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Nonexistence results for some nonlinear inequalities with functional parameters

Evgeny Galakhov[✉1], **Olga Salieva**[2] and **Liudmila Uvarova**[2]

[1]Peoples' Friendship University of Russia, ul. Miklukho-Maklaya 6, Moscow, 117198, Russia
[2]Moscow State Technological University "Stankin", Vadkovsky lane 3a, Moscow, 127994, Russia

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Abstract. We obtain results on nonexistence of nontrivial nonnegative solutions for some elliptic and parabolic inequalities with functional parameters involving the $p(x)$-Laplacian operator. The proof is based on the test function method.

Keywords: nonexistence, functional parameters, $p(x)$-Laplacian, test function.

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1 Introduction

In the recent decades a rich literature has appeared concerning nonlinear elliptic problems with operators such as $\Delta_{p(x)}u \equiv \operatorname{div}(|Du|^{p(x)-2}Du)$. One should note many papers in this field containing extensive results on existence, uniqueness, regularity, and symmetry (see, in particular, [1–3, 8, 11] and references therein). However, sufficient conditions for nonexistence of solutions to such problems are much less studied. Up to our knowledge, they were obtained only for some particular cases of such operators in [10] and [9]. The purpose of the present paper is to fill this gap at least partially.

To obtain our nonexistence results, we use the test function method (also known as the nonlinear capacity one) suggested in [7] and developed more recently in [4–6]. Namely, assuming for contradiction that a solution exists, we multiply both sides of the inequality in question by specially chosen parameter dependent test functions and after partial integration and some algebraic transformations, such as application of the Young inequality, obtain an a priori estimate for a positive functional of the solution. Taking the diameter of the support of the test function or of its derivatives to infinity or to zero, depending on the nature of the problem, we establish the asymptotical behaviour of this estimate, which implies the desired contradiction for a certain range of parameters. This general scheme of the method requires certain changes when applied to problems with operators such as $\Delta_{p(x)}$. In particular, the parameters of the Young inequality in this special case must depend upon the point of the domain where this inequality is applied. Modifying the scheme in this way, we arrive at

✉Corresponding author. Email: galakhov@rambler.ru

nonexistence results in terms of asymptotic behaviour of certain integrals containing $p(x)$ and other functional parameters of the considered problem.

Up to our knowledge, existence theorems for problems with a power-like nonlinearity we study here are not known. However, for constant p, our results coincide with those from [7] that are shown there to be optimal (i.e., both necessary and sufficient) in the scale of parameters under consideration. For example, our Theorem 2.1 implies that the problem

$$-\Delta_p u(x) \geq u^q \quad (x \in \mathbb{R}^n) \tag{1.1}$$

with $1 < p < n$ has no positive weak solutions for $p-1 < q \leq q_{\text{cr}} = \frac{n(p-1)}{n-p}$, and in [7], explicit examples of solutions to (1.1) with $q > q_{\text{cr}}$ are given. The optimality of our results in a more general case should be the subject of future investigation.

The rest of the paper consists of four sections. In Section 2, we formulate and prove nonexistence results for elliptic problems in the whole space. In Section 3, parabolic problems in $\mathbb{R}^n \times \mathbb{R}_+$ are considered. In Sections 4 and 5, respectively, we treat elliptic problems in bounded domains with a singularity near the boundary and similar parabolic problems.

2 Elliptic inequalities in $\mathbb{R}^n$

Let $p(x) \in C^\infty(\mathbb{R}^n)$, $q(x) \in C^\infty(\mathbb{R}^n)$, $g(x) \in C^\infty(\mathbb{R}^n)$ be bounded functions such that

$$\min_{x\in\mathbb{R}^n} p(x) > 1, \quad \min_{x\in\mathbb{R}^n} (p(x) - q(x)) > -1, \quad g(x) > 0.$$

Consider nonlinear elliptic inequalities of the form

$$-\Delta_{p(x)} u \geq u^{q(x)} g(x) \quad (x \in \mathbb{R}^n) \tag{2.1}$$

and

$$-\Delta_{p(x)} u \geq |Du|^{q(x)} g(x) \quad (x \in \mathbb{R}^n). \tag{2.2}$$

Here we use the notation $\Delta_{p(x)} u(x) = \operatorname{div}(|Du(x)|^{p(x)-2} Du(x))$. Denote $\alpha(x) = \log_{1+|x|} g(x)$ and hence $g(x) = (1+|x|)^{\alpha(x)}$.

Theorem 2.1. *If there exists $\lambda \in (1 - \min_{x\in\mathbb{R}^n} p(x), 0)$ such that*

$$\int\limits_{B_{2R}(0)\setminus B_R(0)} R^{-\frac{p(x)(q(x)+\lambda)+\alpha(x)(\lambda+p(x)-1)}{q(x)-p(x)+1}}\,dx \to 0 \quad \text{as } R \to \infty, \tag{2.3}$$

then inequality (2.1) has no nontrivial nonnegative solutions $u \in L^{q(x)}_{\text{loc}}(\mathbb{R}^n)$ in the distributional sense.

