



# Principal solution in Weyl–Titchmarsh theory for second order Sturm–Liouville equation on time scales

*Dedicated to the memory of Professor Ondřej Došlý*

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**Abstract.** A connection between the oscillation theory and the Weyl–Titchmarsh theory for the second order Sturm–Liouville equation on time scales is established by using the principal solution. In particular, it is shown that the Weyl solution coincides with the principal solution in the limit point case, and consequently the square integrability of the Weyl solution is obtained. Moreover, both limit point and oscillatory criteria are derived in the case of real-valued coefficients, while a generalization of the invariance of the limit circle case is proven for complex-valued coefficients. Several of these results are new even in the discrete time case. Finally, some illustrative examples are provided.

**Keywords:** Sturm–Liouville equation, time scale, Weyl solution, principal solution, limit point case, limit circle case, criteria.

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## 1 Introduction

In this paper we continue in the development of the Weyl–Titchmarsh theory for the second order Sturm–Liouville dynamic equation

$$-[p(t) y^\Delta(t, \lambda)]^\Delta + q(t) y^\sigma(t, \lambda) = \lambda w(t) y^\sigma(t, \lambda), \quad t \in [a, \infty)_\mathbb{T}. \quad (E_\lambda)$$

Here  $\lambda \in \mathbb{C}$  and  $[a, \infty)_\mathbb{T} := [a, \infty) \cap \mathbb{T}$ , where  $\mathbb{T}$  denotes a time scale (i.e., any nonempty closed subset of  $\mathbb{R}$ ), which is bounded from below with  $a := \min \mathbb{T}$  and unbounded from above. The coefficients  $p(\cdot)$ ,  $q(\cdot)$ , and  $w(\cdot)$  are (if not specified otherwise) real-valued piecewise rd-continuous functions on  $[a, \infty)_\mathbb{T}$  (i.e., they belong to  $C_{\text{prd}}$ ) and satisfy

$$(i) \quad \inf_{t \in [a, b]_\mathbb{T}} |p(t)| > 0 \quad \text{for all } b \in (a, \infty)_\mathbb{T}, \quad (ii) \quad w(t) > 0 \quad \text{for all } t \in [a, \infty)_\mathbb{T}. \quad (1.1)$$

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Observe that there is no restriction on the sign of  $p(\cdot)$ . Let us emphasize that the first condition in (1.1) cannot be replaced by the weaker assumption  $p(t) \neq 0$  on  $[a, \infty)_{\mathbb{T}}$ , see [11, Remark 2.2]. We also note that  $(E_\lambda)$  includes several equations of particular interest, especially the second order Sturm–Liouville differential and difference equations.

The history of the Weyl–Titchmarsh theory goes back to the celebrated paper [23] devoted to the second order Sturm–Liouville differential equation. Its extension to equation  $(E_\lambda)$  was given by several authors e.g. in [14, 17, 22, 26, 28], see also the references therein. One of the crucial questions of this theory concerns the number of linearly independent solutions of  $(E_\lambda)$ , which are square integrable with respect to the weight  $w(\cdot)$ , i.e., such that  $\int_a^\infty w(t) |y^\sigma(t, \lambda)|^2 \Delta t < \infty$ . It can be shown that there exists at least one square integrable solution for every  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ . Moreover, the situation when all solutions of  $(E_\lambda)$  are square integrable (i.e., the limit circle case) is invariant with respect to  $\lambda \in \mathbb{C}$ . These facts give rise to the dichotomous classification of equation  $(E_\lambda)$  as being in the limit point case (i.e., at least one solution is not square integrable) or in the limit circle case for all  $\lambda \in \mathbb{C}$ , see Section 2 for more details. In the first result of this paper we derive a generalization of the latter invariance in the case of complex-valued coefficients (see Theorem 2.5).

The existence of a square integrable solution remains open only when equation  $(E_\lambda)$  is in the limit point case and  $\lambda \in \mathbb{R}$ . But for  $\lambda \in \mathbb{R}$  equation  $(E_\lambda)$  can be also classified as oscillatory or nonoscillatory and this behavior is partially invariant with respect to  $\lambda$  as a consequence of the Sturmian theory, see e.g. [10]. Moreover, the nonoscillatory case is equivalent with the existence of a solution, which is eventually smaller than any other linearly independent solution. This solution is said to be principal and we show that it plays a significant role in the present problem. In particular, we utilize the principal solution of  $(E_\lambda)$  for a development of a limit point criterion (see Theorem 3.1) and we discuss its connection with the Weyl solution and its square integrability in the limit point case (see Theorem 3.5). These results are new in the case  $\mathbb{T} = \mathbb{Z}$ , while in the case  $\mathbb{T} = \mathbb{R}$  they can be found in [5, Section 2].

The paper is organized as follows. In the next section we derive a generalization of the invariance of the limit circle case, recall several results from the Weyl–Titchmarsh theory equation  $(E_\lambda)$ , and present basic properties of the principal solution. The main results are established in Section 3.

## 2 Preliminaries

For the foundations of the time scale calculus we refer to [3]. For brevity, we write only  $y^{\sigma^2}(t, \lambda)$  instead of  $[y^\sigma(t, \lambda)]^2 = [y(\sigma(t), \lambda)]^2$ . By a *solution* of equation  $(E_\lambda)$  we mean a function  $y(\cdot, \lambda)$  defined on  $[a, \infty)_{\mathbb{T}}$  such that the functions  $y(\cdot, \lambda)$  and  $p(\cdot) y^\Delta(\cdot, \lambda)$  are piecewise rd-continuously delta-differentiable on  $[a, \infty)_{\mathbb{T}}$  and equation  $(E_\lambda)$  is satisfied for all  $t \in [a, \infty)_{\mathbb{T}}$ , see also [18, pg. 4].

In the first part of this section we consider equation  $(E_\lambda)$  with complex-valued coefficients. For simplicity we summarize the assumptions put on the coefficients of equation  $(E_\lambda)$ .

**Hypothesis 2.1.** The functions  $p(\cdot), q(\cdot), w(\cdot) \in C_{\text{prd}}$  are complex-valued and such that inequality (1.1)(i) is satisfied.

