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# Positive Solutions for Singular $\phi-$Laplacian BVPs on the Positive Half-line

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**Abstract**

In this work, we are concerned with the existence of positive solutions for a $\phi$ Laplacian boundary value problem on the half-line. The results are proved using the fixed point index theory on cones of Banach spaces and the upper and lower solution technique. The nonlinearity may exhibit a singularity at the origin with respect to the solution. This singularity is treated by regularization and approximation together with compactness and sequential arguments.

## 1 Introduction

This paper is devoted to the study of the existence of positive solutions to the following boundary value problem (BVP for short) on the positive half-line:

$$\begin{cases} (\phi(x'))'(t) + q(t)f(t, x(t)) = 0, & t \in I, \\ x(0) = 0, \quad \lim\limits_{t\to+\infty} x'(t) = 0 \end{cases} \tag{1.1}$$

where $I := (0, +\infty)$ denotes the set of positive real numbers while $\mathbb{R}^+ = [0, +\infty)$. The function $q : I \longrightarrow I$ is continuous and the function $f : I \times I \longrightarrow \mathbb{R}^+$ is continuous and satisfies $\lim\limits_{x\to 0^+} f(t, x) = +\infty$, i.e. $f(t, x)$ may be singular at $x = 0$, for each $t > 0$. $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous, increasing homeomorphism such that $\phi(0) = 0$, extending the so-called $p-$Laplacian $\varphi_p(s) = |s|^{p-1}s$ $(p > 1)$.

Problem (1.1) with $\phi = I_d$ has been extensively studied in the literature. In [14], D.O'Regan *et al.* established the existence of unbounded solutions. Djebali and Mebarki [5, 6, 7] discussed the solvability and the multiplicity of solutions to the generalized Fisher-like equation associated to the second-order linear operator $-y'' + cy' + \lambda y$ $(c, \lambda > 0)$ with Dirichlet or Neumann limit condition at positive infinity; see also [8] where the nonlinearity may

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change sign and the theory of fixed point index on cones of Banach spaces is used. In [16], the author proved existence of positive solution to a second-order multi-point BVP by application of the Mönch's fixed point theorem. The method of upper and lower solutions together with the fixed point index are employed in [14, 15] to discuss the existence of multiple solutions to a singular BVP on the half line.

Recent papers have also investigated the case of the so-called $p-$Laplacian operator $\varphi(s) = |s|^{p-1}s$ for some $p > 1$. Existence of three positive solutions for singular $p-$Laplacian problems is obtained by means of the three-functional fixed point theorem in [11, 12]. The same method is also used by Guo *et al.* in [10] to prove existence of three positive solutions when the nonlinearity is derivative depending. In [13], the authors prove existence of three positive solutions when the nonlinear operator $\varphi$ generates a $p-$Laplacian operator.

In this paper, our aim is to consider a general homeomorphism $\varphi$ and prove existence of single and twin solutions using fixed point index theory. Existence of at least one positive solution is also proved by application of the method of upper and lower solutions.

This paper has mainly three sections. In section 2, we prove some lemmas which are needed in this work and we gather together some auxiliary results. Section 3 is devoted to establishing existence and multiplicity results; the fixed point theory on a suitable cone in a Banach space is employed to an approximating operator; then a compactness argument allows us to get the desired solution in Theorem 3.1. Finally, in section 4 we use the method of lower and upper solutions to prove the existence of a positive solution of (1.1). For this, a regularization technique both with a sequential argument are considered to overcome the singularity. Theorems 4.1 and 4.2 correspond to the regular problem and singular one respectively. Each existence theorem is illustrated by means of an example of application.

## 2 Preliminaries

In this section, we gather together some definitions and lemmas we need in the sequel.

### 2.1 Auxiliary results

**Definition 2.1.** *A nonempty subset $\mathcal{P}$ of a Banach space $E$ is called a cone if it is convex, closed and satisfies the conditions:*

**(i)** $\alpha x \in \mathcal{P}$ *for all* $x \in \mathcal{P}$ *and* $\alpha \geq 0$,

**(ii)** $x, -x \in \mathcal{P}$ *imply that* $x = 0$.

**Definition 2.2.** *A mapping $A: E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

The following lemmas will be used to prove existence of solutions. More details on the theory of the fixed point index on cones of Banach spaces may be found in [1, 2, 4, 9].

**Lemma 2.1.** *Let $\Omega$ be a bounded open set in a real Banach space $E$, $\mathcal{P}$ a cone of $E$ and $A: \overline{\Omega} \cap \mathcal{P} \to \Omega$ a completely continuous map. Suppose $\lambda Ax \neq x, \forall\, x \in \partial\Omega \cap \mathcal{P}, \lambda \in (0,1]$. Then $i(A, \Omega \cap \mathcal{P}, \mathcal{P}) = 1$.*

**Lemma 2.2.** *Let $\Omega$ be a bounded open set in a real Banach space $E$, $\mathcal{P}$ a cone of $E$ and $A: \overline{\Omega} \cap \mathcal{P} \to \Omega$ a completely continuous map. Suppose $Ax \not\leq x, \forall\, x \in \partial\Omega \cap \mathcal{P}$. Then $i(A, \Omega \cap \mathcal{P}, \mathcal{P}) = 0$.*

Let
$$C_l([0,\infty), \mathbb{R}) = \{x \in C([0,\infty), \mathbb{R}) : \lim_{t\to\infty} x(t) \text{ exists}\}$$
and consider the basic space to study Problem (1.1) namely
$$E = \{x \in C([0,\infty), \mathbb{R}) : \lim_{t\to+\infty} \frac{x(t)}{1+t} \text{ exists}\}.$$
Then $E$ is a Banach space with norm $\|x\| = \sup\limits_{t\in\mathbb{R}^+} \frac{|x(t)|}{1+t}$.

From the following result

**Lemma 2.3.** *([3], p. 62) Let $M \subseteq C_l(\mathbb{R}^+, \mathbb{R})$. Then $M$ is relatively compact in $C_l(\mathbb{R}^+, \mathbb{R})$ if the following conditions hold:*

**(a)** *$M$ is uniformly bounded in $C_l(\mathbb{R}^+, \mathbb{R})$.*

**(b)** *The functions belonging to $M$ are almost equicontinuous on $\mathbb{R}^+$, i.e. equicontinuous on every compact interval of $\mathbb{R}^+$.*

**(c)** *The functions from $M$ are equiconvergent, that is, given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|x(t) - x(+\infty)| < \varepsilon$ for any $t \geq T(\varepsilon)$ and $x \in M$.*

We easily deduce

**Lemma 2.4.** *Let $M \subseteq E$. Then $M$ is relatively compact in $E$ if the following conditions hold:*

**(a)** *$M$ is uniformly bounded in $E$,*

**(b)** *the functions belonging to $\{u:\ u(t) = \frac{x(t)}{1+t},\ x \in E\}$ are almost equicontinuous on $[0, +\infty)$,*

**(c)** *the functions belonging to $\{u:\ u(t) = \frac{x(t)}{1+t},\ x \in E\}$ are equiconvergent at $+\infty$.*

## 2.2 Useful Lemmas

**Definition 2.3.** *A function $x$ is said to be a solution of Problem (1.1) if $x \in C(\mathbb{R}^+, \mathbb{R}) \cap C^1(I, \mathbb{R})$ with $\phi(x') \in C^1(I, \mathbb{R})$ and satisfies (1.1).*

Since $\phi$ is an increasing homeomorphism, it is easy to prove

**Lemma 2.5.** *If $x$ is a solution of Problem (1.1), then $x$ is positive, monotone increasing and concave on $[0, +\infty)$.*

Define the cone

$$\mathcal{P} = \{x \in E : \ x \text{ is nonnegative, concave on } [0, +\infty) \text{ and } \lim_{t \to +\infty} \frac{x(t)}{1+t} = 0\}.$$

**Lemma 2.6.** *If $x \in C(\mathbb{R}^+, \mathbb{R}^+)$ is a positive concave function, then $x$ is nondecreasing on $[0, +\infty)$.*

*Proof.* Let $t, t' \in [0, +\infty)$ be such that $t' \geq t$ and $\lambda := t' - t$. Since $x$ is positive and concave, for all $n \in \mathbb{N}^*$, we have

$$\begin{array}{rcl} x(t') & = & x(t+\lambda) \\ & = & x\left((1-\frac{1}{n})t + \frac{1}{n}(t+n\lambda)\right) \\ & \geq & \left(1-\frac{1}{n}\right)x(t) + \frac{1}{n}x(t+n\lambda) \\ & \geq & \left(1-\frac{1}{n}\right)x(t). \end{array}$$

Therefore

$$x(t') \geq \lim_{n \to +\infty} \left(1 - \frac{1}{n}\right) x(t) = x(t),$$

and our claim follows. □

Moreover, we have

**Lemma 2.7.** *Let $x \in \mathcal{P}$ and $\theta \in (1, +\infty)$. Then*

$$x(t) \geq \frac{1}{\theta}\|x\|, \quad \forall\, t \in [1/\theta, \theta].$$

*Proof.* Since the continuous, positive function $y(t) = \frac{x(t)}{1+t}$ satisfies $y(+\infty) = 0$, then it achieves its maximum at some $t_0 \in [0, +\infty)$. Moreover $x$ is concave and nondecreasing by Lemma 2.6; then for $t \in [\frac{1}{\theta}, \theta]$

$$\begin{array}{rcl} x(t) & \geq & \min\limits_{t \in [\frac{1}{\theta}, \theta]} x(t) = x(\frac{1}{\theta}) = x(\frac{\theta-1+\theta t_0}{\theta+\theta t_0}\frac{1}{\theta-1+\theta t_0} + \frac{1}{\theta+\theta t_0}t_0) \\ & \geq & \frac{\theta-1+\theta t_0}{\theta+\theta t_0}x(\frac{1}{\theta-1+\theta t_0}) + \frac{1}{\theta+\theta t_0}x(t_0) \\ & \geq & \frac{1}{\theta+\theta t_0}x(t_0) = \frac{1}{\theta}\frac{x(t_0)}{1+t_0} = \frac{1}{\theta}\|x\|. \end{array}$$