Proof. Choose a family of nonnegative test functions $\varphi = \varphi_R \in C^1(\mathbb{R}^N; [0,1])$ such that $\varphi_R(x) = \varphi_1\left(\frac{x}{R}\right)$, where

$$\varphi_1(x) = \begin{cases} 1 & (|x| \leq 1), \\ 0 & (|x| \geq 2), \end{cases} \tag{2.4}$$

$$|D\varphi_R(x)| \leq cR^{-1} \quad (x \in \mathbb{R}^N). \tag{2.5}$$

Multiply both parts of (2.1) by $u^\lambda \varphi_R$, where $1 - \min_{x\in\mathbb{R}^n} p(x) < \lambda < 0$. Integrating by parts, we get

$$\begin{aligned}\int_{\mathbb{R}^n} u^{q(x)+\lambda}(1+|x|)^{\alpha(x)}\varphi_R\,dx \int_{\mathbb{R}^n} (|Du|^{p(x)-2}Du, D(u^\lambda\varphi_R))\,dx \\ = \lambda \int_{\mathbb{R}^n} u^{\lambda-1}|Du|^{p(x)}\varphi_R\,dx + \int_{\mathbb{R}^n} u^\lambda |Du|^{p(x)-2}(Du, D\varphi_R)\,dx \\ \leq \lambda \int_{\mathbb{R}^n} u^{\lambda-1}|Du|^{p(x)}\varphi_R\,dx + \int_{\mathbb{R}^n} u^\lambda |Du|^{p(x)-1}|D\varphi_R|\,dx\end{aligned}$$

and by the Young inequality

$$\begin{aligned}\int_{\mathbb{R}^n} u^{q(x)+\lambda}(1+|x|)^{\alpha(x)}\varphi_R\,dx + |\lambda| \int_{\mathbb{R}^n} u^{\lambda-1}|Du|^{p(x)}\varphi_R\,dx \\ \leq |\lambda| \int_{\mathbb{R}^n} u^{\lambda-1}|Du|^{p(x)}\varphi_R\,dx + c(\lambda)\int_{\mathbb{R}^n} u^{\lambda+p(x)-1}|D\varphi_R|^{p(x)}\varphi_R^{1-p(x)}dx,\end{aligned}$$

i.e.,

$$\int_{\mathbb{R}^n} u^{q(x)+\lambda}(1+|x|)^{\alpha(x)}\varphi_R\,dx \leq c(\lambda)\int_{\mathbb{R}^n} u^{\lambda+p(x)-1}|D\varphi_R|^{p(x)}\varphi_R^{1-p(x)}dx.$$

Using the Young inequality again, we arrive at

$$\frac{1}{2}\int_{\mathbb{R}^n} u^{q(x)+\lambda}(1+|x|)^{\alpha(x)}\varphi_R\,dx \leq c(\lambda)\int_{\mathbb{R}^n} |D\varphi_R|^{\frac{p(x)(q(x)+\lambda)}{q(x)-p(x)+1}}(1+|x|)^{-\frac{\alpha(x)\cdot(\lambda+p(x)-1)}{q(x)-p(x)+1}}\varphi_R^{1-\frac{p(x)(q(x)+\lambda)}{q(x)-p(x)+1}}dx.$$

Restricting the domain of integration and making use of (2.4) and (2.5), we obtain

$$\int_{B_R(0)} u^{q(x)+\lambda}(1+|x|)^{\alpha(x)}\,dx \leq c \int_{B_{2R}(0)\setminus B_R(0)} R^{-\frac{p(x)(q(x)+\lambda)+\alpha(x)\cdot(\lambda+p(x)-1)}{q(x)-p(x)+1}}dx,$$

which leads to a contradiction as $R\to\infty$ under our assumptions. □

Remark 2.2. This result can be extended to a wider class of quasilinear problems, including systems of the form

$$\begin{cases} -\Delta_{p(x)}u \geq v^{s(x)}f(x) & (x\in\mathbb{R}^n), \\ -\Delta_{q(x)}v \geq u^{z(x)}g(x) & (x\in\mathbb{R}^n) \end{cases}$$

with appropriate functional parameters.

Theorem 2.3. *If*

$$\int_{B_{2R}(0)\setminus B_R(0)} R^{-\frac{\alpha(x)\cdot(p(x)-1)+q(x)}{q(x)-p(x)+1}}\,dx \to 0 \quad \textit{as } R\to\infty, \tag{2.6}$$

then inequality (2.2) has no solutions $u \in W^{1,q(x)}_{\mathrm{loc}}(\mathbb{R}^n)$ in the distributional sense that are distinct from a constant a.e.