The following lemma guarantees the existence and uniqueness of the solution of any initial value problem associated with equation  $(E_\lambda)$ , see [3, Theorem 5.8]. Moreover, it shows an intimate connection between equation  $(E_\lambda)$  with real-valued coefficients and the scalar symplectic

dynamic system, i.e., the system of the form

$$z^\Delta(t, \lambda) = \mathbf{S}(t, \lambda) z(t, \lambda), \quad \mathbf{S}(t, \lambda) := \mathcal{S}(t) + \lambda \mathcal{V}(t), \quad (\mathbf{S}_\lambda)$$

where  $\mathbf{S}(\cdot, \lambda) : [a, \infty)_{\mathbb{T}} \rightarrow \mathbb{C}^{2 \times 2}$  is a piecewise rd-continuous function satisfying for all  $\lambda \in \mathbb{C}$  and all  $t \in [a, \infty)_{\mathbb{T}}$  the symplectic-type identity

$$\mathbf{S}^*(t, \lambda) \mathcal{J} + \mathcal{J} \mathbf{S}(t, \bar{\lambda}) + \mu(t) \mathbf{S}^*(t, \lambda) \mathcal{J} \mathbf{S}(t, \bar{\lambda}) = 0, \quad \mathcal{J} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (2.1)$$

see also [18, Theorem 3.4]. Here  $\mathbf{S}^*(t, \lambda) = [\mathbf{S}(t, \lambda)]^* = \overline{[\mathbf{S}(t, \lambda)]^\top}$ , i.e.,  $*$  stands for the conjugate transpose. The later fact was used e.g. in [17], where some results of the Weyl–Titchmarsh theory for equation  $(\mathbf{E}_\lambda)$  were obtained as a special case of general results for system  $(\mathbf{S}_\lambda)$  established in [18], see also [19, 20]. Finally, we note that system  $(\mathbf{S}_\lambda)$  is closely related to the linear Hamiltonian dynamic system, which leads to system  $(\mathbf{S}_\lambda)$  with polynomial dependence on  $\lambda$ , see [21]. In addition, system  $(\mathbf{S}_\lambda)$  reduces to the linear Hamiltonian differential system if  $\mathbb{T} = \mathbb{R}$ .

**Lemma 2.2.** *Let Hypothesis 2.1 be satisfied. Equation  $(\mathbf{E}_\lambda)$  is equivalent with the the first order system of the form as in  $(\mathbf{S}_\lambda)$ , where*

$$z(t, \lambda) = \begin{pmatrix} y(t, \lambda) \\ p(t) y^\Delta(t, \lambda) \end{pmatrix}, \quad \mathcal{S}(t) = \begin{pmatrix} 0 & 1/p(t) \\ q(t) & \mu(t) q(t)/p(t) \end{pmatrix},$$

$$\mathcal{V}(t) = - \begin{pmatrix} 0 & 0 \\ w(t) & \mu(t) w(t)/p(t) \end{pmatrix}.$$

The matrix-valued function  $\mathbf{S}(\cdot, \lambda)$  is regressive on  $[a, \infty)_{\mathbb{T}}$  for all  $\lambda \in \mathbb{C}$ . In addition,  $\mathbf{S}(\cdot, \lambda)$  satisfies identity (2.1) for all  $\lambda \in \mathbb{C}$  and all  $t \in [a, \infty)_{\mathbb{T}}$  if and only if the coefficients  $p(\cdot)$ ,  $q(\cdot)$ ,  $w(\cdot)$  are real-valued functions.

*Proof.* The proof follows by straightforward calculations. The regressivity is a consequence of the equality  $\det[I + \mu(t) \mathbf{S}(t, \lambda)] \equiv 1$  on  $[a, \infty)_{\mathbb{T}} \times \mathbb{C}$ .  $\square$

We denote by  $\Phi(t, \lambda)$  the fundamental matrix of systems of the form as in  $(\mathbf{S}_\lambda)$  determined by the initial value condition  $\Phi(a, \lambda) = I$ , i.e.,

$$\Phi(t, \lambda) = \begin{pmatrix} \phi_1(t, \lambda) & \phi_2(t, \lambda) \\ p(t) \phi_1^\Delta(t, \lambda) & p(t) \phi_2^\Delta(t, \lambda) \end{pmatrix},$$

where  $\phi_1(t, \lambda)$  and  $\phi_2(t, \lambda)$  are linearly independent solutions of equation  $(\mathbf{E}_\lambda)$  such that

$$\phi_1(a, \lambda) = 1, \quad \phi_1^\Delta(a, \lambda) = 0 \quad \text{and} \quad \phi_2(a, \lambda) = 0, \quad \phi_2^\Delta(a, \lambda) = \frac{1}{p(a)}.$$

The following lemma extends [25, Theorem 7.2.1] to any time scale. This result is new even in the case  $\mathbb{T} = \mathbb{Z}$ . Observe that its proof does not rely on the symplectic-type identity (2.1), which may be violated under Hypothesis 2.1, compare with the proof of [19, Theorem 6.1].

**Lemma 2.3.** *Let Hypothesis 2.1 be satisfied and  $\lambda, \nu \in \mathbb{C}$  be arbitrary. Then the matrix-valued function  $Y(t, \lambda, \nu) := \Phi^{-1}(t, \nu) \Phi(t, \lambda)$  solves the first order dynamic system*

$$Y^\Delta(t, \lambda, \nu) = (\nu - \lambda) \Omega(t, \nu) Y(t, \lambda, \nu), \quad (2.2)$$

where

$$\Omega(t, \nu) := \begin{pmatrix} -w(t) \phi_1^\sigma(t, \nu) \phi_2^\sigma(t, \nu) & -w(t) \phi_2^{\sigma^2}(t, \nu) \\ w(t) \phi_1^{\sigma^2}(t, \nu) & w(t) \phi_1^\sigma(t, \nu) \phi_2^\sigma(t, \nu) \end{pmatrix} \quad (2.3)$$

and  $\Omega(\cdot, \nu)$  is regressive on  $[a, \infty)_{\mathbb{T}}$ . In addition, if there exists  $\lambda_0 \in \mathbb{C}$  such that all solutions of  $(E_{\lambda_0})$  satisfy

$$\int_a^\infty |w(t)| |y^\sigma(t, \lambda_0)|^2 \Delta t < \infty, \quad (2.4)$$

then  $\int_a^\infty |\Omega(t, \lambda_0)| \Delta t < \infty$ .