□

**Lemma 2.8.** *Define the function* $\rho$ *by*

$$\rho(t) = \begin{cases} t, & t \in [0,1] \\ \frac{1}{t}, & t \in [1,+\infty) \end{cases} \tag{2.1}$$

*and let* $x \in \mathcal{P}$*. Then*

$$x(t) \geq \rho(t)\|x\|, \quad \forall\, t \in [0,+\infty).$$

*Proof.* Let $t \in [0,+\infty)$ and distinguish between four cases:

- If $t = 0$, then $x(0) \geq 0 = \rho(0)\|x\|$.
- If $t \in (0,1)$, then $\frac{1}{t} \in (1,+\infty)$. By lemma 2.7, we have $x(s) \geq t\|x\|, \forall\, s \in [t, \frac{1}{t}]$. In particular for $s = t$, $x(t) \geq t\|x\| = \rho(t)\|x\|$.
- If $t \in (1,+\infty)$, then by lemma 2.7, we have $x(s) \geq \frac{1}{t}\|x\|, \forall\, s \in [\frac{1}{t}, t]$. In particular for $s = t$, $x(t) \geq \frac{1}{t}\|x\| = \rho(t)\|x\|$.
- If $t = 1$, then let $\{t_n\}_n$ be a real sequence such that $t_n > 1$ and $t_n \to 1$. By the latter case, we have $x(t_n) \geq \frac{1}{t_n}\|x\|$, $\forall\, n \geq 1$. Then

$$x(1) = \lim_{n\to+\infty} x(t_n) \geq \lim_{n\to+\infty} \frac{1}{t_n}\|x\| = \|x\| = \rho(1)\|x\|.$$

□

**Lemma 2.9.** *Let* $g \in C(\mathbb{R}^+, \mathbb{R}^+)$ *be such that* $\int_0^{+\infty} g(s)ds < +\infty$ *and let* $x(t) = \int_0^t \phi^{-1}\left(\int_s^{+\infty}(g(\tau))d\tau\right) ds$*. Then*

$$\begin{cases} (\phi(x'))'(t) + g(t) = 0, & t > 0, \\ x(0) = 0, & \lim\limits_{t\to+\infty} x'(t) = 0, \end{cases}$$

*hence* $x \in \mathcal{P}$*.*

*Proof.* It is easy to check that

$$\begin{cases} (\phi(x'))'(t) + g(t) = 0, & t > 0, \\ x(0) = 0, & \lim\limits_{t\to+\infty} x'(t) = 0. \end{cases}$$

Moreover, $x$ is positive, concave on $[0,+\infty)$, hence nondecreasing by Lemma 2.6. Therefore

$$\begin{cases} \text{If } \lim\limits_{t\to+\infty} x(t) < \infty, & \text{then } \lim\limits_{t\to+\infty} \frac{x(t)}{1+t} = 0. \\ \text{If } \lim\limits_{t\to+\infty} x(t) = +\infty, & \text{then } \lim\limits_{t\to+\infty} \frac{x(t)}{1+t} = \lim\limits_{t\to+\infty} x'(t) = 0, \end{cases}$$

proving the lemma. □

## 3 A fixed point index argument

Let $\widetilde{\rho}(t) = \frac{\rho(t)}{1+t}$, $F(t,x) = f(t,(1+t)x)$ and assume that

$(\mathcal{H}_1)$ There exist $m \in C(I,I)$ and $p \in C(\mathbb{R}^+,\mathbb{R}^+)$ such that

$$F(t,x) \leq m(t)p(x), \ \ \forall\,(t,x) \in I^2. \tag{3.1}$$

There exists a decreasing function $h \in C(I,I)$ such that $\frac{p(x)}{h(x)}$ is an increasing function and for each $c, c' > 0$,

$$\int_0^{+\infty} q(\tau)m(\tau)h(c\widetilde{\rho}(\tau))d\tau < +\infty, \tag{3.2}$$

$$\int_0^{+\infty} \phi^{-1}\left(\frac{p(c')}{h(c')}\int_s^{+\infty} q(\tau)m(\tau)h(c\widetilde{\rho}(\tau))d\tau\right) ds < +\infty. \tag{3.3}$$

$(\mathcal{H}_2)$ For any $c > 0$, there exists $\psi_c \in C(I,I)$ such that

$$F(t,x) \geq \psi_c(t), \quad \forall\, t \in I,\ \forall\, x \in (0,c]$$

with

$$\int_0^{+\infty} q(\tau)\psi_c(\tau)d\tau < +\infty \text{ and } \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} q(\tau)\psi_c(\tau)d\tau\right) ds < +\infty. \tag{3.4}$$

$(\mathcal{H}_3)$

$$\sup_{c>0} \frac{c}{\int_0^{+\infty} \phi^{-1}\left(\frac{p(c)}{h(c)}\int_s^{+\infty} q(\tau)m(\tau)h(c\widetilde{\rho}(\tau))d\tau\right) ds} > 1.$$

### 3.1 Existence of a single solution

We first consider a family of regular problems which approximate Problem (1.1). Given $f \in C(I^2,\mathbb{R}^+)$, define a sequence of functions $\{f_n\}_{n\geq 1}$ by

$$f_n(t,x) = f(t, \max\{(1+t)/n, x\}), \quad n \in \{1,2,\ldots\}$$

and for $x \in \mathcal{P}$, define a sequence of operators by

$$A_n x(t) = \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)f_n(\tau,x(\tau))d\tau\right) ds, \quad n \in \{1,2,\ldots\}.$$

We have

**Lemma 3.1.** *Assume $(\mathcal{H}_1)$ holds. Then, for each $n \geq 1$, the operator $A_n$ sends $\mathcal{P}$ into $\mathcal{P}$ and is completely continuous.*

*Proof.* **(a)** $A_n\mathcal{P} \subseteq \mathcal{P}$. For $x \in \mathcal{P}$, we have $A_n x(t) \geq 0$, $\forall\, t \in \mathbb{R}^+$. Moreover

$$(A_n x)'(t) = \phi^{-1}\left(\int_t^{+\infty} q(\tau) f_n(\tau, x(\tau)) d\tau\right) \geq 0,$$

$$\lim_{t \to +\infty} (A_n x)'(t) = \phi^{-1}(0) = 0,$$

and

$$(\phi(A_n x)')' = -q(t) f_n(t, x(t)) \leq 0,$$

which implies that $A_n x$ is concave, nondecreasing on $[0, +\infty)$ and $\lim\limits_{t \to +\infty} \frac{(A_n x)(t)}{1+t} = 0$. Then $A_n \mathcal{P} \subseteq \mathcal{P}$.

**(b)** $A_n$ is continuous. Let $x, x_0 \in E$. By the continuity of $f$ and the Lebesgue dominated convergence theorem, we have for all $s \in \mathbb{R}^+$,

$$\begin{array}{l} \left|\int_s^{+\infty} q(\tau) f_n(\tau, x(\tau)) d\tau - \int_s^{+\infty} q(\tau) f_n(\tau, x_0(\tau)) d\tau\right| \\ \leq \int_s^{+\infty} q(\tau) |f_n(\tau, x(\tau)) - f_n(\tau, x_0(\tau))| d\tau \longrightarrow 0, \quad \text{as } x \to x_0 \end{array}$$

i.e.

$$\int_s^{+\infty} q(\tau) f_n(\tau, x(\tau)) d\tau \to \int_s^{+\infty} q(\tau) f_n(\tau, x_0(\tau)) d\tau, \quad \text{as } x \to x_0.$$

Moreover, the continuity of $\phi^{-1}$ implies that

$$\phi^{-1}\left(\int_s^{+\infty} q(\tau) f_n(\tau, x(\tau)) d\tau\right) \to \phi^{-1}\left(\int_s^{+\infty} q(\tau) f_n(\tau, x_0(\tau)) d\tau\right),$$

as $x \to x_0$. Thus

$$\begin{array}{rl} & \|A_n x - A_n x_0\| \\ = & \sup\limits_{t \in \mathbb{R}^+} \frac{|A_n x(t) - A_n x_0(t)|}{1+t} \\ = & \sup\limits_{t \in \mathbb{R}^+} \frac{\left|\int_0^t (\phi^{-1}(\int_s^{+\infty} q(\tau) f_n(\tau, x(\tau)) d\tau)) ds - \int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau) f_n(\tau, x_0(\tau)) d\tau) ds\right|}{1+t} \\ \leq & \sup\limits_{t \in \mathbb{R}^+} \frac{\int_0^t \left|\phi^{-1}(\int_s^{+\infty} q(\tau) f_n(\tau, x(\tau))) - \phi^{-1}(\int_s^{+\infty} q(\tau) f_n(\tau, x_0(\tau)) d\tau)\right| ds}{1+t} \to 0, \\ & \text{as } x \to x_0, \end{array}$$

and our claim follows.