Proof. Multiplying both parts (2.2) by φ_R and integrating by parts, we obtain

$$\int\limits_{B_{2R}(0)} |Du|^{q(x)}(1+|x|)^{\alpha(x)}\varphi_R\,dx \leq \int\limits_{B_{2R}(0)\setminus B_R(0)} (|Du|^{p(x)-2}Du, D\varphi_R)\,dx$$

and by the Young inequality

$$\begin{aligned}
&\int\limits_{B_{2R}(0)\setminus B_R(0)} (|Du|^{p(x)-2}Du, D\varphi_R)\,dx \\
&\leq \int\limits_{B_{2R}(0)\setminus B_R(0)} |Du|^{p(x)-1}\cdot|D\varphi_R|\,dx \leq \varepsilon \int\limits_{B_{2R}(0)\setminus B_R(0)} |Du|^{q(x)}(1+|x|)^{\alpha(x)}\varphi_R\,dx \\
&\quad + c(\varepsilon)\int\limits_{B_{2R}(0)\setminus B_R(0)} (1+|x|)^{-\frac{\alpha(x)\cdot(p(x)-1)}{q(x)-p(x)+1}}|D\varphi_R|^{\frac{q(x)}{q(x)-p(x)+1}}\varphi_R^{-\frac{p(x)-1}{q(x)-p(x)+1}}\,dx,
\end{aligned}$$

whence for $\varepsilon < 1/2$ one has

$$\int\limits_{B_R(0)} |Du|^{q(x)}(1+|x|)^{\alpha(x)}\,dx \leq 2c(\varepsilon)\int\limits_{B_{2R}(0)\setminus B_R(0)} (1+|x|)^{-\frac{\alpha(x)\cdot(p(x)-1)}{q(x)-p(x)+1}}|D\varphi_R|^{\frac{q(x)}{q(x)-p(x)+1}}\varphi_R^{-\frac{p(x)-1}{q(x)-p(x)+1}}\,dx$$

and due to assumptions (2.4)–(2.5)

$$\int\limits_{B_R(0)} |Du|^{q(x)}(1+|x|)^{\alpha(x)}\,dx \leq c\int\limits_{B_{2R}(0)\setminus B_R(0)} R^{-\frac{\alpha(x)\cdot(p(x)-1)+q(x)}{q(x)-p(x)+1}}\,dx,$$

which implies the claim of the theorem as $R\to\infty$. □

Remark 2.4. This result can be also extended to a wider class of quasilinear problems, including systems of the form

$$\begin{cases} -\Delta_{p(x)}u \geq |Dv|^{s(x)}f(x) & (x\in\mathbb{R}^n),\\ -\Delta_{q(x)}v \geq |Du|^{z(x)}g(x) & (x\in\mathbb{R}^n) \end{cases}$$

with appropriate functional parameters.

3 Parabolic inequalities in $\mathbb{R}^n\times\mathbb{R}_+$

A nonexistence result also takes place for a parabolic inequality

$$u_t - \Delta_{p(x)}u \geq u^{q(x)}f(x) \quad (x\in\mathbb{R}^n; t\in\mathbb{R}_+) \tag{3.1}$$

with initial condition

$$u(x,0) = u_0(x) \geq 0 \quad (x\in\mathbb{R}^n). \tag{3.2}$$

We assume that $u_0\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Here we introduce for the Cauchy problem (3.1)–(3.2) two families of test functions, namely $\varphi_R(x)$ with respect to spatial variables and $T_\tau(t)$ w.r.t. time. Here $\varphi_R(x)$ is defined as in previous sections, and $T_\tau\in C^1(\mathbb{R}_+;[0,1])$ with $\tau>0$ is such that

$$T_\tau(t) = \begin{cases} 1 & (0\leq t\leq\tau),\\ 0 & (t\geq 2\tau) \end{cases}$$

and there exists a $\tau_0 > 0$ such that for any $\tau > \tau_0$ and $x \in \mathbb{R}^n$

$$\int_{2\tau}^{\tau} \frac{|T'_\tau|^{q'(x)}}{|T_\tau|^{q'(x)-1}}\, dt \le c_0 \tau^{1-q'(x)} \tag{3.3}$$

holds with some constant $c_0 > 0$ independent of x, where $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$. A typical result for problem (3.1)–(3.2) can be formulated as follows.

