alpha\xi},$$

which is in contradiction to (2.6). Hence $u(t) \geq 0$ for all $t \in [0,1]$. □

3 Main result

In this section, we shall apply the method of upper and lower solutions and monotone iterative technique to consider the existence of extremal solutions for problem (1.1).

Theorem 3.1. *Assume that the following conditions hold.*

(A_1) $f(t,u)$ *is increasing with respect to* u.

(A_2) $v_0(t), w_0(t) \in C[0,1]$ *are lower and upper solutions of problem* (1.1)*, respectively, and* $v_0(t) \leq w_0(t),\ t \in [0,1]$.

(A_3) *There exist constants* ς, τ *such that* $0 < \tau\xi < \varsigma$ *and for* $v_0(1) \leq x \leq y \leq w_0(1)$, $v_0(\xi) \leq \bar{x} \leq \bar{y} \leq w_0(\xi)$,

$$p(y,\bar{y}) + \tau(\bar{y} - \bar{x}) \leq p(x,\bar{x}) + \varsigma(y - x).$$

Then problem (1.1) *has extremal solutions in the sector* $[v_0, w_0]$*, where*

$$[v_0, w_0] = \{u \in C[0,1] : v_0(t) \leq u(t) \leq w_0(t),\ t \in [0,1]\}.$$

Proof. For $n = 0, 1, \ldots,$ define

$$v_{n+1}(t) = \frac{b_n}{1-\alpha\xi}t + \frac{\alpha t}{1-\alpha\xi}\int_0^1 G(\xi,s)f(s,v_n(s))ds + \int_0^1 G(t,s)f(s,v_n(s))ds,$$
$$w_{n+1}(t) = \frac{c_n}{1-\alpha\xi}t + \frac{\alpha t}{1-\alpha\xi}\int_0^1 G(\xi,s)f(s,w_n(s))ds + \int_0^1 G(t,s)f(s,w_n(s))ds,$$

where

$$\alpha = \frac{\tau}{\varsigma}, \quad b_n = v_n(1) - \alpha v_n(\xi) - \frac{1}{\varsigma}g(v_n(1), v_n(\xi)), \quad c_n = w_n(1) - \alpha w_n(\xi) - \frac{1}{\varsigma}g(w_n(1), w_n(\xi)).$$

Then due to Lemma 2.2, it is easy to show that $v_{n+1}(t)$, $w_{n+1}(t)$ are solutions of the following boundary value problems, respectively:

$$\begin{cases} -v'''_{n+1}(t) = f(t, v_n(t)), \quad t \in [0,1], \\ v_{n+1}(0) = v''_{n+1}(0) = 0, \\ 0 = p\left(v_n(1), v_n(\xi)\right) + \varsigma(v_{n+1}(1) - v_n(1)) - \tau(v_{n+1}(\xi) - v_n(\xi)), \end{cases} \tag{3.1}$$

and

$$\begin{cases} -w'''_{n+1}(t) = f(t, w_n(t)), \quad t \in [0,1], \\ w_{n+1}(0) = w''_{n+1}(0) = 0, \\ 0 = p\left(w_n(1), w_n(\xi)\right) + \varsigma(w_{n+1}(1) - w_n(1)) - \tau(w_{n+1}(\xi) - w_n(\xi)). \end{cases} \tag{3.2}$$

Moreover, from (2.3) we have

$$v'_{n+1}(t) = \frac{b_n}{1-\alpha\xi} + \frac{\alpha}{1-\alpha\xi}\int_0^1 G(\xi,s)f(s,v_n(s))ds + \int_0^1 G'(t,s)f(s,v_n(s))ds, \tag{3.3}$$
$$w'_{n+1}(t) = \frac{c_n}{1-\alpha\xi} + \frac{\alpha}{1-\alpha\xi}\int_0^1 G(\xi,s)f(s,w_n(s))ds + \int_0^1 G'(t,s)f(s,w_n(s))ds, \tag{3.4}$$

where $G'(t,s)$ is given as in (2.4).

Claim 1. The sequences $v_n(t)$, $w_n(t)(n \geq 1)$ are lower and upper solutions of problem (1.1), respectively and the following relation holds

$$v_0(t) \leq v_1(t) \leq \cdots \leq v_n(t) \leq \cdots \leq w_n(t) \leq \cdots \leq w_1(t) \leq w_0(t), \qquad t \in [0,1]. \tag{3.5}$$

First, we prove that

$$v_0(t) \le v_1(t) \le w_1(t) \le w_0(t), \qquad t \in [0,1].$$

Let $x(t) = w_0(t) - w_1(t)$. From (3.2) and (A_1), we have

$$\begin{cases} x'''(t) \le 0, & t \in [0,1], \\ x(0) = x''(0) = 0, & \\ x(1) \ge \frac{\tau}{\varsigma} x(\xi), & 0 < \tau\xi < \varsigma. \end{cases}$$

In view of Lemma 2.5, we have $x(t) \ge 0, t \in [0,1]$, that is $w_0(t) \ge w_1(t)$. Similarly, it can be obtained that $v_0(t) \le v_1(t)$, $t \in [0,1]$.