*Proof.* From the definition of  $Y(t, \lambda, \nu)$ , rules for the time scale differentiation, see [3, Theorems 1.20 and 5.3], and the form of system  $(S_\lambda)$  we obtain

$$\begin{aligned} Y^\Delta(t, \lambda, \nu) &= [\Phi^\sigma(t, \nu)]^{-1} [-\Phi^\Delta(t, \nu) \Phi^{-1}(t, \nu) + \mathcal{S}(t) + \lambda \mathcal{V}(t)] \Phi(t, \nu) Y(t, \lambda, \nu) \\ &= [\Phi^\sigma(t, \nu)]^{-1} [-\mathcal{S}(t) - \nu \mathcal{V}(t) + \mathcal{S}(t) + \lambda \mathcal{V}(t)] \Phi(t, \nu) Y(t, \lambda, \nu) \\ &= (\lambda - \nu) [\Phi^\sigma(t, \nu)]^{-1} \mathcal{V}(t) \Phi(t, \nu) Y(t, \lambda, \nu). \end{aligned}$$

Simultaneously the Liouville formula, see [3, Theorem 5.28], yields

$$\det \Phi(t, \nu) = e_{r(\cdot)}(t, a) \det \Phi(a, \nu) = e_{r(\cdot)}(t, a),$$

where  $r(t) := \operatorname{tr} \mathcal{S}(t, \nu) + \mu(t) \det \mathcal{S}(t, \nu)$ . But  $r(t) \equiv 0$  on the interval  $[a, \infty)_{\mathbb{T}}$ , which implies  $\det \Phi(t, \nu) = e_0(t, a) \equiv 1$  for all  $t \in [a, \infty)_{\mathbb{T}}$ , see [3, Theorem 2.36]. Hence  $\Omega(t, \nu)$  defined in (2.3) corresponds to  $-\Phi^\sigma(t, \nu)^{-1} \mathcal{V}(t) \Phi(t, \nu)$ , which proves (2.2). The regressivity of  $\Omega(\cdot, \nu)$  is a simple consequence of the relation  $\det[I + \mu(t) \Omega(t, \nu)] \equiv 1$  on  $[a, \infty)_{\mathbb{T}}$ , which is obtained by a straightforward calculation. Finally, the inequality  $\int_a^\infty |\Omega(t, \lambda_0)| \Delta t < \infty$  follows directly from assumption (2.4) and from the Cauchy–Schwarz inequality, see [3, Theorem 6.15],

$$\begin{aligned} \int_a^\infty |w(t)| |\phi_1^\sigma(t, \lambda_0) \phi_2^\sigma(t, \lambda_0)| \Delta t &= \int_a^\infty \sqrt{|w(t)| |\phi_1^\sigma(t, \lambda_0)|} \sqrt{|w(t)| |\phi_2^\sigma(t, \lambda_0)|} \Delta t \\ &\leq \left( \int_a^\infty |w(t)| |\phi_1^\sigma(t, \lambda_0)|^2 \Delta t \right)^{1/2} \times \left( \int_a^\infty |w(t)| |\phi_2^\sigma(t, \lambda_0)|^2 \Delta t \right)^{1/2} < \infty, \end{aligned} \quad (2.5)$$

which completes the proof.  $\square$

**Remark 2.4.** Let us denote by  $\|\cdot\|_1$  the Hölder (or  $\ell^1$ ) matrix norm on  $\mathbb{C}^{2 \times 2}$ , i.e.,  $\|A\|_1 := \sum_{i,j=1}^2 |a_{ij}|$  for any  $A \in \mathbb{C}^{2 \times 2}$ . Then the additional assumption in (2.4) and the conclusion of Lemma 2.3 imply

$$\int_a^\infty \|\Omega(t)\|_1 \Delta t < \infty, \quad (2.6)$$

where  $\Omega(t) := \Omega(t, \lambda_0)$ . Since  $(\lambda_0 - \lambda) \Omega(\cdot)$  is regressive by Lemma 2.3 and the norm  $\|\cdot\|_1$  is submultiplicative, i.e.,  $\|AB\|_1 \leq \|A\|_1 \|B\|_1$ , see [2, Proposition 9.3.5], it follows from inequality (2.6) and [19, Lemma 3.1] that there exists  $K > 0$  such that

$$\|Y(t, \lambda)\|_1 = \|e_{(\lambda_0 - \lambda) \Omega(\cdot)}(t, a)\|_1 \leq K < \infty, \quad \text{for all } t \in [a, \infty)_{\mathbb{T}}, \quad (2.7)$$

i.e.,  $Y(t, \lambda) := Y(t, \lambda, \lambda_0)$  is bounded in the norm  $\|\cdot\|_1$ . In addition, if  $\{t_k\}_{k=1}^\infty \subseteq \mathbb{T}$  is a strictly increasing sequence such that  $t_k \rightarrow \infty$  for  $k \rightarrow \infty$ , then we obtain from the submultiplicativity of  $\|\cdot\|_1$  and (2.7) that

$$\|Y(t_i, \lambda) - Y(t_j, \lambda)\|_1 \leq |\lambda_0 - \lambda| \int_{t_i}^{t_j} \|\Omega(t) Y(t, \lambda)\|_1 \Delta t \leq K |\lambda_0 - \lambda| \int_{t_i}^{t_j} \|\Omega(t)\|_1 \Delta t$$

for any  $i, j \in \mathbb{N}$ ,  $i < j$ . Since the improper integral  $\int_a^\infty \|\Omega(t)\|_1 \Delta t$  is convergent by (2.6), it follows from [4, Theorem 5.49] that for any  $\varepsilon > 0$  there exists  $k \in \mathbb{N}$  such that

$$\|Y(t_i, \lambda) - Y(t_j, \lambda)\|_1 < \varepsilon$$

for any  $i, j \in \mathbb{N}$ ,  $k < i < j$ . This means that  $\{Y(t_k, \lambda)\}_{k=1}^\infty$  is a Cauchy sequence. Therefore the limit  $\lim_{t \rightarrow \infty} Y(t, \lambda)$  exists and, by (2.7), is finite.