Theorem 3.1. *Let there exist constants $\tau_0 > 0$ and $\lambda \in (1 - \min_{x\in\mathbb{R}^n} p(x), 0)$ such that for any $\tau > \tau_0$ and $x \in \mathbb{R}^n$ (3.3) holds, and, moreover,*

$$\begin{aligned} &\tau \int_{B_{2R}(0)\setminus B_R(0)} R^{\frac{\alpha(x)\cdot(p(x)+\lambda-1)-p(x)(q(x)+\lambda)}{q(x)-p(x)+1}}\, dx \\ &\quad + \int_{B_{2R}(0)} (1+|x|)^{-\frac{\alpha(x)\cdot(\lambda+1)}{q(x)-1}} \tau^{1-q'(x)} dx \to 0 \quad \text{as } R \to \infty \text{ and } \tau \to \infty. \end{aligned} \tag{3.4}$$

Then problem (3.1)–(3.2) has no nonnegative global solutions $u \in L^{q(x)}_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R}_+)$ in the distributional sense.

Proof. Multiplying both parts of (3.1) by $u^\lambda \varphi_R(x) T_\tau(t)$, we get

$$\begin{aligned} &\int_{\mathbb{R}_+} T_\tau dt \int_{\mathbb{R}^n} u^{q(x)+\lambda}(1+|x|)^{\alpha(x)} \varphi_R dx + \int_{\mathbb{R}^n} u_0^{1+\lambda} \varphi_R dx \\ &\quad \le \lambda \int_{\mathbb{R}^n} u^{\lambda-1}|Du|^{p(x)} \varphi_R\, dx \int_{\mathbb{R}_+} T_\tau dt + \int_{\mathbb{R}^n} u^\lambda |Du|^{p(x)-2}(Du, D\varphi_R) dx \int_{\mathbb{R}_+} T_\tau dt \\ &\qquad - \frac{1}{1+\lambda} \int_{\mathbb{R}^n} u^{1+\lambda} \varphi_R dx \int_{\mathbb{R}_+} T'_\tau dt \le \lambda \int_{\mathbb{R}^n} u^{\lambda-1}|Du|^{p(x)} \varphi_R\, dx \int_{\mathbb{R}_+} T_\tau dt \\ &\qquad + \int_{\mathbb{R}^n} u^{p(x)+\lambda} |Du|^{p(x)-1} |D\varphi_R| dx \int_{\mathbb{R}_+} T_\tau dt + \frac{1}{1+\lambda} \int_{\mathbb{R}^n} u^{1+\lambda} \varphi_R dx \int_{\mathbb{R}_+} |T'_\tau| dt. \end{aligned}$$

Applying the Young parametric inequality to the second and third terms on the right-hand side of this formula, we arrive at

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}^n} u^{q(x)+\lambda}(1+|x|)^{\alpha(x)} \varphi_R dx \int_{\mathbb{R}_+} T_\tau dt + \int_{\mathbb{R}^n} u_0^{1+\lambda} \varphi_R dx \\ &\quad \le \frac{\lambda}{2} \int_{\mathbb{R}^n} u^{\lambda-1}|Du|^{p(x)} \varphi_R\, dx \int_{\mathbb{R}_+} T_\tau dt + c_1 \int_{\mathbb{R}^n} u^{\lambda+p(x)-1} |D\varphi_R|^{p(x)} \varphi_R^{1-p(x)} dx \int_{\mathbb{R}_+} T_\tau dt \\ &\qquad + c_2 \int_{\mathbb{R}^n} (1+|x|)^{-\frac{\alpha(x)\cdot(\lambda+1)}{q-1}} \varphi_R dx \int_{\mathbb{R}_+} |T'_\tau|^{\frac{q(x)+\lambda}{q(x)-1}} T_\tau^{-\frac{\lambda+1}{q(x)-1}} dt \end{aligned}$$

with some constants $c_1, c_2 > 0$ dependent only on λ.

Making use of the parametric Young inequality once more, removing the first nonnegative term on the right and restricting integration on both sides to smaller domains due to the choice of $\varphi_R(x)$ and $T_\tau(t)$, we get

$$\begin{aligned}
&\frac{1}{4}\int\limits_{B_R(0)} u^{q(x)+\lambda}(1+|x|)^{\alpha(x)}\varphi_R dx\int\limits_0^\tau dt+\int\limits_{B_R(0)} u_0^{1+\lambda}\varphi_R dx\\
&\quad\le c_3\int\limits_{B_{2R}(0)\setminus B_R(0)} |D\varphi_R|^{\frac{q(x)}{q(x)-p(x)+1}}(1+|x|)^{\frac{\alpha(x)\cdot(p(x)+\lambda-1)}{q(x)-p(x)+1}}\varphi_R^{-\frac{p(x)(q(x)+\lambda)}{q(x)-p(x)+1}}dx\int\limits_0^{2\tau} T_\tau dt\\
&\qquad+c_2\int\limits_{B_{2R}(0)}(1+|x|)^{-\frac{\alpha(x)\cdot(\lambda+1)}{q(x)-1}}\varphi_R\,dx\int\limits_\tau^{2\tau}|T_\tau'|^{q'(x)}T_\tau^{-\frac{1}{q(x)-1}}dt
\end{aligned}$$

with some constant $c_3>0$.