Now, let $x(t) = w_1(t) - v_1(t)$. From (A_1) and (A_2), it follows that

$$x'''(t) = f(t, v_0(t)) - f(t, w_0(t)) \le 0.$$

Also, $x(0) = x''(0) = 0$ and

$$\begin{aligned} 0 &= p\left(w_0(1), w_0(\xi)\right) - p\left(v_0(1), v_0(\xi)\right) + \varsigma(w_1(1) - w_0(1) - v_1(1) + v_0(1)) \\ &\quad - \tau(w_1(\xi) - w_0(\xi) - v_1(\xi) + v_0(\xi)) \\ &\le \varsigma(w_0(1) - v_0(1)) - \tau(w_0(\xi) - v_0(\xi)) + \varsigma(w_1(1) - w_0(1) - v_1(1) + v_0(1)) \\ &\quad - \tau(w_1(\xi) - w_0(\xi) - v_1(\xi) + v_0(\xi)) \\ &= \varsigma x(1) - \tau x(\xi). \end{aligned}$$

That is,

$$x(0) = x''(0) = 0, \qquad x(1) \ge \frac{\tau}{\varsigma} x(\xi).$$

By Lemma 2.5, we have $w_1(t) \ge v_1(t), t \in [0,1]$. And then, by induction, (3.5) holds.

In what follows, we show that $v_1(t), w_1(t)$ are lower and upper solutions of problem (1.1), respectively. From (3.1), (3.2) and $(A_1), (A_2)$, it follows that

$$-v_1'''(t) = f(t, v_0(t)) \le f(t, v_1(t)).$$

Also, $v_1(0) = v_1''(0) = 0$ and

$$\begin{aligned} 0 &= -p\left(v_0(1), v_0(\xi)\right) + p\left(v_1(1), v_1(\xi)\right) - p\left(v_1(1), v_1(\xi)\right) - \varsigma[v_1(1) - v_0(1)] + \tau[v_1(\eta) - v_0(\xi)] \\ &\le \varsigma[v_1(1) - v_0(1)] - \tau[v_1(\xi) - v_0(\xi)] - p\left(v_1(1), v_1(\xi)\right) - \varsigma[v_1(1) - v_0(1)] + \tau[v_1(\xi) - v_0(\xi)] \\ &= -p\left(v_1(1), v_1(\xi)\right), \end{aligned}$$

which prove that $v_1(t)$ is a lower solution of problem (1.1). Similarly, it can be obtained that $w_1(t)$ is an upper solution of problem (1.1).

Analogously to the above arguments, using the induction method, we can show that the sequences $v_n(t)$, $w_n(t)$ $(n \ge 1)$ are lower and upper solutions of problem (1.1), respectively and the following relation holds

$$v_0(t) \le v_1(t) \le \cdots \le v_n(t) \le \cdots \le w_n(t) \le \cdots \le w_1(t) \le w_0(t), \qquad t \in [0,1].$$

Claim 2. The sequences $\{v_n(t)\}$, $\{w_n(t)\}$ uniformly converge to their limit functions $v(t), w(t)$, respectively.

We need to show that the sequences are bounded and equicontinuous on $[0,1]$. Indeed,

$$C_1 \leq v_0(t) \leq \cdots \leq v_n(t) \leq \cdots \leq \cdots \leq w_n(t) \cdots \leq w_0(t) \leq C_2 \tag{3.6}$$

for $t \in [0,1]$ and $n = 1, 2, \ldots$. That is to say that the sequences $\{v_n(t)\}$, $\{w_n(t)\}$ are uniformly bounded with respect to t. Note that $\{v_n'(t)\}$, $\{w_n'(t)\}$ are bounded on $[0,1]$ by $C_3 > 0$ because (3.3), (3.4), (3.6) and $|f(t, v_n)|$, $|p(v_m, v_n)|$ is bounded. Hence $\{v_n(t)\}$, $\{w_n(t)\}$ are equicontinuous because for $\forall \varepsilon > 0$, $t_1, t_2 \in [0,1]$ such that $|t_1 - t_2| < \frac{\varepsilon}{M_3}$, we have

$$|v_n(t_1) - v_n(t_2)| = |v_n'(\gamma)||t_1 - t_2| < \varepsilon, \qquad |w_n(t_1) - w_n(t_2)| < \varepsilon, \qquad \gamma \in [0,1].$$

Therefore, by the Arzelà–Ascoli theorem, the sequences $\{v_n(t)\}$, $\{w_n(t)\}$ have subsequences $\{v_{n_k}(t)\}$, $\{w_{n_k}(t)\}$ which uniformly converge to their continuous limit functions $v(t), w(t)$, respectively.