The following theorem generalizes [25, Theorem 7.2.2] to any time scale, see also [19, Theorem 6.1 and Remark 6.2(ii)], [26, Theorem 3.2], and more generally [20]. In its proof we utilize the Euclidean (or  $\ell^2$ ) vector norm on  $\mathbb{C}^2$ , i.e.,  $\|\xi\|_2 = (\sum_{i=1}^2 |\xi_i|^2)^{1/2}$  for  $\xi \in \mathbb{C}^2$ , and also the spectral matrix norm on  $\mathbb{C}^{2 \times 2}$ , i.e., for  $A \in \mathbb{C}^{2 \times 2}$  we put

$$\|A\|_s := \max \{ \sqrt{v}, v \text{ is an eigenvalue of } A^* A \}.$$

It is well known that

$$|\xi^* \zeta| \leq \|\xi\|_2 \|\zeta\|_2 \quad (2.8)$$

for any  $\xi, \zeta \in \mathbb{C}^2$ , see [2, Fact 9.7.4(xii)]. In addition, the norms  $\|\cdot\|_2$  and  $\|\cdot\|_s$  are compatible, i.e.,  $\|A\xi\|_2 \leq \|A\|_s \|\xi\|_2$ , while the norms  $\|\cdot\|_1$  and  $\|\cdot\|_s$  satisfy the inequality

$$\|A\|_s \leq \|A\|_1 \quad (2.9)$$

for any matrix  $A \in \mathbb{C}^{2 \times 2}$ , see [2, Fact 9.8.12(v)]. For brevity, we also employ the condensed notation  $M^{\sigma*}(t) := [M^\sigma(t)]^* = [M^*(t)]^\sigma$  for any matrix-valued function  $M(\cdot)$ .

**Theorem 2.5.** *Let Hypothesis 2.1 be satisfied and assume that there exists  $\lambda_0 \in \mathbb{C}$  such that all solutions of  $(E_{\lambda_0})$  satisfy (2.4). Then equation  $(E_\lambda)$  possesses the same property for any  $\lambda \in \mathbb{C}$ .*

*Proof.* Let  $\lambda \in \mathbb{C} \setminus \{\lambda_0\}$  be arbitrary and put  $\Psi(t) := \begin{pmatrix} w(t) & 0 \\ 0 & 0 \end{pmatrix}$ . Then by the assumptions and inequality (2.5) we get

$$\begin{aligned} & \int_a^\infty \|\Phi^{\sigma*}(t, \lambda_0) \Psi(t) \Phi^\sigma(t, \lambda_0)\|_1 \Delta t \\ &= \int_a^\infty [ |w(t)| |\phi_1^\sigma(t, \lambda_0)|^2 + |w(t)| |\phi_2^\sigma(t, \lambda_0)|^2 \\ & \quad + 2|w(t)| |\phi_1^\sigma(t, \lambda_0) \phi_2^\sigma(t, \lambda_0)| ] \Delta t \leq L < \infty, \end{aligned} \quad (2.10)$$

for some  $L > 0$ . Therefore with  $Y(\cdot, \lambda) := Y(t, \lambda, \lambda_0)$  as in Lemma 2.3 it follows

$$\begin{aligned} & \int_a^\infty \|\Phi^{\sigma*}(t, \lambda) \Psi(t) \Phi^\sigma(t, \lambda)\|_1 \Delta t \\ &= \int_a^\infty \|Y^{\sigma*}(t, \lambda) \Phi^{\sigma*}(t, \lambda_0) \Psi(t) \Phi^\sigma(t, \lambda_0) Y^\sigma(t, \lambda)\|_1 \Delta t \\ &\leq \int_a^\infty \|Y^\sigma(t, \lambda)\|_1^2 \|\Phi^{\sigma*}(t, \lambda_0) \Psi(t) \Phi^\sigma(t, \lambda_0)\|_1 \Delta t \leq K^2 L < \infty, \end{aligned} \quad (2.11)$$

where we used (2.10), (2.7), and the submultiplicativity and self-adjointness of the norm  $\|\cdot\|_1$ , i.e.,  $\|A\|_1 = \|A^*\|_1$ . Since any nontrivial solution  $y(t, \lambda)$  of  $(E_\lambda)$  can be obtained as

$$y(t, \lambda) = (1, 0) \Phi(t, \lambda) \xi, \quad t \in [a, \infty)_T,$$

for some  $\zeta \in \mathbb{C}^2 \setminus \{0\}$ , we have

$$\begin{aligned} \int_a^\infty |w(t)| |y^\sigma(t, \lambda)|^2 \Delta t &= \int_a^\infty |\zeta^* \Phi^{\sigma*}(t, \lambda) \Psi(t) \Phi^\sigma(t, \lambda) \zeta| \Delta t \\ &\leq \int_a^\infty \|\zeta\|_2 \|\Phi^{\sigma*}(t, \lambda) \Psi(t) \Phi^\sigma(t, \lambda) \zeta\|_2 \Delta t \leq \int_a^\infty \|\zeta\|_2^2 \|\Phi^{\sigma*}(t, \lambda) \Psi(t) \Phi^\sigma(t, \lambda)\|_s \Delta t \\ &\leq \int_a^\infty \|\zeta\|_2^2 \|\Phi^{\sigma*}(t, \lambda) \Psi(t) \Phi^\sigma(t, \lambda)\|_1 \Delta t \leq \|\zeta\|_2^2 K^2 L < \infty, \end{aligned}$$

where we used (2.8), (2.9), (2.11), and the compatibility of  $\|\cdot\|_2$  and  $\|\cdot\|_s$ . This shows that any solution of  $(E_\lambda)$  satisfies  $\int_a^\infty |w(t)| |y^\sigma(t, \lambda)|^2 \Delta t < \infty$  and the proof is complete.  $\square$

**Remark 2.6.** If we replace  $\Psi(t)$  by  $\tilde{\Psi}(t) := \begin{pmatrix} 0 & 0 \\ 0 & w(t) \end{pmatrix}$  in the proof of Theorem 2.5, we obtain the following statement: if there exists  $\lambda_0 \in \mathbb{C}$  such that every quasi-derivative  $y^{[1]}(t, \lambda_0) := p(t) y^\Delta(t, \lambda_0)$  of any nontrivial solution  $y(t, \lambda_0)$  of equation  $(E_{\lambda_0})$  satisfies

$$\int_a^\infty |w(t)| |[y^{[1]}(t, \lambda_0)]^\sigma|^2 \Delta t < \infty,$$

then equation  $(E_\lambda)$  possesses this property for any  $\lambda \in \mathbb{C}$ .

Moreover, Theorem 2.5 and Remark 2.6 immediately yield the following sufficient condition for the invariance concerning solutions of  $(E_\lambda)$  and their quasi-derivatives.