Note that on the left-hand side of the inequality the second term is nonnegative and $\varphi_R(x)\equiv 1$ in the whole domain of integration. Making use of assumptions (2.4) and (2.5), we get

$$\begin{aligned}
&\int\limits_{B_R(0)} u^{q(x)+\lambda}(1+|x|)^{\alpha(x)}\,dx\int\limits_0^\tau dt\\
&\quad\le 8c_3\tau\int\limits_{B_{2R}(0)\setminus B_R(0)} R^{\frac{\alpha(x)\cdot(p(x)+\lambda-1)-p(x)(q(x)+\lambda)}{q(x)-p(x)+1}}dx\\
&\qquad+4c_0c_2\int\limits_{B_{2R}(0)}(1+|x|)^{-\frac{\alpha(x)\cdot(\lambda+1)}{q(x)-1}}\tau^{1-q'(x)}dx.
\end{aligned}\tag{3.5}$$

Taking $R\to\infty$ and $\tau\to\infty$, due to assumption (3.4) we arrive at a contradiction. □

Remark 3.2. Here, as well as in Section 5 below, the functional parameters may also depend on t in an appropriate way.

4 Elliptic inequalities in a bounded domain Ω

Now let Ω be a bounded domain with a smooth boundary. Consider nonlinear elliptic inequalities of the form

$$-\Delta_{p(x)}u\ge u^{q(x)}f(x)\quad(x\in\Omega)\tag{4.1}$$

and

$$-\Delta_{p(x)}u\ge |Du|^{q(x)}f(x)\quad(x\in\Omega).\tag{4.2}$$

Here we will use the notation $\alpha(x)=-\log_{\rho(x)}f(x)$, where $\rho(x)=\mathrm{dist}(x,\partial\Omega)$, and $\partial\Omega^{k\varepsilon}=\{x\in\Omega:\ \mathrm{dist}(x,\partial\Omega)\le k\varepsilon\}$, $k=1,2$.

Theorem 4.1. *If there exists $\lambda\in(1-\min_{x\in\mathbb{R}^n}p(x),0)$ such that*

$$\int\limits_{\partial\Omega^{2\varepsilon}\setminus\partial\Omega^{\varepsilon}}\varepsilon^{\frac{\alpha(x)\cdot(\lambda+p(x)-1)-p(x)(q(x)+\lambda)}{q(x)-p(x)+1}}dx\to 0\quad \text{as } \varepsilon\to 0_+,\tag{4.3}$$

then inequality (4.1) has no nontrivial nonnegative solutions $u\in L^{q(x)}(\Omega)$ in the distributional sense.

Proof. Choose a family of nonnegative test functions $\varphi = \varphi_\varepsilon \in C^1(\Omega; [0,1])$ such that

$$\varphi_\varepsilon(x) = \begin{cases} 1 & (x \in \Omega \setminus \partial\Omega^{2\varepsilon}), \\ 0 & (x \in \partial\Omega^\varepsilon), \end{cases} \tag{4.4}$$

$$|D\varphi_\varepsilon(x)| \le c\varepsilon^{-1} \quad (x \in \Omega). \tag{4.5}$$

Multiply both parts of inequality (4.1) by $u^\lambda \varphi_\varepsilon$ with $1 - \min_{x\in\Omega} p(x) < \lambda < 0$. Integrating by parts, we get

$$\begin{aligned} \int_\Omega u^{q(x)+\lambda}\rho^{-\alpha(x)}\varphi_\varepsilon\,dx &\le \int_\Omega (|Du|^{p(x)-2}Du, D(u^\lambda\varphi_\varepsilon))\,dx \\ &= \lambda\int_\Omega u^{\lambda-1}|Du|^{p(x)}\varphi_\varepsilon\,dx + \int_\Omega u^\lambda|Du|^{p(x)-2}(Du, D\varphi_\varepsilon)\,dx \\ &\le \lambda\int_\Omega u^{\lambda-1}|Du|^{p(x)}\varphi_\varepsilon\,dx + \int_\Omega u^\lambda|Du|^{p(x)-1}|D\varphi_\varepsilon|\,dx \end{aligned}$$

and by the Young inequality,

$$\begin{aligned} &\int_\Omega u^{q(x)+\lambda}\rho^{-\alpha(x)}\varphi_\varepsilon\,dx + |\lambda|\int_\Omega u^{\lambda-1}|Du|^{p(x)}\varphi_\varepsilon\,dx \\ &\qquad \le |\lambda|\int_\Omega u^{\lambda-1}|Du|^{p(x)}\varphi_\varepsilon\,dx + c(\lambda)\int_\Omega u^{\lambda+p(x)-1}|D\varphi_\varepsilon|^{p(x)}\varphi_\varepsilon^{1-p(x)}dx, \end{aligned}$$

i.e.,

$$\int_\Omega u^{q(x)+\lambda}\rho^{-\alpha(x)}\varphi_\varepsilon\,dx \le c(\lambda)\int_\Omega u^{\lambda+p(x)-1}|D\varphi_\varepsilon|^{p(x)}\varphi_\varepsilon^{1-p(x)}dx.$$