**(c)** $A_n(B)$ is relatively compact, where $B = \{x \in E : \|x\| \leq R\}$ is a bounded subset of $\mathcal{P}$. Indeed:

• $A_n(B)$ is uniformly bounded. Let $x \in B$. By the monotonicity of $h$ and $\frac{p}{h}$, we have the estimates:

$$\begin{aligned}
&\|A_n x\|_E = \sup_{t\in\mathbb{R}^+} \tfrac{|A_n x(t)|}{1+t} \\
= \ & \sup_{t\geq 0} \tfrac{1}{1+t} \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau) f_n(\tau, x(\tau)) d\tau\right) ds, \\
\leq \ & \sup_{t\geq 0} \tfrac{1}{1+t} \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau) m(\tau) p(\max\{\tfrac{1}{n}, \tfrac{x(\tau)}{1+\tau}\}) d\tau\right) ds, \\
\leq \ & \sup_{t\geq 0} \tfrac{1}{1+t} \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau) m(\tau) h(\max\{\tfrac{1}{n}, \tfrac{x(\tau)}{1+\tau}\}) \tfrac{p(\max\{\frac{1}{n}, \frac{x(\tau)}{1+\tau}\})}{h(\max\{\frac{1}{n}, \frac{x(\tau)}{1+\tau}\})} d\tau\right) ds \\
\leq \ & \int_0^{+\infty} \phi^{-1}\left(\tfrac{p(\max\{1/n,R\})}{h(\max\{1/n,R\})} \int_s^{+\infty} q(\tau) m(\tau) h(\tfrac{\widetilde{\rho}(\tau)}{n}) d\tau\right) ds < +\infty.
\end{aligned}$$

• $\frac{A_n(B)}{1+t}$ is almost equicontinuous. For given $T > 0$, $x \in B$, and $t, t' \in [0, T]$ ($t' < t$), we have

$$\begin{aligned}
&\left|\tfrac{A_n x(t)}{1+t} - \tfrac{A_n x(t')}{1+t'}\right| \\
= \ & \left|\tfrac{\int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau) f_n(\tau, x(\tau)) d\tau) ds}{1+t} - \tfrac{\int_0^{t'} \phi^{-1}(\int_s^{+\infty} q(\tau) f_n(\tau, x(\tau)) d\tau) ds}{1+t'}\right| \\
\leq \ & \left|\tfrac{1}{1+t} - \tfrac{1}{1+t'}\right| \int_0^{+\infty} \phi^{-1}(\int_s^{+\infty} q(\tau) f_n(\tau, x(\tau)) d\tau) ds \\
& + \left|\tfrac{\int_{t'}^{+\infty} \phi^{-1}(\int_s^{+\infty} q(\tau) f_n(\tau, x(\tau)) d\tau) ds}{1+t'} - \tfrac{\int_t^{+\infty} \phi^{-1}(\int_s^{+\infty} q(\tau) f_n(\tau, x(\tau)) d\tau) ds}{1+t}\right| \\
\leq \ & 2\left|\tfrac{1}{1+t} - \tfrac{1}{1+t'}\right| \int_0^{+\infty} \phi^{-1}(\int_s^{+\infty} q(\tau) f_n(\tau, x(\tau)) d\tau) ds \\
& + \tfrac{1}{1+t'} \int_{t'}^t \phi^{-1}(\int_s^{+\infty} q(\tau) f_n(\tau, x(\tau)) d\tau) ds \\
\leq \ & 2\left|\tfrac{1}{1+t} - \tfrac{1}{1+t'}\right| \int_0^{+\infty} \phi^{-1}\left(\tfrac{p(\max\{1/n,R\})}{h(\max\{1/n,R\})} \int_s^{+\infty} q(\tau) m(\tau) h(\tfrac{\widetilde{\rho}(\tau)}{n}) d\tau\right) ds \\
& + \tfrac{1}{1+t'} \int_{t'}^t \phi^{-1}\left(\tfrac{p(\max\{1/n,R\})}{h(\max\{1/n,R\})} \int_s^{+\infty} q(\tau) m(\tau) h(\tfrac{\widetilde{\rho}(\tau)}{n}) d\tau\right) ds.
\end{aligned}$$

Then, for any $\varepsilon > 0$ and $T > 0$, there exists $\delta > 0$ such that $\left|\frac{Ax(t)}{1+t} - \frac{Ax(t')}{1+t'}\right| < \varepsilon$ for all $t, t' \in [0, T]$ with $|t - t'| < \delta$, proving our claim.

• $\frac{A_n(B)}{1+t}$ is equiconvergent at $+\infty$. Since

$$\lim_{t\to+\infty} \frac{A_n x(t)}{1+t} = \lim_{t\to+\infty} \frac{\int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau) f_n(\tau, x(\tau)) d\tau) ds}{1+t} = 0,$$

then

$$\begin{aligned}
& \sup_{x\in B}|\tfrac{Ax(t)}{1+t}-\lim_{t\to+\infty}\tfrac{A_nx(t)}{1+t}| \\
= & \sup_{x\in B}\left|\tfrac{\int_0^t\phi^{-1}(\int_s^{+\infty}q(\tau)f_n(\tau,x(\tau))d\tau)ds}{1+t}\right| \\
\leq & \tfrac{1}{1+t}\sup_{x\in B}\int_0^{+\infty}\phi^{-1}\left(\tfrac{p(\max\{1/n,R\})}{h(\max\{1/n,R\})}\int_s^{+\infty}q(\tau)m(\tau)h(\tfrac{\widetilde{\rho}(\tau)}{n})d\tau\right)ds \\
\leq & \tfrac{1}{1+t}\sup_{x\in B}\int_0^{+\infty}\phi^{-1}\left(\tfrac{p(\max\{1/n,R\})}{h(\max\{1/n,R\})}\int_s^{+\infty}q(\tau)m(\tau)h(\tfrac{\widetilde{\rho}(\tau)}{n})d\tau\right)ds
\end{aligned}$$

which implies that $\lim_{t\to+\infty}\sup_{x\in B}|\frac{Ax(t)}{1+t}-\lim_{t\to+\infty}\frac{Ax(t)}{1+t}|=0.$

□

**Theorem 3.1.** *Assume that Assumptions $(\mathcal{H}_1)-(\mathcal{H}_3)$ hold. Then Problem (1.1) has at least one positive solution.*

*Proof.*

Step 1: an approximating solution. From condition $(\mathcal{H}_3)$, there exists $R>0$ such that:

$$\frac{R}{\int_0^{+\infty}\phi^{-1}\left(\frac{p(R)}{h(R)}\int_s^{+\infty}q(\tau)m(\tau)h(R\widetilde{\rho}(\tau))d\tau\right)ds}>1. \tag{3.5}$$

Let

$$\Omega_1=\{x\in E:\ \|x\|<R\}.$$

We claim that $x\neq\lambda A_nx$ for any $x\in\partial\Omega_1\cap\mathcal{P},\lambda\in(0,1]$ and $n\geq n_0>1/R$. On the contrary, suppose that there exist $n\geq n_0$, $x_0\in\partial\Omega_1\cap\mathcal{P}$ and $\lambda_0\in(0,1]$ such that $x_0=\lambda_0A_nx_0$. By Lemma 2.8, we have $x_0(t)\geq\rho(t)\|x_0\|=\rho(t)R,\forall\,t\in\mathbb{R}^+$. Then $\frac{x_0(t)}{1+t}\geq\widetilde{\rho}(t)\|x_0\|=\widetilde{\rho}(t)R$. Therefore, for $n$ large enough, we have

$$\begin{aligned}
R= & \|x_0\| \\
= & \|\lambda_0A_nx_0\| \\
\leq & \sup_{t\geq0}\tfrac{1}{1+t}\int_0^t\phi^{-1}\left(\int_s^{+\infty}q(\tau)f_n(\tau,x_0(\tau))d\tau\right)ds, \\
\leq & \sup_{t\geq0}\tfrac{1}{1+t}\int_0^t\phi^{-1}\left(\int_s^{+\infty}q(\tau)m(\tau)p(\max\{\tfrac{1}{n},\tfrac{x_0(\tau)}{1+\tau}\})d\tau\right)ds, \\
\leq & \sup_{t\geq0}\tfrac{1}{1+t}\int_0^t\phi^{-1}\left(\int_s^{+\infty}q(\tau)m(\tau)h(\max\{\tfrac{1}{n},\tfrac{x_0(\tau)}{1+\tau}\})\tfrac{p(\max\{\frac{1}{n},\frac{x_0(\tau)}{1+\tau}\})}{h(\max\{\frac{1}{n},\frac{x_0(\tau)}{1+\tau}\})}d\tau\right)ds \\
\leq & \int_0^{+\infty}\phi^{-1}\left(\tfrac{p(R)}{h(R)}\int_s^{+\infty}q(\tau)m(\tau)h(R\widetilde{\rho}(\tau))d\tau\right)ds
\end{aligned}$$

which is a contradiction to (3.5). Then by Lemma 2.1, we deduce that

$$i(A_n,\Omega_1\cap\mathcal{P},\mathcal{P})=1,\ \text{ for all }\ n\in\{n_0,n_0+1,\dots\}. \tag{3.6}$$

Hence there exists an $x_n\in\Omega_1\cap\mathcal{P}$ such that $A_nx_n=x_n$, $\forall\,n\geq n_0$.