**Corollary 2.7.** *Let Hypothesis 2.1 be satisfied and assume that*

$$\int_a^\infty (|1/p(t)| + |q(t)| + \mu(t) |q(t)/p(t)|) \Delta t < \infty, \quad \int_a^\infty |w(t)| \Delta t < \infty. \quad (2.12)$$

*Then all solutions of  $(E_\lambda)$  and their quasi-derivatives satisfy*

$$\int_a^\infty |w(t)| |y^\sigma(t, \lambda)|^2 \Delta t < \infty \quad \text{and} \quad \int_a^\infty |w(t)| |[y^{[1]}(t, \lambda)]^\sigma|^2 \Delta t < \infty, \quad (2.13)$$

*respectively, for any  $\lambda \in \mathbb{C}$ .*

*Proof.* According to Theorem 2.5 and Remark 2.6 it suffices to show that all solutions of equation  $(E_0)$  and their quasi-derivatives satisfy (2.13) with  $\lambda = 0$ . From the first condition in (2.12) we get  $\int_a^\infty \|\mathcal{S}(t)\|_1 \Delta t < \infty$ . Therefore [19, Lemma 3.1] implies that  $\|\Phi(t, 0)\|_1 \leq \alpha < \infty$  on  $[a, \infty)_\mathbb{T}$  for some  $\alpha > 0$ . Upon using similar arguments as in the proof of Theorem 2.5 with  $\Psi(t)$  and  $\tilde{\Psi}(t)$ , respectively, we obtain the conclusion.  $\square$

Henceforward we restrict our attention only to equation  $(E_\lambda)$  with the coefficients satisfying the following hypothesis, although we will not repeat it explicitly.

**Hypothesis 2.8.** The functions  $p(\cdot), q(\cdot), w(\cdot) \in C_{\text{prd}}$  are real-valued and satisfy (1.1).

Now we recall several results from the Weyl–Titchmarsh theory for equation  $(E_\lambda)$ . The results can be easily obtained as in [17], i.e., as a consequence of Lemma 2.2, Hypothesis 2.8, and general statements for symplectic dynamic systems established in [18, 19]. On the other hand, some of these results were derived also directly in [14, 22, 26, 28].

We denote by  $\mathcal{L}_w^2$  and  $\mathcal{N}(\lambda)$  the linear spaces consisting of all square integrable functions with respect to the weight  $w(\cdot)$  and of all square integrable solutions of  $(E_\lambda)$ , respectively, i.e.,

$$\mathcal{L}_w^2 := \left\{ y : [a, \infty)_{\mathbb{T}} \rightarrow \mathbb{C}, \int_a^\infty w(t) |y^\sigma(t)|^2 \Delta t < \infty \right\},$$

$$\mathcal{N}(\lambda) := \{ y(\cdot, \lambda) \in \mathcal{L}_w^2, y(\cdot, \lambda) \text{ solves } (E_\lambda) \}.$$

Moreover, for brevity, by  $n(\lambda)$  we mean the number of (nontrivial) linearly independent square integrable solutions of equation  $(E_\lambda)$ , i.e.,

$$n(\lambda) := \dim \mathcal{N}(\lambda).$$

Then obviously  $n(\lambda) = n(\bar{\lambda})$ , because Hypothesis 2.8 implies that  $y(\cdot, \bar{\lambda}) = \overline{y(\cdot, \lambda)}$ , i.e., the function  $y(\cdot, \lambda)$  solves  $(E_\lambda)$  if and only if  $\overline{y(\cdot, \lambda)}$  solves  $(E_{\bar{\lambda}})$ . In addition, from Theorem 2.5 we obtain immediately the following statement, see also [19, Section 6] and [26, Theorem 3.2].

**Theorem 2.9.** *If there exists  $\lambda_0 \in \mathbb{C}$  such such that  $n(\lambda_0) = 2$ , then  $n(\lambda) = 2$  for all  $\lambda \in \mathbb{C}$ .*

More precisely, the number  $n(\lambda)$  satisfies  $1 \leq n(\lambda) \leq 2$  for any  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  by [17, Theorem 3.10], which upon combining with Theorem 2.9 yields the famous dichotomy for equation  $(E_\lambda)$  as stated in Theorem 2.10 below. The latter estimate is obtained by using the so-called Weyl circles, which are nested and converge to a circle ( $n(\lambda) = 2$ ) or a point ( $n(\lambda) = 1$ ), see e.g. [19, Sections 3 and 4]. This geometrical background naturally motivates the *limit circle* and *limit point* terminology. Finally, we note that Theorem 2.9 is known as the *invariance of the limit circle case* and Theorem 2.10 below as the *Weyl alternative*.

**Theorem 2.10.** *Only one of the following statements is true.*

- (i) *For any  $\lambda \in \mathbb{C}$  equation  $(E_\lambda)$  is in the limit circle case, i.e.,  $n(\lambda) \equiv 2$ .*
- (ii) *For any  $\lambda \in \mathbb{C}$  equation  $(E_\lambda)$  is in the limit point case, i.e.,  $n(\lambda) \leq 1$ . In this case,  $n(\lambda) = 1$  for all  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  and  $n(\lambda) \in \{0, 1\}$  for  $\lambda \in \mathbb{R}$ .*

If equation  $(E_\lambda)$  is in the limit point case and  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ , then the unique square integrable solution (up to a constant multiple) corresponds to the so-called *Weyl solution*  $\mathcal{X}(\cdot, \lambda)$ , which is of the form

$$\mathcal{X}(t, \lambda) = \varphi(t, \lambda) + m_+(\lambda) \psi(t, \lambda), \quad t \in [a, \infty)_{\mathbb{T}}, \quad (2.14)$$

where  $\varphi(\cdot, \lambda)$  and  $\psi(\cdot, \lambda)$  are linearly independent solutions of  $(E_\lambda)$  determined by the initial conditions

$$\left. \begin{aligned} \varphi(a, \lambda) &= \sin \alpha, & \psi(a, \lambda) &= -\cos \alpha, \\ [p(t) \varphi^\Delta(t, \lambda)]_{t=a} &= \cos \alpha, & [p(t) \psi^\Delta(t, \lambda)]_{t=a} &= \sin \alpha, \end{aligned} \right\} \quad (2.15)$$

for  $\alpha \in [0, \pi)$  and  $m_+(\lambda)$  can be defined as the limit

$$m_+(\lambda) = -\lim_{t \rightarrow \infty} \frac{\varphi(t, \lambda)}{\psi(t, \lambda)}. \quad (2.16)$$