Making use of the Young inequality once more, we arrive at

$$\frac{1}{2}\int_\Omega u^{q(x)+\lambda}\rho^{-\alpha(x)}\varphi_\varepsilon\,dx \le c(\lambda)\int_\Omega |D\varphi_\varepsilon|^{\frac{p(x)(q(x)+\lambda)}{q(x)-p(x)+1}}\rho^{\frac{\alpha(x)\cdot(\lambda+p(x)-1)}{q(x)-p(x)+1}}\varphi_\varepsilon^{1-\frac{p(x)(q(x)+\lambda)}{q(x)-p(x)+1}}dx.$$

Restricting the domain of integration and making use of (4.4) and (4.5), we obtain

$$\int_{\Omega\setminus\partial\Omega^{2\varepsilon}} u^{q(x)+\lambda}\rho^{-\alpha(x)}\,dx \le c\int_{\partial\Omega^{2\varepsilon}\setminus\partial\Omega^\varepsilon} \varepsilon^{\frac{\alpha(x)\cdot(\lambda+p(x)-1)-p(x)(q(x)+\lambda)}{q(x)-p(x)+1}}dx,$$

which leads to a contradiction as $\varepsilon \to 0_+$ under our assumptions. ☐

Remark 4.2. This result can be extended to a wider class of quasilinear problems, including systems of the form

$$\begin{cases} -\Delta_{p(x)}u \ge v^{s(x)}f(x) & (x \in \Omega), \\ -\Delta_{q(x)}v \ge u^{z(x)}g(x) & (x \in \Omega) \end{cases}$$

with appropriate functional parameters p, q, s, z, f, g.

Theorem 4.3. *If*

$$\int_{\partial\Omega^{2\varepsilon}\setminus\partial\Omega^\varepsilon} \varepsilon^{\frac{\alpha(x)\cdot(p(x)-1)+q(x)}{q(x)-p(x)+1}}dx \to 0 \quad \text{as } \varepsilon \to 0_+, \tag{4.6}$$

then inequality (4.2) has no solutions $u \in W^{1,q(x)}(\Omega)$ in the distributional sense that are distinct from a constant a.e.

Proof. Multiplying both parts of (4.2) by φ_ε and integrating by parts, we get

$$\int\limits_{\Omega\setminus\partial\Omega^{2\varepsilon}} |Du|^{q(x)}\rho^{-\alpha(x)}\varphi_\varepsilon\,dx \le \int\limits_{\partial\Omega^{2\varepsilon}\setminus\partial\Omega^{\varepsilon}} (|Du|^{p(x)-2}Du, D\varphi_\varepsilon)\,dx$$

and by the Young inequality

$$\int\limits_{\partial\Omega^{2\varepsilon}\setminus\partial\Omega^{\varepsilon}} (|Du|^{p(x)-2}Du, D\varphi_\varepsilon)\,dx \le \int\limits_{\partial\Omega^{2\varepsilon}\setminus\partial\Omega^{\varepsilon}} |Du|^{p(x)-1}\cdot|D\varphi_\varepsilon|\,dx$$

$$\le \eta \int\limits_{\partial\Omega^{2\varepsilon}\setminus\partial\Omega^{\varepsilon}} |Du|^{q(x)}\rho^{-\alpha(x)}\varphi_\varepsilon\,dx + c(\eta)\int\limits_{\partial\Omega^{2\varepsilon}\setminus\partial\Omega^{\varepsilon}} \rho^{\frac{\alpha(x)\cdot(p(x)-1)}{q(x)-p(x)+1}}|D\varphi_\varepsilon|^{\frac{q(x)}{q(x)-p(x)+1}}\varphi_\varepsilon^{-\frac{p(x)-1}{q(x)-p(x)+1}}\,dx,$$

whence for $\eta < 1/2$ one has

$$\int\limits_{\Omega\setminus\partial\Omega^{2\varepsilon}} |Du|^{q(x)}\rho^{-\alpha(x)}\,dx \le 2c(\varepsilon)\int\limits_{\partial\Omega^{2\varepsilon}\setminus\partial\Omega^{\varepsilon}} \rho^{\frac{\alpha(x)\cdot(p(x)-1)}{q(x)-p(x)+1}}|D\varphi_\varepsilon|^{\frac{q(x)}{q(x)-p(x)+1}}\varphi_\varepsilon^{-\frac{p(x)-1}{q(x)-p(x)+1}}\,dx$$