Claim 3. The limit functions $v(t), w(t)$ are the minimal solution and maximal solution of problem (1.1), respectively.

Let $u(t) \in [v_0(t), w_0(t)]$ be any solution of problem (1.1). We assume that the following relation holds for some $k \in N$:

$$v_k(t) \leq u(t) \leq w_k(t), \qquad t \in [0,1].$$

Let $y(t) = u(t) - v_{k+1}(t), z(t) = w_{k+1}(t) - u(t)$. Then

$$\begin{cases} y'''(t) \leq 0, & t \in [0,1], \\ y(0) = y''(0) = 0, \\ y(1) \geq \frac{\tau}{\varsigma} y(\xi), & 0 < \tau\xi < \varsigma, \end{cases}$$

and

$$\begin{cases} z'''(t) \leq 0, & t \in [0,1], \\ z(0) = z''(0) = 0, \\ z(1) \geq \frac{\tau}{\varsigma} z(\xi), & 0 < \tau\xi < \varsigma. \end{cases}$$

This and Lemma 2.5 show $v_{k+1}(t) \leq u(t) \leq w_{k+1}(t)$. By induction, $v_n(t) \leq u(t) \leq w_n(t)$, $t \in [0,1]$ for all $n \in N$. Taking the limit as $n \to \infty$, we get $v(t) \leq u(t) \leq w(t)$, $t \in [0,1]$. □

Theorem 3.2. *Assume that all assumptions of Theorem 3.1 hold. In addition, we assume that there exists $q(t) > 0\ (\in C[0,1])$ such that*

$$f(t,x) - f(t,y) \leq q(t)(x - y), \qquad v_0(t) \leq x(t) \leq y(t) \leq w_0(t), \qquad t \in [0,1]$$

and

$$\int_0^1 (1+s)^2 (1-s)^2 q(s) ds < \frac{8(\varsigma - \tau\xi)}{\varsigma + \tau - \tau\xi}.$$

Then the sequences $\{v_n(t)\}$, $\{w_n(t)\}$ $(n \geq 1)$ satisfying

$$\begin{cases} -v_{n+1}'''(t) = f(t, v_n(t)) + q(t)[v_{n+1}(t) - v_n(t),], & t \in (0,1), \\ v_{n+1}(0) = v_{n+1}''(0) = 0, \\ 0 = p\left(v_n(1), v_n(\xi)\right) + \varsigma(v_{n+1}(1) - v_n(1)) - \tau(v_{n+1}(\xi) - v_n(\xi)), \end{cases}$$

and

$$\begin{cases} -w'''_{n+1}(t) = f(t, w_n(t)) + q(t)[w_{n+1}(t) - w_n(t),], & t \in (0,1), \\ w_{n+1}(0) = w''_{n+1}(0) = 0, \\ 0 = p\,(w_n(1), w_n(\xi)) + \varsigma(w_{n+1}(1) - w_n(1)) - \tau(w_{n+1}(\xi) - w_n(\xi)), \end{cases}$$

also uniformly converge to their continuous limit functions $v(t), w(t)$*, respectively. That is,* $v(t), w(t)$ *are also extremal solutions for problem* (1.1)*.*

Proof. Using Lemmas 2.6 and 2.7, we can complete the proof by the same way as in Theorem 3.1. □

Example 3.3. Consider the following third-order boundary value problem

$$\begin{cases} -u'''(t) = u(t)\sin t - \frac{113}{8}t^{\frac{1}{2}}, & t \in [0,1], \\ u(0) = u''(0) = 0, \\ u(1) - u(\frac{1}{2}) - \frac{1}{8}u(1)u(\frac{1}{2}) = 0, \end{cases} \tag{3.7}$$

where

$$f(x,y) = y\sin x - \frac{113}{8}x^{\frac{1}{2}}, \qquad p(x,y) = x - y - \frac{1}{8}xy, \qquad \xi = \frac{1}{2}.$$

It is not difficult to show that $v_0 = 0$, $w_0(t) = t^{\frac{7}{2}}$ are lower and upper solutions of problem (3.7), respectively. Moreover, for $t \in [0,1]$, $f(t,u)$ is increasing with respect to u, and for $v_0(1) \le x \le y \le w_0(1)$, $v_0(\eta) \le \bar{x} \le \bar{y} \le w_0(\eta)$,

$$p(y,\bar{y}) - p(x,\bar{x}) = (y-x) - (\bar{y}-\bar{x}) - \frac{1}{8}(y\bar{y} - x\bar{x}) \le (y-x) - (\bar{y}-\bar{x}).$$

Choose $\varsigma = \tau = 1$ in Theorem 3.1 or $\varsigma = \tau = 1$, $q(t) = \sin t$ in Theorem 3.2, problem (3.7) has extremal solutions in $[v_0(t), w_0(t)]$.

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