The functions  $\varphi(\cdot, \lambda)$  and  $\psi(\cdot, \lambda)$  are analytic with respect to  $\lambda$ , see [12, Section 4], and  $\psi(t, \lambda) \neq 0$  for all  $t \in (a, \infty)_{\mathbb{T}}$ . The latter fact follows from the Lagrange identity, see e.g. [19, Theorem 2.3],

$$W[x(t, \lambda), y(t, \nu)] = W[x(a, \lambda), y(a, \nu)] + (\lambda - \nu) \int_a^t w(\tau) x^\sigma(\tau, \lambda) y^\sigma(\tau, \nu) \Delta \tau,$$



where  $x(\cdot, \lambda)$  and  $y(\cdot, \nu)$  are solutions of  $(E_\lambda)$  and  $(E_\nu)$  with  $\lambda, \nu \in \mathbb{C}$ , respectively, and

$$W[x(t, \lambda), y(t, \nu)] := p(t)[x(t, \lambda) y^\Delta(t, \nu) - y(t, \nu) x^\Delta(t, \lambda)]$$

represents the Wronskian of  $x(\cdot, \lambda)$  and  $y(\cdot, \nu)$ . Moreover,  $m_+(\lambda)$  is an analytic function in the half-planes  $\mathbb{C}_+$  and  $\mathbb{C}_-$  as a limit of a family of locally (uniformly) bounded analytic functions.

If  $y(\cdot, \lambda)$  is a solution of  $(E_\lambda)$  such that  $y(t, \lambda) \neq 0$  for all  $t \geq t_0$  with  $t_0 \in [a, \infty)_\mathbb{T}$ , then the function  $z(\cdot, \lambda)$  given by

$$z(t, \lambda) = y(t, \lambda) \left( c_1 + c_2 \int_{t_0}^t \frac{1}{p(\tau) y^\sigma(\tau, \lambda) y(\tau, \lambda)} \Delta\tau \right), \quad t \geq t_0, \quad (2.17)$$

satisfies equation  $(E_\lambda)$  for all  $t \geq t_0$  and any  $c_1, c_2 \in \mathbb{C}$ , see [7, Remark 6]. Moreover, we have  $W[y(t, \lambda), z(t, \lambda)] \equiv c_2$  and  $c_1 = z(t_0, \lambda)/y(t_0, \lambda)$ .

If  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  and  $\alpha \neq 0$ , then  $\varphi(t, \lambda) \neq 0$  for all  $t \in [a, \infty)_\mathbb{T}$  and (2.17) yields

$$\psi(t, \lambda) = \varphi(t, \lambda) \left( -\cotan \alpha + \int_a^t \frac{1}{p(\tau) \varphi^\sigma(\tau, \lambda) \varphi(\tau, \lambda)} \Delta\tau \right). \quad (2.18)$$

Similarly for  $\alpha \neq \pi/2$  we get

$$\varphi(t, \lambda) = \psi(t, \lambda) \left( -\tan \alpha - \int_a^t \frac{1}{p(\tau) \psi^\sigma(\tau, \lambda) \psi(\tau, \lambda)} \Delta\tau \right). \quad (2.19)$$

Upon combining identities (2.16), (2.18), and (2.19) we obtain for  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  that

$$m_+(\lambda) = \begin{cases} \tan \alpha + \int_a^\infty \frac{1}{p(t) \psi^\sigma(t, \lambda) \psi(t, \lambda)} \Delta t, & \alpha \in [0, \pi) \setminus \{\pi/2\}, \\ \left( \cotan \alpha - \int_a^\infty \frac{1}{p(t) \varphi^\sigma(t, \lambda) \varphi(t, \lambda)} \Delta t \right)^{-1}, & \alpha \in (0, \pi), \end{cases} \quad (2.20)$$

compare with [5, Formula (2.53)] and see also identity (3.7). The latter formula is illustrated in the following example.

**Example 2.11.** Let  $[a, \infty)_\mathbb{T} = [0, \infty)$  and consider the second order Sturm–Liouville differential equation

$$-y''(t, \lambda) = \lambda y(t, \lambda).$$

If  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  and  $\alpha = \pi/2$ , then the two linearly independent solutions determined by the initial conditions (2.15) are

$$\varphi(t, \lambda) = (e^{\sqrt{-\lambda}t} + e^{-\sqrt{-\lambda}t})/2 \quad \text{and} \quad \psi(t, \lambda) = (e^{\sqrt{-\lambda}t} - e^{-\sqrt{-\lambda}t})/(2\sqrt{-\lambda}).$$

Therefore we obtain from (2.16) that  $m_+(\lambda) = -\sqrt{-\lambda}$ . The same follows from the calculation

$$m_+(\lambda) = \left( \cotan \pi/2 - \int_0^\infty \frac{1}{p(t) \varphi^2(t, \lambda)} dt \right)^{-1} = - \left( \int_0^\infty \frac{1}{\cosh^2 \sqrt{-\lambda}t} dt \right)^{-1} = -\sqrt{-\lambda}.$$

Similarly, in the case  $\alpha = 0$  we have

$$\varphi(t, \lambda) = (e^{\sqrt{-\lambda}t} - e^{-\sqrt{-\lambda}t})/(2\sqrt{-\lambda}) \quad \text{and} \quad \psi(t, \lambda) = -(e^{\sqrt{-\lambda}t} + e^{-\sqrt{-\lambda}t})/2,$$



which yields

$$m_+(\lambda) = \tan 0 + \int_0^\infty \frac{1}{p(t)\psi^2(t,\lambda)} dt = \int_0^\infty \frac{1}{\cosh^2 \sqrt{-\lambda} t} dt = \frac{1}{\sqrt{-\lambda}}.$$

Moreover, according to (2.14), we get the Weyl solution  $\mathcal{X}(t,\lambda) = e^{-\sqrt{-\lambda}t}$  if  $\alpha = \pi/2$ , and  $\mathcal{X}(t,\lambda) = -e^{-\sqrt{-\lambda}t}/\sqrt{-\lambda}$  if  $\alpha = 0$ . Nevertheless one easily observes that these two expressions differ only by a constant multiple and they both satisfy  $\mathcal{X}(\cdot,\lambda) \in \mathcal{L}_w^2$ . Finally, we point out that the Weyl solution and  $m_+(\lambda)$  are well defined even for any  $\lambda \in \mathbb{C} \setminus (0,\infty)$  and the property  $\mathcal{X}(\cdot,\lambda) \in \mathcal{L}_w^2$  remains valid on  $\lambda \in \mathbb{C} \setminus [0,\infty)$ , see Theorem 3.5 for more details.