Step 2: a compactness argument. (a) Since $\|x_n\| < R$, from $(\mathcal{H}_2)$ there exists $\psi_R \in C(I, I)$ such that

$$f_n(t, x_n(t)) \geq \psi_R(t), \quad \forall\, t \in I$$

with

$$\int_0^{+\infty} q(s)\psi_R(s)ds < +\infty \text{ and } \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} q(\tau)\psi_R(\tau)d\tau\right) ds < +\infty.$$

Then

$$\begin{array}{rcl} x_n(t) & = & A_n x_n(t) \\ & = & \int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau) f_n(\tau, x_n(\tau))d\tau)\, ds \\ & \geq & \int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau)\psi_R(\tau)d\tau)ds. \end{array}$$

Let

$$c^* = \phi^{-1}\left(\int_1^{+\infty} q(\tau)\psi_R(\tau)d\tau\right),$$

and distinguish between two cases.

- If $t \in [0, 1]$, then

$$\begin{array}{rcl} x_n(t) & \geq & t\phi^{-1}(\int_t^{+\infty} q(\tau) f_n(\tau, x_n(\tau))d\tau)\, ds \\ & \geq & t\phi^{-1}(\int_1^{+\infty} q(\tau)\psi_R(\tau)d\tau)ds = \rho(t)c^*. \end{array}$$

- If $t \in (1, +\infty)$, then

$$\begin{array}{rcl} x_n(t) & \geq & \int_0^1 \phi^{-1}(\int_s^{+\infty} q(\tau)\psi_R(\tau)d\tau)ds \\ & \geq & \int_0^1 \phi^{-1}(\int_1^{+\infty} q(\tau)\psi_R(\tau)d\tau)ds \\ & \geq & \frac{1}{t}\phi^{-1}(\int_1^{+\infty} q(\tau)\psi_R(\tau)d\tau \geq \rho(t)c^*. \end{array}$$

We infer that $\frac{x_n(t)}{1+t} \geq c^*\widetilde{\rho}(t),\ \forall\, t \in \mathbb{R}^+$.

(b) $\{x_n\}_{n \geq n_0}$ is almost equicontinuous. For any $T > 0$ and $t, t' \in [0, T]$ $(t > t')$, we have

$$\begin{array}{rcl} \left|\frac{x_n(t)}{1+t} - \frac{x_n(t')}{1+t'}\right| & \leq & \left|\frac{\int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau)f_n(\tau,x_n(\tau))d\tau)ds}{1+t} - \frac{\int_0^{t'} \phi^{-1}(\int_s^{+\infty} q(\tau)f_n(\tau,x_n(\tau))d\tau)ds}{1+t'}\right| \\ & \leq & 2\left|\frac{1}{1+t} - \frac{1}{1+t'}\right| \int_0^{+\infty} \phi^{-1}(\int_s^{+\infty} q(\tau)m(\tau)h(c^*\widetilde{\rho}(\tau))\frac{p(R)}{h(R)}d\tau)ds \\ & & +\frac{1}{1+t'}\int_{t'}^t \phi^{-1}(\int_s^{+\infty} q(\tau)m(\tau)h(c^*\widetilde{\rho}(\tau))\frac{p(R)}{h(R)}d\tau)ds. \end{array}$$

Then, for any $\varepsilon > 0$ and $T > 0$, there exists $\delta > 0$ such that $|\frac{x_n(t)}{1+t} - \frac{x_n(t')}{1+t'}| < \varepsilon$ for all $t, t' \in [0, T]$ with $|t - t'| < \delta$.

(c) $\{x_n\}$ is equiconvergent at $+\infty$:

$$\begin{array}{rcl}
\sup\limits_{n\geq n_0}\left|\frac{x_n(t)}{1+t}-\lim\limits_{t\to+\infty}\frac{x_n(t)}{1+t}\right| & = & \sup\limits_{n\geq n_0}\frac{\int_0^t\phi^{-1}(\int_s^{+\infty}q(\tau)f_n(\tau,x_n(\tau))d\tau)ds}{1+t}\\
& \leq & \frac{\int_0^{+\infty}\phi^{-1}(\int_s^{+\infty}q(\tau)m(\tau)h(c^*\widetilde{\rho}(\tau))\frac{p(R)}{h(R)}d\tau)ds}{1+t}\\
& & \to 0,\ \text{as } t\to+\infty.
\end{array}$$

Therefore $\{x_n\}_{n\geq n_0}$ is relatively compact and hence there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ with $\lim\limits_{k\to+\infty}x_{n_k}=x_0$. Since $x_{n_k}(t)\geq\widetilde{\rho}(t)c^*, \forall\, k\geq 1$, we have $x_0(t)\geq\widetilde{\rho}(t)c^*, \forall\, t\in\mathbb{R}^+$. Consequently, the continuity of $f$ implies that for all $s\in I$

$$\begin{array}{rcl}
\lim\limits_{k\to+\infty}f_{n_k}(s,x_{n_k}(s)) & = & \lim\limits_{k\to+\infty}f(s,\max\{(1+s)/n_k,x_{n_k}(s)\})\\
& = & f(s,\max\{0,x_0(s)\})\\
& = & f(s,x_0(s)).
\end{array}$$

By the Lebesgue dominated convergence theorem, we deduce that

$$\begin{array}{rcl}
x_0(t) & = & \lim\limits_{k\to+\infty}x_{n_k}(t)\\
& = & \lim\limits_{k\to+\infty}\int_0^t\phi^{-1}(\int_s^{+\infty}q(\tau)f_{n_k}(\tau,x_{n_k}(\tau))d\tau)ds\\
& = & \int_0^t\phi^{-1}(\int_s^{+\infty}q(\tau)f(\tau,x_0(\tau))d\tau)ds.
\end{array}$$

Then $x_0$ is a positive nontrivial solution of Problem (1.1). □

**Example 3.1.** *Consider the singular boundary value problem*

$$\left\{\begin{array}{l}
((x'(t))^5)'+e^{-t}\frac{m(t)(x^2+(1+t)^2)}{(1+t)^{\frac{3}{2}}\sqrt{x}}=0,\quad t>0,\\
x(0)=0,\ \lim\limits_{t\to+\infty}x'(t)=0,
\end{array}\right.\tag{3.7}$$

*where*

$$m(t)=\left\{\begin{array}{ll}
\frac{t}{1+t} & t\in(0,1]\\
\frac{1}{t(1+t)} & t\in(1,+\infty).
\end{array}\right.$$

*Here* $f(t,x)=\frac{m(t)(x^2+(1+t)^2)}{(1+t)^{\frac{3}{2}}\sqrt{x}}$, $\phi(t)=t^5$ *and* $q(t)=e^{-t}$. *Then* $\phi$ *is continuous, increasing and* $\phi(0)=0$. *Moreover* $F(t,x)=f(t,(1+t)x)=\frac{m(t)(x^2+1)}{\sqrt{x}}$.

($\mathcal{H}_1$) *Let* $p(x)=\frac{x^2+1}{\sqrt{x}}$, $h(x)=\frac{1}{x}$. *Then* $h$ *is a decreasing function,* $\frac{p}{h}$ *is an increasing one and* $F(t,x)\leq m(t)p(x)$, $\forall\,(t,x)\in I^2$. *In addition, for any* $c,c'>0$, *we have*

$$\int_0^{+\infty}q(\tau)m(\tau)h(c\widetilde{\rho}(\tau))d\tau=\frac{1}{c}\int_0^{+\infty}e^{-\tau}d\tau=\frac{1}{c}<+\infty$$

*and*

$$\begin{aligned}\int_0^{+\infty}\phi^{-1}\Big(\int_s^{+\infty}q(\tau)m(\tau)h(c\widetilde{\rho}(\tau))\tfrac{p(c')}{h(c')}d\tau\Big)ds &= \int_0^{+\infty}\phi^{-1}\left(\tfrac{p(c')}{ch(c')}e^{-s}\right)ds\\ &= \left(\tfrac{p(c')}{ch(c')}\right)^{\frac{1}{5}}\int_0^{+\infty}e^{\frac{-s}{5}}ds\\ &= 5\left(\tfrac{p(c')}{ch(c')}\right)^{\frac{1}{5}} < +\infty.\end{aligned}$$

($\mathcal{H}_2$) *For any $c > 0$, there exists $\psi_c(t) = \frac{m(t)}{\sqrt{c}}$ such that*

$$F(t,x) \geq \psi_c(t),\ \forall\, t \in I,\ \forall\, x \in (0,c].$$

($\mathcal{H}_3$)

$$\begin{aligned}\sup_{c>0}\frac{c}{\int_0^{+\infty}\phi^{-1}\big(\int_s^{+\infty}q(\tau)m(\tau)h(c\widetilde{\rho}(\tau))\frac{p(c)}{h(c)}d\tau\big)ds} &= \sup_{c>0}\frac{c}{5(\frac{p(c)}{ch(c)})^{\frac{1}{5}}}\\ &= \frac{1}{5}\sup_{c>0}\frac{cc^{\frac{1}{10}}}{(c^2+1)^{\frac{1}{5}}}\\ &= \frac{1}{5}\sup_{c>0}\frac{c^{\frac{11}{10}}}{(c^2+1)^{\frac{1}{5}}} > 1.\end{aligned}$$

*Then all conditions of Theorem 3.1 are met, yielding that Problem (3.7) has at least one positive solution.*

### 3.2 Two positive solutions

In this section, we suppose further that the nonlinear function $\phi$ is such that the inverse $\phi^{-1}$ is super-multiplicative, that is:

$$\phi^{-1}(xy) \geq \phi^{-1}(x)\phi^{-1}(y),\ \forall\, x,y > 0.$$

**Remark 3.1. (a)** *If $\phi$ is sub-multiplicative, say*

$$\forall\, x,y \in \mathbb{R}^+, \quad \phi(xy) \leq \phi(x)\phi(y), \tag{3.8}$$

*then $\phi^{-1}$ is super-multiplicative.*
**(b)** *The $p-$Laplacian operator is super-multiplicative and sub-multiplicative, hence a multiplicative mapping.*

Consider the additional hypothesis:

($\mathcal{H}_4$) there exist $m_1 \in C(I,I)$ and $p_1 \in C(I,I)$ such that

$$F(t,x) \geq m_1(t)p_1(x), \quad \forall\, t > 0,\ \forall\, x > 0, \tag{3.9}$$

with $\lim\limits_{x\to+\infty}\frac{p_1(x)}{\phi(x)} = +\infty$ and $\int_0^{+\infty}q(\tau)m_1(\tau)d\tau < +\infty$.