and due to assumptions (4.3)–(4.4)

$$\int\limits_{\Omega\setminus\partial\Omega^{2\varepsilon}} |Du|^{q(x)}\rho^{-\alpha(x)}\,dx \le c\int\limits_{\partial\Omega^{2\varepsilon}\setminus\partial\Omega^{\varepsilon}} \varepsilon^{\frac{\alpha(x)\cdot(p(x)-1)+q(x)}{q(x)-p(x)+1}}\,dx,$$

which implies the claim as $\varepsilon \to 0_+$. □

Remark 4.4. This result can be also extended to a wider class of quasilinear problems, including systems of the form

$$\begin{aligned} -\Delta_{p(x)}u &\ge |Dv|^{s(x)}f(x) \quad (x\in\Omega),\\ -\Delta_{q(x)}v &\ge |Du|^{z(x)}g(x) \quad (x\in\Omega) \end{aligned}$$

with appropriate functional parameters p, q, s, z, f, g.

5 Parabolic inequalities in a cylindrical domain $\Omega\times\mathbb{R}_+$

A nonexistence result also takes place for a parabolic inequality

$$u_t - \Delta_{p(x)}u \ge u^{q(x)}\rho^{-\alpha(x)} \quad (x\in\Omega; t\in\mathbb{R}_+) \tag{5.1}$$

with initial condition

$$u(x,0) = u_0(x) \ge 0 \quad (x\in\Omega). \tag{5.2}$$

We assume that $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is distinct from the identical zero a.e. Here we define $T_\tau \in C^1(\mathbb{R}_+;[0,1])$ so that

$$T_\tau(t) = \begin{cases} 1 & (0\le t\le\tau),\\ 0 & (t\ge 2\tau) \end{cases}$$

and for some $\tau_0 > 0$, for all $0<\tau<\tau_0$ and for all $x\in\Omega$ one has

$$\int\limits_{\tau}^{2\tau} \frac{|T_\tau'|^{q'(x)}}{|T_\tau|^{q'(x)-1}}\,dt \le c_0\tau^{1-q'(x)} \tag{5.3}$$

with some constant $c_0 > 0$ independent of x and τ, where $\frac{1}{q(x)}+\frac{1}{q'(x)} = 1$.

A typical result for problem (5.1)–(5.2) can be formulated as follows.

Theorem 5.1. *Let there exist a constant $\tau_0 > 0$ such that for any $0 < \tau < \tau_0$ and $x \in \Omega$ there holds (5.3), and*

$$\tau \int\limits_{\partial\Omega^{2\varepsilon}\setminus\partial\Omega^{\varepsilon}} \varepsilon^{\frac{\alpha(x)\cdot(p(x)+\lambda-1)-p(x)(q(x)+\lambda)}{q(x)-p(x)+1}}\,dx + \int\limits_{\Omega\setminus\partial\Omega^{2\varepsilon}} \rho^{\frac{\alpha(x)\cdot(\lambda+1)}{q(x)-1}}\tau^{1-q'(x)}dx \to 0 \quad \text{as } \varepsilon \to 0_+ \text{ and } \tau \to 0_+. \tag{5.4}$$

Then problem (5.1)–(5.2) has no nonnegative solutions $u \in L^{q(x)}_{\mathrm{loc}}(\Omega\times[0,T])$ distinct from identical zero for any $T > 0$.

Proof. Multiplying both parts of (3.1) by $u^{\lambda}\varphi_\varepsilon(x)T_\tau(t)$, where φ_ε and λ are chosen as in Theorem 4.1, and integrating by parts, we get

$$\begin{aligned}&\int\limits_{\Omega} u^{q(x)+\lambda}\rho^{-\alpha(x)}\varphi_\varepsilon dx\int\limits_{\mathbb{R}_+}T_\tau dt + \int\limits_{\Omega}u_0^{1+\lambda}\varphi_\varepsilon dx\\
&\quad\le \lambda\int\limits_{\Omega}u^{\lambda-1}|Du|^{p(x)}\varphi_\varepsilon\,dx\int\limits_{\mathbb{R}_+}T_\tau dt + \int\limits_{\Omega}u^{\lambda}|Du|^{p(x)-2}(Du,D\varphi_\varepsilon)dx\int\limits_{\mathbb{R}_+}T_\tau\,dt\\
&\qquad-\frac{1}{1+\lambda}\int\limits_{\Omega}u^{1+\lambda}\varphi_\varepsilon dx\int\limits_{\mathbb{R}_+}T_\tau' dt \le \lambda\int\limits_{\Omega}u^{\lambda-1}|Du|^{p(x)}\varphi_\varepsilon\,dx\int\limits_{\mathbb{R}_+}T_\tau dt\\
&\qquad+\int\limits_{\Omega}u^{p(x)+\lambda}|Du|^{p(x)-1}|D\varphi_\varepsilon|dx\int\limits_{\mathbb{R}_+}T_\tau\,dt + \frac{1}{1+\lambda}\int\limits_{\Omega}u^{1+\lambda}\varphi_\varepsilon dx\int\limits_{\mathbb{R}_+}|T_\tau'|dt.\end{aligned}$$