In the last part of this section we focus on the principal solution of equation  $(E_\lambda^{\mathbb{R}})$ , i.e., equation  $(E_\lambda)$  with  $\lambda \in \mathbb{R}$ . In addition, without loss of generality, we consider only real-valued solutions of  $(E_\lambda^{\mathbb{R}})$ . A solution  $y(\cdot,\lambda)$  of  $(E_\lambda^{\mathbb{R}})$  has a *generalized zero* at  $t \in [a,\infty)_{\mathbb{T}}$  if  $p(t)y^\sigma(t,\lambda)y(t,\lambda) \leq 0$ . Then equation  $(E_\lambda^{\mathbb{R}})$  is said to be *disconjugate on an interval*  $[b,c]_{\mathbb{T}} \subset [a,\infty)_{\mathbb{T}}$  if every nontrivial solution of  $(E_\lambda^{\mathbb{R}})$  has at most one generalized zero in  $[b,c]_{\mathbb{T}}$ , and it is said to be *disconjugate on*  $[b,\infty)_{\mathbb{T}}$  if it is disconjugate on  $[b,c]_{\mathbb{T}}$  for every  $c \in (b,\infty)_{\mathbb{T}}$ . Equation  $(E_\lambda^{\mathbb{R}})$  is called *oscillatory* on  $[a,\infty)_{\mathbb{T}}$  if some nontrivial solution has infinitely many generalized zeros on  $[a,\infty)_{\mathbb{T}}$ . As a consequence of the Sturmian theory, see e.g. [10], it follows that in the latter case every solution does as well. In the opposite case equation  $(E_\lambda^{\mathbb{R}})$  is said to be *nonoscillatory*, i.e., if there exists a solution such that  $p(t)y^\sigma(t,\lambda)y(t,\lambda) > 0$  for all  $t \in [a,\infty)_{\mathbb{T}}$  large enough. In other words,  $(E_\lambda^{\mathbb{R}})$  is nonoscillatory if it is eventually disconjugate.

**Remark 2.12.** As a consequence of the Sturmian theory it also follows that if  $(E_{\lambda_0})$  is nonoscillatory for some  $\lambda_0 \in \mathbb{R}$ , then  $(E_\lambda)$  is nonoscillatory for all  $\lambda \leq \lambda_0$ . The simple equation  $-y^{\Delta\Delta}(t,\lambda) = \lambda y(t,\lambda)$  illustrates that equation  $(E_\lambda)$  can be oscillatory for some values of  $\lambda \in \mathbb{R}$  and nonoscillatory for another values  $\lambda \in \mathbb{R}$ . On the other hand, it is well known in the special case  $\mathbb{T} = \mathbb{R}$  that the oscillatory/nonoscillatory behavior is invariant in the limit circle case, i.e., equation  $(E_\lambda)$  being in the limit circle case is either oscillatory or nonoscillatory for all  $\lambda \in \mathbb{R}$ . An elegant proof based on the existence of the finite limit of  $Y(t,\lambda,\nu)$  discussed in Remark 2.4 can be found in [25, Theorem 7.3.1]. A similar statement on a general time scale remains open and its solution is closely connected with the problem discussed in Remark 3.6(ii), see also Corollary 3.3.

Following [6], a nontrivial solution  $y(\cdot,\lambda)$  of  $(E_\lambda^{\mathbb{R}})$  is called *principal* if there exists  $t_0 \in [a,\infty)_{\mathbb{T}}$  such that  $p(t)y^\sigma(t,\lambda)y(t,\lambda) > 0$  for all  $t \in [t_0,\infty)_{\mathbb{T}}$ , and it satisfies

$$\lim_{t \rightarrow \infty} \frac{y(t,\lambda)}{\tilde{y}(t,\lambda)} = 0$$

for any solution  $\tilde{y}(\cdot,\lambda)$  of  $(E_\lambda^{\mathbb{R}})$  which is linearly independent of  $y(\cdot,\lambda)$ . Any solution linearly independent of the principal solution is said to be *nonprincipal*, see also [1]. The existence of the principal solution of  $(E_\lambda^{\mathbb{R}})$  is equivalent with its nonoscillatory behavior, see [6, Theorem 3.1]. Moreover, the principal solution is determined uniquely up to a nonzero constant multiple and satisfies

$$\int_{t_0}^\infty \frac{1}{p(\tau)y^\sigma(\tau,\lambda)y(\tau,\lambda)} \Delta\tau = \infty, \quad (2.21)$$

while for any nonprincipal solution  $\tilde{y}(\cdot,\lambda)$  we have

$$\int_{t_1}^\infty \frac{1}{p(\tau)\tilde{y}^\sigma(\tau,\lambda)\tilde{y}(\tau,\lambda)} \Delta\tau < \infty, \quad (2.22)$$

where  $t_0, t_1 \in [a, \infty)_{\mathbb{T}}$  are such that the denominators are positive on the intervals of integration. The following statement will be useful in the proof of Theorem 3.1 and it can be verified by direct calculations.

**Theorem 2.13.** *Let  $\lambda \in \mathbb{R}$  and assume that equation  $(E_\lambda)$  is nonoscillatory. If  $y(\cdot, \lambda)$  is a nontrivial solution of  $(E_\lambda)$ , then*

$$\tilde{y}(t, \lambda) := y(t, \lambda) \int_{t_0}^t \frac{1}{p(\tau) y^\sigma(\tau, \lambda) y(\tau, \lambda)} \Delta\tau, \quad t \in [t_0, \infty)_{\mathbb{T}},$$

is a nonprincipal solution, where  $t_0 \in [a, \infty)_{\mathbb{T}}$  is such that the denominator is positive on  $[t_0, \infty)_{\mathbb{T}}$ . On the other hand, if  $\tilde{y}(\cdot, \lambda)$  is a nonprincipal solution of  $(E_\lambda)$ , then

$$\hat{y}(t, \lambda) := \tilde{y}(t, \lambda) \int_t^\infty \frac{1}{p(\tau) \tilde{y}^\sigma(\tau, \lambda) \tilde{y}(\tau, \lambda)} \Delta\tau, \quad t \in [t_1, \infty)_{\mathbb{T}},$$

is the principal solution of  $(E_\lambda)$ , where  $t_1 \in [a, \infty)_{\mathbb{T}}$  is such that the denominator is positive on  $[t_1, \infty)_{\mathbb{T}}$ .