Then, we have

**Theorem 3.2.** *Under Assumptions* $(\mathcal{H}_1)-(\mathcal{H}_4)$, *Problem (1.1) has at least two positive solutions.*

*Proof.* Choosing the same $R$ as in the proof of Theorem 3.1, we get

$$i(A_n, \Omega_1 \cap \mathcal{P}, \mathcal{P}) = 1, \quad \text{for all} \;\; n \in \{n_0, n_0+1, \ldots\} \tag{3.10}$$

and there exists $x_0$ solution of Problem (1.1) in $\Omega_1 = \{x \in E : \|x\| < R\}$.

Let $0 < a < b-1 < b < +\infty$ and $N = 1 + \frac{\phi(\frac{1}{c^2})}{\int_{b-1}^{b} q(s) m_1(s)\, ds}$ where $c = \min\limits_{t\in[a,b]} \widetilde{\rho}(t)$. By $(\mathcal{H}_4)$, there exists an $R' > R$ such that

$$p_1(x) > N\phi(x), \quad \forall\, x \geq R'.$$

Define

$$\Omega_2 = \left\{ x \in E : \; \|x\| < \frac{R'}{c} \right\}.$$

Without loss of generality, we may assume $R' > \max\{1, R\}$ and show that $A_n x \not\leq x$ for all $x \in \partial\Omega_2 \cap \mathcal{P}$ and $n \in \{1, 2, \ldots\}$. Suppose on the contrary that there exist an $n \in \{1, 2, \ldots\}$ and $x_0 \in \partial\Omega_2 \cap \mathcal{P}$ such that $A_n x_0 \leq x_0$. Since $x_0 \in \mathcal{P}$, we have $\frac{x_0(t)}{1+t} \geq \widetilde{\rho}(t)\|x_0\| \geq \min_{t\in[a,b]} \widetilde{\rho}(t)\frac{R'}{c} \geq R'$, $\forall\, t \in [a,b]$. Then for any $t \in [a, b-1]$, we have the lower bounds:

$$\begin{aligned}
\tfrac{x_0(t)}{1+t} &\geq \tfrac{A_n x_0(t)}{1+t} = \tfrac{\int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau) F(\tau, \frac{x_0(\tau)}{1+\tau}) d\tau\right) ds}{1+t} \\
&\geq \tfrac{\int_0^t \phi^{-1}\left(\int_t^{+\infty} q(\tau) F(\tau, \frac{x_0(\tau)}{1+\tau}) d\tau\right) ds}{1+t} \\
&\geq \tfrac{t}{1+t}\phi^{-1}\left(\int_{b-1}^{b} q(\tau) m_1(\tau) p_1\left(\tfrac{x_0(\tau)}{1+\tau}\right) d\tau\right) \\
&> \tfrac{t}{1+t}\phi^{-1}\left(\int_{b-1}^{b} q(\tau) m_1(\tau) N\phi\left(\tfrac{x_0(\tau)}{1+\tau}\right) d\tau\right) \\
&\geq \tfrac{t}{1+t}\phi^{-1}\left(\phi(R') N \int_{b-1}^{b} q(\tau) m_1(\tau) d\tau\right) \\
&\geq \widetilde{\rho}(t)\phi^{-1}\left(\phi(R')\right)\phi^{-1}(N \int_{b-1}^{b} q(\tau) m_1(\tau) d\tau\right) \\
&\geq cR'\phi^{-1}\left(N \int_{b-1}^{b} q(\tau) m_1(\tau) d\tau\right) \\
&> \tfrac{R'}{c},
\end{aligned}$$

contradicting $\|x_0\| = \frac{R'}{c}$. Finally, Lemma 2.2 yields

$$i(A_n, \Omega_2 \cap \mathcal{P}, \mathcal{P}) = 0, \quad \forall\, n \in \mathbb{N}^* \tag{3.11}$$

while (3.10) and (3.11) imply that

$$i(A_n, (\Omega_2 \setminus \overline{\Omega}_1) \cap \mathcal{P}, \mathcal{P}) = -1, \quad \forall\, n \geq n_0. \tag{3.12}$$

This shows that $A_n$ has another fixed point $y_n \in (\Omega_2 \setminus \overline{\Omega}_1) \cap \mathcal{P}$, $\forall\, n \geq n_0$. Consider the sequence $\{y_n\}_{n \geq n_0}$. Then $y_n(t) \geq \rho(t)R$, $\forall\, t \in \mathbb{R}^+$ and $\|y_n\| < \frac{R'}{c}$, $\forall\, n \geq n_0$. Arguing as above, we can show that $\{y_n\}_{n\geq n_0}$ has a convergent subsequence $\{y_{n_j}\}_{j\geq 1}$ with $\lim\limits_{j\to+\infty} y_{n_j} = y_0$ and $y_0$ is a solution of Problem (1.1). Moreover $R < \|y_0\| < \frac{R'}{c}$. Hence $x_0$ and $y_0$ are two distinct nontrivial positive solutions of Problem (1.1). $\square$

**Example 3.2.** *Consider the singular boundary value problem*

$$\begin{cases} (a(x'(t))^{\frac{3}{5}})' + e^{-t}\frac{m(t)(x^2+(1+t)^2)}{(1+t)^{\frac{3}{2}}\sqrt{x}} = 0, \quad t > 0 \\ x(0) = 0, \ \lim\limits_{t\to+\infty} x'(t) = 0, \end{cases} \tag{3.13}$$

*where $m$ is as in Example 3.1, $\phi(t) = at^{\frac{3}{5}}$ and $a > 1$ is a large parameter. Then $\phi$ is continuous, increasing, $\phi(0) = 0$ and for all $x, y > 0$ we have*

$$\phi^{-1}(xy) \geq \phi^{-1}(x)\phi^{-1}(y).$$

*Moreover $F(t,x) = \frac{m(t)(x^2+1)}{\sqrt{x}}$. Let $m_1(t) = m(t)$, $h(x) = \frac{1}{x}$ and $p_1(x) = p(x) = \frac{x^2+1}{\sqrt{x}}$; then it is easy to show ($\mathcal{H}_1$) and ($\mathcal{H}_2$).*

($\mathcal{H}_3$)

$$\begin{aligned} \sup_{c>0} \frac{c}{\int_0^{+\infty} \phi^{-1}(\int_s^{+\infty} q(\tau)m(\tau)h(c\widetilde{\rho}(\tau))\frac{p(c)}{h(c)}d\tau)ds} &= \sup_{c>0} \frac{c}{\frac{3}{5}(\frac{p(c)}{a})^{\frac{5}{3}}} \\ &= \frac{5}{3}a^{\frac{5}{3}} \sup_{c>0} \frac{cc^{\frac{5}{6}}}{(c^2+1)^{\frac{5}{3}}}. \end{aligned}$$

*If we choose $a$ large enough, say $a > \max\{1, (\sup\limits_{c>0} \frac{cc^{\frac{5}{6}}}{(c^2+1)^{\frac{5}{3}}})^{-1}\}$, then condition ($\mathcal{H}_3$) hold.*

($\mathcal{H}_4$) *It is clear that*

$$F(t,x) \geq m_1(t)p_1(x), \ \forall\, t > 0, \ \forall\, x > 0.$$

*and* $\lim\limits_{x\to+\infty} \frac{p_1(x)}{\phi(x)} = \lim\limits_{x\to+\infty} \frac{x^2+1}{ax^{\frac{3}{5}}\sqrt{x}} = \lim\limits_{x\to+\infty} \frac{x^2+1}{ax^{\frac{11}{10}}} = +\infty.$

*Then all conditions of Theorem 3.2 hold which implies that Problem (3.13) has at least two positive solutions.*

# 4 Upper and Lower solutions

## 4.1 Regular Problem

For some real positive number $k_1$, consider the regular boundary value problem

$$\begin{cases} -(\phi(x'))'(t) = q(t)f(t,x(t)), & t>0, \\ x(0)=k_1, & \lim_{t\to+\infty} x'(t)=0. \end{cases} \tag{4.1}$$

**Definition 4.1.** *A function $\alpha \in C(\mathbb{R}^+, I) \cap C^1(I,\mathbb{R})$ is called lower solution of (4.1) if $\phi \circ \alpha' \in C^1(I,\mathbb{R})$ and satisfies*

$$\begin{cases} -(\phi(\alpha'(t)))' \leq q(t)f(t,\alpha(t)), & t>0 \\ \alpha(0)\leq k_1, & \lim_{t\to+\infty} \alpha'(t)\leq 0. \end{cases}$$