Applying the Young parametric inequality to the second and third terms on the right-hand side of this formula, we arrive at

$$\begin{aligned}&\frac{1}{2}\int\limits_{\Omega} u^{q(x)+\lambda}\rho^{-\alpha(x)}\varphi_\varepsilon dx\int\limits_{\mathbb{R}_+}T_\tau dt + \int\limits_{\Omega}u_0^{1+\lambda}\varphi_\varepsilon dx\\
&\quad\le \frac{\lambda}{2}\int\limits_{\Omega}u^{\lambda-1}|Du|^{p(x)}\varphi_\varepsilon\,dx\int\limits_{\mathbb{R}_+}T_\tau dt + c_1\int\limits_{\Omega}u^{\lambda+p(x)-1}|D\varphi_\varepsilon|^{p(x)}\varphi_\varepsilon^{1-p(x)}dx\int\limits_{\mathbb{R}_+}T_\tau dt\\
&\qquad+c_2\int\limits_{\Omega}\rho^{\frac{\alpha(x)\cdot(\lambda+1)}{q(x)-1}}\varphi_\varepsilon dx\int\limits_{\mathbb{R}_+}|T_\tau'|^{\frac{q(x)+\lambda}{q(x)-1}}T_\tau^{-\frac{\lambda+1}{q(x)-1}}dt\end{aligned}$$

with some constants $c_1, c_2 > 0$ dependent only on λ.

Making use of the parametric Young inequality once more, removing the first nonnegative term on the right and restricting integration on both sides to smaller domains due to the choice of $\varphi_\varepsilon(x)$ and $T_\tau(t)$, we obtain

$$\begin{aligned}&\frac{1}{4}\int\limits_{\Omega\setminus\partial\Omega^{2\varepsilon}} u^{q(x)+\lambda}\rho^{-\alpha(x)}\varphi_\varepsilon dx\int\limits_0^{\tau}dt + \int\limits_{\Omega\setminus\partial\Omega^{2\varepsilon}}u_0^{1+\lambda}\varphi_\varepsilon dx\\
&\quad\le c_3\int\limits_{\partial\Omega^{2\varepsilon}\setminus\partial\Omega^{\varepsilon}}|D\varphi_\varepsilon|^{\frac{p(x)(q(x)+\lambda)}{q(x)-p(x)+1}}\rho^{\frac{\alpha(x)\cdot(p(x)+\lambda-1)}{q(x)-p(x)+1}}\varphi_\varepsilon^{1-\frac{p(x)(q(x)+\lambda)}{q(x)-p(x)+1}}dx\int\limits_0^{2\tau}T_\tau dt\\
&\qquad+c_2\int\limits_{\Omega\setminus\partial\Omega^{2\varepsilon}}\rho^{\frac{\alpha(x)\cdot(\lambda+1)}{q(x)-1}}\varphi_\varepsilon\,dx\int\limits_{\tau}^{2\tau}\frac{|T_\tau'|^{q'(x)}}{T_\tau^{q'(x)-1}}dt\end{aligned}$$

with some constant $c_3 > 0$.

Note that on the left-hand side of the inequality the second term is nonnegative and $\varphi_\varepsilon(x) \equiv 1$ in the whole domain of integration. Making use of assumptions (4.4) and (4.5), we get

$$\begin{aligned} &\int\limits_{\Omega\setminus\partial\Omega^{2\varepsilon}} u^{q(x)+\lambda}\rho^{-\alpha(x)}\,dx \int\limits_0^\tau dt + \int\limits_{\Omega\setminus\partial\Omega^{2\varepsilon}} u_0^{1+\lambda} dx \\ &\qquad \le 8c_3\tau \int\limits_{\partial\Omega^{2\varepsilon}\setminus\partial\Omega^{\varepsilon}} \varepsilon^{\frac{\alpha(x)(p(x)+\lambda-1)-p(x)(q(x)+\lambda)}{q(x)-p(x)+1}}\,dx + 4c_0c_2 \int\limits_{\Omega\setminus\partial\Omega^{2\varepsilon}} \rho^{\frac{\alpha(x)(\lambda+1)}{q(x)-1}}\tau^{1-q'(x)} dx. \end{aligned} \tag{5.5}$$

Taking $\varepsilon \to 0_+$ and $\tau \to 0_+$, by (5.4) we arrive at a contradiction. □

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