*A function $\beta \in C(\mathbb{R}^+, I) \cap C^1(I,\mathbb{R})$ is called upper solution of (4.1) if $\phi\circ\beta' \in C^1(I,\mathbb{R})$ and satisfies*

$$\begin{cases} -(\phi(\beta'(t)))' \geq q(t)f(t,\beta(t)), & t>0 \\ \beta(0)\geq k_1, & \lim_{t\to+\infty} \beta'(t)\geq 0. \end{cases}$$

If there exist two functions $\beta$ and $\alpha$ such that $\alpha(t) \leq \beta(t)$ for all $t \in \mathbb{R}^+$, then we can define the closed set

$$D_\alpha^\beta(t) = \{x \in \mathbb{R} :\ \alpha(t) \leq x \leq \beta(t)\},\ t \geq 0.$$

**Theorem 4.1.** *Assume that $\alpha, \beta$ are lower and upper solutions of Problem (4.1) respectively with $\alpha(t) \leq \beta(t)$ for all $t \in \mathbb{R}^+$. Furthermore, suppose that there exists some $\delta \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that*

$$\sup_{x\in D_\alpha^\beta(t)} |f(t,x)| \leq \delta(t),\ \ \forall\, t \in I$$

*and*

$$\int_0^{+\infty} q(\tau)\delta(\tau)d\tau < +\infty, \quad \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} q(\tau)\delta(\tau)d\tau\right) ds < +\infty. \tag{4.2}$$

*Then Problem (4.1) has at least one solution $x^* \in E$ with*

$$\alpha(t) \leq x^*(t) \leq \beta(t),\ \ t \in \mathbb{R}^+.$$

*Proof.* Consider the truncation function

$$f^*(t,x) = \begin{cases} f(t,\alpha(t)), & x<\alpha(t) \\ f(t,x), & \alpha(t)\leq x\leq \beta(t) \\ f(t,\beta(t)), & x>\beta(t) \end{cases}$$

and the modified problem

$$\begin{cases} -(\phi(x'))'(t) = q(t)f^*(t,x(t)), & t>0, \\ x(0)=k_1, & \lim_{t\to+\infty} x'(t)=0. \end{cases} \tag{4.3}$$

Step 1. To show that Problem (4.3) has at least one solution $x$, let the operator defined on $E$ by

$$Ax(t) = k_1 + \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)f^*(\tau, x(\tau))d\tau\right) ds.$$

**(a)** $A(E) \subseteq E$. For $x \in E$ and $t \in \mathbb{R}^+$, we have

$$\lim_{t\to+\infty} \frac{Ax(t)}{1+t} = \lim_{t\to+\infty} \frac{k_1}{1+t} + \frac{\int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau)f^*(\tau, x(\tau))d\tau)ds}{1+t} = 0,$$

then $A(E) \subseteq E$.

**(b)** $A$ is continuous. Let some sequence $\{x_n\}_{n\geq 1} \subseteq E$ be such that $\lim\limits_{n\to+\infty} x_n = x_0 \in E$. By the continuity of $f^*$ and the Lebesgue dominated convergence theorem, we have for all $s \in \mathbb{R}^+$,

$$\begin{aligned}&\left|\int_s^{+\infty} q(\tau)f^*(\tau, x_n(\tau))d\tau - \int_s^{+\infty} q(\tau)f^*(\tau, x_0(\tau))d\tau\right| \\ &\leq \int_s^{+\infty} q(\tau)|f^*(\tau, x_n(\tau)) - f^*(\tau, x_0(\tau))|d\tau \longrightarrow 0, \quad \text{as } n \to +\infty\end{aligned}$$

i.e.

$$\int_s^{+\infty} q(\tau)f^*(\tau, x_n(\tau))d\tau \to \int_s^{+\infty} q(\tau)f^*(\tau, x_0(\tau))d\tau, \quad \text{as } n \to +\infty.$$

Moreover, the continuity of $\phi^{-1}$ implies that

$$\phi^{-1}\left(\int_s^{+\infty} q(\tau)f^*(\tau, x_n(\tau))d\tau\right) \to \phi^{-1}\left(\int_s^{+\infty} q(\tau)f^*(\tau, x_0(\tau))d\tau\right),$$

as $n \to +\infty$. Thus

$$\begin{aligned}\|Ax_n - Ax_0\| &= \sup_{t\in\mathbb{R}^+} \frac{|Ax_n(t)-Ax_0(t)|}{1+t} \\ &= \sup_{t\in\mathbb{R}^+} \frac{\left|\int_0^t(\phi^{-1}(\int_s^{+\infty} q(\tau)f^*(\tau,x_n(\tau))d\tau))ds - \int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau)f^*(\tau,x_0(\tau))d\tau)ds\right|}{1+t} \\ &\leq \sup_{t\in\mathbb{R}^+} \frac{\int_0^t \left|\phi^{-1}(\int_s^{+\infty} q(\tau)f^*(\tau,x_n(\tau))) - \phi^{-1}(\int_s^{+\infty} q(\tau)f^*(\tau,x_0(\tau))d\tau)\right|ds}{1+t} \\ &\to 0, \quad \text{as } n \to +\infty,\end{aligned}$$

and our claim follows.

**(c)** $A(E)$ is relatively compact. Indeed

- $A(E)$ is uniformly bounded. For $x \in E$, we have

$$\begin{aligned}\|Ax\| &= \sup_{t\in\mathbb{R}^+} \frac{|Ax(t)|}{1+t} \\ &\leq \sup_{t\in\mathbb{R}^+} \frac{k_1}{1+t} + \frac{\int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau)f^*(\tau,x(\tau))d\tau)ds}{1+t} \\ &\leq \sup_{t\in\mathbb{R}^+} \frac{k_1}{1+t} + \frac{\int_0^t(\phi^{-1}(\int_s^{+\infty} q(\tau)\delta(\tau)d\tau))ds}{1+t} < \infty.\end{aligned}$$

- $\frac{A(E)}{1+t}$ is almost equicontinuous. For a given $T > 0$, $x \in E$, and $t, t' \in [0, T]$ $(t > t')$, we have

$$
\begin{aligned}
& \left|\tfrac{Ax(t)}{1+t} - \tfrac{Ax(t')}{1+t'}\right| \\
\le\ & k_1 \left|\tfrac{1}{1+t} - \tfrac{1}{1+t'}\right| \\
& + \left|\tfrac{\int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau)f^*(\tau,x(\tau))d\tau)ds}{1+t} - \tfrac{\int_0^{t'} \phi^{-1}(\int_s^{+\infty} q(\tau)f^*(\tau,x(\tau))d\tau)ds}{1+t'}\right| \\
\le\ & k_1 \left|\tfrac{1}{1+t} - \tfrac{1}{1+t'}\right| + \left|\tfrac{1}{1+t} - \tfrac{1}{1+t'}\right| \int_0^{+\infty} \phi^{-1}(\int_s^{+\infty} q(\tau)f^*(\tau, x(\tau))d\tau)ds \\
& + \left|\tfrac{\int_{t'}^{+\infty} \phi^{-1}(\int_s^{+\infty} q(\tau)f^*(\tau,x(\tau))d\tau)ds}{1+t'} - \tfrac{\int_t^{+\infty} \phi^{-1}(\int_s^{+\infty} q(\tau)f^*(\tau,x(\tau))d\tau)ds}{1+t}\right| \\
\le\ & k_1 \left|\tfrac{1}{1+t} - \tfrac{1}{1+t'}\right| + 2\left|\tfrac{1}{1+t} - \tfrac{1}{1+t'}\right| \int_0^{+\infty} \phi^{-1}(\int_s^{+\infty} q(\tau)f^*(\tau, x(\tau))d\tau)ds \\
& + \tfrac{1}{1+t'} \int_{t'}^{t} \phi^{-1}(\int_s^{+\infty} q(\tau)f^*(\tau, x(\tau))d\tau)ds \\
\le\ & k_1 \left|\tfrac{1}{1+t} - \tfrac{1}{1+t'}\right| + 2\left|\tfrac{1}{1+t} - \tfrac{1}{1+t'}\right| \int_0^{+\infty} \phi^{-1}(\int_s^{+\infty} q(\tau)\delta(\tau)d\tau)ds \\
& + \tfrac{1}{1+t'} \int_{t'}^{t} \phi^{-1}(\int_s^{+\infty} q(\tau)\delta(\tau)d\tau)ds.
\end{aligned}
$$

Then, for any $\varepsilon > 0$ and $T > 0$, there exists $\delta > 0$ such that $\left|\frac{Ax(t)}{1+t} - \frac{Ax(t')}{1+t'}\right| < \varepsilon$ for all $t, t' \in [0, T]$ with $|t - t'| < \delta$. Hence $\{\frac{A(E)}{1+t}\}$ are almost equicontiuous.

- $\frac{A(E)}{1+t}$ is equiconvergent at $+\infty$. Since

$$
\lim_{t\to+\infty} \frac{Ax(t)}{1+t} = \lim_{t\to+\infty} \frac{k_1}{1+t} + \frac{\int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau)f^*(\tau, x(\tau))d\tau)ds}{1+t} = 0,
$$

then

$$
\begin{aligned}
\sup_{x\in E} \left|\tfrac{Ax(t)}{1+t} - \lim_{t\to+\infty} \tfrac{Ax(t)}{1+t}\right| &= \sup_{x\in E} \left|\tfrac{k_1}{1+t} + \tfrac{\int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau)f^*(\tau,x(\tau))d\tau)ds}{1+t}\right| \\
&\le \sup_{x\in E} \tfrac{k_1}{1+t} + \tfrac{\int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau)\delta(\tau)d\tau)ds}{1+t}
\end{aligned}
$$

which implies that $\lim\limits_{t\to+\infty} \sup\limits_{x\in E} \left|\frac{Ax(t)}{1+t} - \lim\limits_{t\to+\infty} \frac{Ax(t)}{1+t}\right| = 0$.

By Lemma 2.4, $A(E)$ is relatively compact. Finally by the Schauder fixed point theorem, $A$ has at least one fixed point $x \in E$, which is a solution of Problem (4.3).

Step 2. We show that $\alpha(t) \leq x(t) \leq \beta(t), \forall\, t \in \mathbb{R}^+$, in which case $x$ is also a solution of (4.1). On the contrary, suppose that some point $t^* \in \mathbb{R}^+$ exists and satisfies $x(t^*) > \beta(t^*)$ and let $z(t) = x(t) - \beta(t)$. Define

$$t_1 = \inf\{t < t^* : x(t) > \beta(t), \quad \forall\, t \in [t, t^*]\},$$

$$t_1' = \inf\{t > t^* : x(t) > \beta(t), \quad \forall\, t \in [t^*, t]\}.$$

Then $z(t) > 0$ on $(t_1, t_1')$, $z(t_1) = 0$ and for all $t \in [t_1, t_1')$, we have

$$\begin{array}{rcl}(\phi(x'(t))' - (\phi(\beta'(t))' & \geq & -q(t)f^*(t, x(t)) + q(t)f^*(t, \beta(t)) \\ & = & q(t)[f(t, \beta(t)) - f(t, \beta(t))] = 0.\end{array}$$

Hence $\phi(x'(t)) - \phi(\beta'(t))$ is nondecreasing on $[t_1, t_1')$.

If $t_1' < \infty$, then $z(t_1) = z(t_1') = 0$ and there exists $t_0 \in [t_1, t_1']$ such that $z(t_0) = \max\limits_{t \in [t_1, t_1']} z(t) > 0$. Hence

$$\phi(x'(t)) - \phi(\beta'(t)) \leq \phi(x'(t_0)) - \phi(\beta'(t_0)) = 0, \ \forall\, t \in [t_1, t_0].$$

Then $x'(t) \leq \beta'(t)$ on $[t_1, t_0]$, i.e. $z$ is nonincreasing on $[t_1, t_0]$; therefore $0 = z(t_1) \geq z(t_0)$, which is a contradiction.

If $t_1' = \infty$, then

$$\phi(x'(t)) - \phi(\beta'(t)) \leq \phi(x'(\infty)) - \phi(\beta'(\infty)) \leq 0, \ \forall\, t \in [t_1, +\infty).$$

Then $x'(t) \leq \beta'(t)$ on $[t_1, +\infty)$, i.e. $z$ is nonincreasing on $[t_1, \infty)$; therefore $z(t) \leq z(t_1) = 0, \quad \forall\, t \in [t_1, +\infty)$, which is a contradiction.

In the same way, we can prove that $\alpha(t) \leq x^*(t)$. The proof is complete. □

### 4.2 The singular problem

Using Theorem 4.1, our main existence result in this section is:

**Theorem 4.2.** *Further to Assumptions* $(\mathcal{H}_1), (\mathcal{H}_2)$, *assume that*

$(\mathcal{H}_5)$ *There exist a constant* $M > 0$ *and a function* $k \in C(I, I)$ *such that*

$$f(t, x) \leq k(t), \ \forall\, (t, x) \in I \times [M, +\infty) \tag{4.4}$$

*with*

$$\int_0^{+\infty} q(\tau)k(\tau)\, d\tau < +\infty \text{ and } \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} q(\tau)k(\tau) d\tau\right) ds < +\infty. \tag{4.5}$$

*Then Problem* (1.1) *has at least one positive solution.*

*Proof.* Choose a decreasing sequence $\{\varepsilon_n\}_{n\geq 1}$ with $\lim\limits_{n\to+\infty}\varepsilon_n = 0$ and $\varepsilon_1 < M$, then consider the sequence of boundary value problems

$$\begin{cases} -(\phi(x'))'(t) = q(t)f(t,x(t)), & t > 0, \\ x(0) = \varepsilon_n, & \lim\limits_{t\to+\infty} x'(t) = 0. \end{cases} \tag{4.6}$$

Step 1. For each $n \geq 1$, (4.6) has at least one solution $x_n$.

(a) Let $\alpha_n(t) = \varepsilon_n$, $t \geq 0$. Then

$$\begin{cases} -(\phi(\alpha_n'(t)))' = -\phi(0) = 0 \leq q(t)f(t,\alpha_n(t)), & t > 0 \\ \alpha(0) \leq \varepsilon_n, \quad \lim\limits_{t\to+\infty}\alpha_n'(t) \leq 0. \end{cases}$$

Let $\beta$ be a solution of the boundary value problem

$$\begin{cases} \phi(x'(t))' + q(t)k(t) = 0, & t > 0 \\ x(0) = M, & \lim\limits_{t\to+\infty} x'(t) = 0, \end{cases}$$

that is

$$\beta(t) = M + \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)k(\tau)d\tau\right) ds;$$

then $\beta(t) \geq M, \forall\, t \in \mathbb{R}^+$ which implies that for any $t > 0$, $f(t,\beta(t)) \leq k(t)$. Hence

$$\begin{cases} -(\phi(\beta'(t)))' = q(t)k(t) \geq q(t)f(t,\beta(t)), & t > 0 \\ \beta(0) \geq \varepsilon_n, \quad \lim\limits_{t\to+\infty}\beta'(t) \geq 0. \end{cases}$$

For any $n \geq 1$, $\alpha_n$ and $\beta$ are lower and upper solution of (4.6) respectively; moreover

$$\alpha_n(t) \leq \beta(t), \quad \forall\, t > 0.$$

(b) For all $t \in \mathbb{R}^+$, by the monotonicity of $h$ and $\frac{p}{h}$, the following estimates hold

$$\begin{aligned} \sup_{x\in D^{\beta}_{\alpha_n}(t)} f(t,x) &= \sup_{\alpha_n\leq x\leq\beta} F\left(t, \tfrac{x}{1+t}\right) \\ &\leq \sup_{\alpha_n\leq x\leq\beta} m(t)p\left(\tfrac{x}{1+t}\right) \\ &\leq \sup_{\alpha_n\leq x\leq\beta} m(t)h\left(\tfrac{x}{1+t}\right)\frac{p\left(\frac{x}{1+t}\right)}{h\left(\frac{x}{1+t}\right)} \\ &\leq m(t)h\left(\varepsilon_n\widetilde{\rho}(t)\right)\frac{p(\|\beta\|)}{h(\|\beta\|)} := \delta(t). \end{aligned}$$

Using $(\mathcal{H}_1)$, we have

$$\int_0^{+\infty} q(\tau)\delta(\tau)d\tau < +\infty, \quad \int_0^{+\infty}\phi^{-1}\left(\int_s^{+\infty} q(\tau)\delta(\tau)d\tau\right) ds < +\infty.$$

Then all conditions of Theorem 4.1 are satisfied. Hence for any $n \geq 1$, Problem (4.6) has at least one positive solution $x_n \in E$ with

$$\alpha_n(t) \leq x_n(t) \leq \beta(t), \ \forall\, t \in \mathbb{R}^+.$$

Step 2. The sequence $\{x_n\}_{n\geq 1}$ is relatively compact in $E$.

**(a)** The sequence $\{x_n\}_{n\geq 1}$ is bounded in $E$. By Step 1, we have

$$\|x_n\| = \sup_{t\in\mathbb{R}^+} \frac{x_n(t)}{1+t} \leq \sup_{t\in\mathbb{R}^+} \frac{\beta(t)}{1+t} = \|\beta\|,\ \forall\, n \geq 1.$$

From condition $(\mathcal{H}_2)$, there exists $\psi_{\|\beta\|} \in C(\mathbb{R}^+, (0, +\infty))$ such that

$$|F(t,x)| \geq \psi_{\|\beta\|}(t), \quad \text{for } t \in I \text{ and } 0 < x \leq \|\beta\| \tag{4.7}$$

with
$$\int_0^{+\infty} q(\tau)\psi_{\|\beta\|}(\tau)d\tau < +\infty.$$

Let
$$c^{**} = \phi^{-1}\left(\int_1^{+\infty} q(\tau)\psi_{\|\beta\|}(\tau)d\tau\right).$$

Then we have the discussion:

- If $t \in [0,1]$, then

$$\begin{array}{rcl} x_n(t) & \geq & \int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau)f(\tau, x_n(\tau))d\tau)ds \\ & \geq & \int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau)F(\tau, \frac{x_n(\tau)}{1+\tau})d\tau)ds \\ & \geq & \int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau)\psi_{\|\beta\|}(\tau)d\tau)ds \\ & \geq & t\phi^{-1}(\int_t^{+\infty} q(\tau)\psi_{\|\beta\|}(\tau)d\tau) \\ & \geq & t\phi^{-1}(\int_1^{+\infty} q(\tau)\psi_{\|\beta\|}(\tau)d\tau) \geq \rho(t)c^{**}. \end{array}$$

- If $t \in (1, +\infty)$, then

$$\begin{array}{rcl} x_n(t) & \geq & \int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau)\psi_{\|\beta\|}(\tau)d\tau)ds \\ & \geq & \int_0^1 \phi^{-1}(\int_s^{+\infty} q(\tau)\psi_{\|\beta\|}(\tau)d\tau)ds \\ & \geq & \phi^{-1}(\int_1^{+\infty} q(\tau)\psi_{\|\beta\|}(\tau)d\tau)ds \\ & \geq & c^{**} \geq \frac{1}{t}c^{**} = \rho(t)c^{**}. \end{array}$$

Then, for any $t \in \mathbb{R}^+$, and $n \geq 1$, $x_n(t) \geq \rho(t)c^{**}$. Using $(\mathcal{H}_1)$ and the monotonicity of $h$ and $\frac{p}{h}$, we obtain the upper bounds

$$\begin{array}{rcl} q(s)f(s, x_n(s)) & = & q(s)F(s, \frac{x_n(s)}{1+s}) \\ & \leq & q(s)m(s)h(\frac{x_n(s)}{1+s})\frac{p(\frac{x_n(s)}{1+s})}{h(\frac{x_n(s)}{1+s})} \\ & \leq & q(s)m(s)h(c^{**}\widetilde{\rho}(s))\frac{p(\|\beta\|)}{h(\|\beta\|)}. \end{array}$$

**(b)** The sequence $\{x_n\}_{n\geq 1}$ is almost equicontinuous. For any $T > 0$ and $t, t' \in [0, T]$ $(t > t')$, we have the estimates

$$\begin{aligned}
& |\tfrac{x_n(t)}{1+t} - \tfrac{x_n(t')}{1+t'}| \\
\leq\ & \varepsilon_n \left|\tfrac{1}{1+t} - \tfrac{1}{1+t'}\right| \\
& + \left| \tfrac{\int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau)f(\tau,x_n(\tau))d\tau)ds}{1+t} - \tfrac{\int_0^{t'} \phi^{-1}(\int_s^{+\infty} q(\tau)f(\tau,x_n(\tau))d\tau)ds}{1+t'} \right| \\
\leq\ & \varepsilon_n \left|\tfrac{1}{1+t} - \tfrac{1}{1+t'}\right| \\
& +2|\tfrac{1}{1+t} - \tfrac{1}{1+t'}| \int_0^{+\infty} \phi^{-1}(\int_s^{+\infty} q(\tau)f(\tau, x_n(\tau))d\tau)ds \\
& +\tfrac{1}{1+t'} \int_{t'}^t \phi^{-1}(\int_s^{+\infty} q(\tau)f(\tau, x_n(\tau))d\tau)ds \\
\leq\ & M \left|\tfrac{1}{1+t} - \tfrac{1}{1+t'}\right| \\
& +2|\tfrac{1}{1+t} - \tfrac{1}{1+t'}| \int_0^{+\infty} \phi^{-1}(\int_s^{+\infty} q(\tau)m(\tau)h(c^{**}\widetilde{\rho}(s))\tfrac{p(\|\beta\|)}{h(\|\beta\|)}d\tau)ds \\
& +\tfrac{1}{1+t'} \int_{t'}^t \phi^{-1}(\int_s^{+\infty} q(\tau)m(\tau)h(c^{**}\widetilde{\rho}(s))\tfrac{p(\|\beta\|)}{h(\|\beta\|)}d\tau)ds.
\end{aligned}$$

Then, for any $\varepsilon > 0$ and $T > 0$, there exists $\delta > 0$ such that $|\frac{x_n(t)}{1+t} - \frac{x_n(t')}{1+t'})| < \varepsilon$ for all $t, t' \in [0, T]$ with $|t - t'| < \delta$.

**(c)** $\{x_n\}$ is equiconvergent at $+\infty$:

$$\begin{aligned}
\sup_{n\geq 1} |\tfrac{x_n(t)}{1+t} - 0| =\ & \sup_{n\geq 1} \tfrac{\varepsilon_n + \int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau)f(\tau,x_n(\tau))d\tau)ds}{1+t} \\
\leq\ & \tfrac{\varepsilon_n + \int_0^{+\infty} \phi^{-1}(\int_s^{+\infty} q(\tau)m(\tau)h(c^{**}\widetilde{\rho}(\tau))\frac{p(\|\beta\|)}{h(\|\beta\|)}d\tau)ds}{1+t} \\
& \to 0, \text{ as } t \to +\infty.
\end{aligned}$$

Consequently $\{x_n\}$ is relatively compact in $E$ by Lemma 2.4. Therefore $\{x_n\}_{n\geq 1}$ has a subsequence $\{x_{n_k}\}_{k\geq 1}$ converging to some limit $x_0$, as $k \to +\infty$. The continuity of $f$, $\phi^{-1}$ and the Lebesgue dominated convergence theorem, imply that, for every $t \in \mathbb{R}^+$,

$$\int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)f(\tau, x_{n_k}(\tau))d\tau\right) ds \to \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)f(\tau, x_0(\tau))d\tau\right) ds,$$

as $\to +\infty$. Then

$$\begin{aligned}
x_0(t) =\ & \lim_{k\to+\infty} x_{n_k}(t) \\
=\ & \lim_{k\to+\infty} \left(\varepsilon_{n_k} + \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)f(\tau, x_{n_k}(\tau))d\tau\right) ds\right) \\
=\ & \int_0^t \phi^{-1}(\int_s^{+\infty} q(\tau)f(\tau, x_0(\tau))d\tau)ds.
\end{aligned}$$

Hence $x_0$ is a positive nontrivial solution of Problem (1.1). $\square$

**Example 4.1.** *Consider the singular boundary value problem*

$$\begin{cases} ((x'(t))^3)' + e^{-t}\frac{m(t)}{\sqrt{x}} = 0, \\ x(0) = 0, \ \lim_{t\to+\infty} x'(t) = 0, \end{cases} \tag{4.8}$$

*where*

$$m(t) = \begin{cases} \frac{t}{\sqrt{1+t}} & t \in (0,1] \\ \frac{1}{t\sqrt{1+t}} & t \in (1,+\infty). \end{cases}$$

*Here* $f(t,x) = \frac{m(t)}{\sqrt{x}}$, $\phi(t) = t^3$ *and* $q(t) = e^{-t}$. *Then* $\phi$ *is continuous, increasing and* $\phi(0) = 0$. *Therefore for each* $M > 0$, *there exists* $k(t) = \frac{m(t)}{\sqrt{M}} \in C(I,I)$ *such that* $f(t,x) \le k(t)$, $\forall\, t > 0, \forall\, x \ge M$ *with*

$$\int_0^{+\infty} q(\tau)k(\tau)d\tau \le \frac{1}{\sqrt{M}}\int_0^{+\infty} e^{-\tau}d\tau = \frac{1}{\sqrt{M}} < +\infty$$

*and*

$$\begin{aligned} \int_0^{+\infty}\phi^{-1}\left(\int_s^{+\infty} q(\tau)k(\tau)d\tau\right)ds &\le \int_0^{+\infty}\phi^{-1}\left(\frac{1}{\sqrt{M}}e^{-s}\right)ds \\ &= \left(\frac{1}{\sqrt{M}}\right)^{\frac{1}{3}}\int_0^{+\infty} e^{\frac{-s}{3}}ds = 3\left(\frac{1}{\sqrt{M}}\right)^{\frac{1}{3}} < +\infty. \end{aligned}$$

*Moreover* $F(t,x) = f(t,(1+t)x) = \frac{m(t)}{\sqrt{x}}$, *for* $(t,x) \in I^2$.

($\mathcal{H}_1$) *Let* $p(x) = \frac{x^2+1}{\sqrt{x}}$, *and* $h(x) = \frac{1}{x}\cdot$ *Then* $h$ *is a decreasing function,* $\frac{g}{h}$ *is increasing and* $F(t,x) \le m(t)p(x)$, $\forall\, t > 0$, $\forall\, x > 0$. *In addition, for any* $c, c' > 0$, *we have*

$$\int_0^{+\infty} q(\tau)m(\tau)h(c\widetilde{\rho}(\tau))d\tau = \frac{1}{c}\int_0^{+\infty} e^{-\tau}d\tau = \frac{1}{c} < +\infty$$

*and*

$$\begin{aligned} \int_0^{+\infty}\phi^{-1}\left(\int_s^{+\infty} q(\tau)m(\tau)h(c\widetilde{\rho}(\tau))\frac{g(c')}{h(c')}d\tau\right)ds &= \int_0^{+\infty}\phi^{-1}\left(\frac{p(c')}{ch(c')}e^{-s}\right)ds \\ &= \left(\frac{p(c')}{ch(c')}\right)^{\frac{1}{3}}\int_0^{+\infty} e^{\frac{-s}{3}})ds \\ &= 3\left(\frac{p(c')}{ch(c')}\right)^{\frac{1}{3}} < +\infty. \end{aligned}$$

($\mathcal{H}_2$) *For any* $c > 0$, *there exists* $\psi_c(t) = \frac{m(t)}{\sqrt{c}}$ *such that*

$$F(t,x) \ge \psi_c(t), \ \forall\, t \in I, \ \forall\, x \in (0,c].$$

*As a consequence, all conditions of Theorem 4.2 hold and then Problem (4.8) has at least one positive solution.*

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