### 3 Main results

As a simple consequence of the existence of the principal solution we obtain the following limit point criterion. If  $\mathbb{T} = \mathbb{R}$  it reduces to [16, Theorem 4.1], see also [9] and [8, Theorem 11.6], while in the case  $\mathbb{T} = \mathbb{Z}$  and  $w(t) \equiv 1$  it can be found in [15, Theorem 5].

**Theorem 3.1.** *Let us assume that there exists  $\nu \in \mathbb{R}$  such that equation  $(E_\nu)$  is nonoscillatory and the corresponding principal solution  $\hat{y}(\cdot, \nu)$  satisfies  $\int_{t_0}^\infty w^\rho(t) \hat{y}^2(t, \nu) \Delta t < \infty$  for some  $t_0 \in (a, \infty)_{\mathbb{T}}$  whenever  $\hat{y}(\cdot, \nu) \in \mathcal{L}_w^2$ . If there exists  $t_1 \in (a, \infty)_{\mathbb{T}}$  such that*

$$\int_{t_1}^\infty \frac{[w(t) w^\rho(t)]^{1/4}}{|p(t)|^{1/2}} \Delta t = \infty, \quad (3.1)$$

then equation  $(E_\lambda)$  is in the limit point case for all  $\lambda \in \mathbb{C}$ .

*Proof.* Let (3.1) hold and  $\nu \in \mathbb{R}$  be such that the assumptions are satisfied. With respect to Theorem 2.10 it suffices to show that there exists a solution  $y(\cdot, \nu) \notin \mathcal{L}_w^2$ . Since  $(E_\nu)$  is nonoscillatory, it possesses the principal solution  $\hat{y}(\cdot, \nu)$  and we define

$$\tilde{y}(t, \nu) := \hat{y}(t, \nu) \int_{t_2}^t \frac{1}{p(\tau) \hat{y}^\sigma(\tau, \nu) \hat{y}(\tau, \nu)} \Delta\tau, \quad t \in [t_2, \infty)_{\mathbb{T}},$$

where  $t_2 \in [a, \infty)_{\mathbb{T}}$  is such that the denominator is positive on  $[t_2, \infty)_{\mathbb{T}}$ . Then for any  $t_3 \in (t_2, \infty)_{\mathbb{T}}$  we have

$$\int_{t_3}^\infty \frac{1}{p(\tau) \tilde{y}^\sigma(\tau, \nu) \tilde{y}(\tau, \nu)} \Delta\tau < \infty,$$

because  $\tilde{y}(\cdot, \nu)$  is a nonprincipal solution of  $(E_\nu)$  by Theorem 2.13. Suppose that the linearly independent solutions  $\hat{y}(\cdot, \nu)$  and  $\tilde{y}(\cdot, \nu)$  belong to  $\mathcal{L}_w^2$ . Then by the assumptions also

$\int_{t_0}^{\infty} w^\rho(t) \hat{y}^2(t, \nu) \Delta t < \infty$  for some  $t_0 \in (a, \infty)_{\mathbb{T}}$  and for  $t_4 \geq \max_{i=0,1,2,3} \{t_i\}$  we have

$$\begin{aligned} \int_{t_4}^{\infty} \frac{[w(t) w^\rho(t)]^{1/4}}{|p(t)|^{1/2}} \Delta t &= \int_{t_4}^{\infty} \frac{[w(t) \hat{y}^{\sigma^2}(t, \nu)]^{1/4} [w^\rho(t) \hat{y}^2(t, \nu)]^{1/4}}{[p(t) \hat{y}^\sigma(t, \nu) \hat{y}(t, \nu)]^{1/2}} \Delta t \\ &\leq \left( \int_{t_4}^{\infty} \frac{1}{p(t) \hat{y}^\sigma(t, \nu) \hat{y}(t, \nu)} \Delta t \right)^{1/2} \left( \int_{t_4}^{\infty} [w(t) \hat{y}^{\sigma^2}(t, \nu)]^{1/2} [w^\rho(t) \hat{y}^2(t, \nu)]^{1/2} \Delta t \right)^{1/2} \\ &\leq \left( \int_{t_4}^{\infty} \frac{1}{p(t) \hat{y}^\sigma(t, \nu) \hat{y}(t, \nu)} \Delta t \right)^{1/2} \left( \int_{t_4}^{\infty} w(t) \hat{y}^{\sigma^2}(t, \nu) \Delta t \right)^{1/4} \left( \int_{t_4}^{\infty} w^\rho(t) \hat{y}^2(t, \nu) \Delta t \right)^{1/4} < \infty, \end{aligned}$$

where we used the Cauchy–Schwarz inequality in the last two steps, see [3, Theorem 6.15]. But this yields a contradiction with the assumption (3.1). Hence there exists a nontrivial solution of  $(E_\nu)$ , which is not in  $\mathcal{L}_w^2$ , i.e., equation  $(E_\nu)$  is in the limit point case. Therefore  $(E_\lambda)$  is in the limit point case for all  $\lambda \in \mathbb{C}$  by Theorem 2.10.  $\square$

**Remark 3.2.** The additional assumption concerning the convergence of  $\int_{t_0}^{\infty} w^\rho(t) \hat{y}^2(t, \nu) \Delta t$  is trivially satisfied if  $\mathbb{T} = \mathbb{R}$  or if  $\mathbb{T}$  consists only of isolated points, especially when  $\mathbb{T} = h\mathbb{Z}$  or  $\mathbb{T} = q^{\mathbb{N}}$ . On the other hand, it does not mean  $y(\cdot, \nu) \in \mathcal{L}_w^2$ , because  $\sigma(\rho(t)) \neq t$  for  $t \in [a, \infty)_{\mathbb{T}}$ , which are left-dense and right scattered simultaneously. In particular, it can be shown that one of the integrals  $\int_a^\infty f(t) \Delta t$  and  $\int_a^\infty f^\sigma(t) \Delta t$  can be convergent, while the other is divergent, compare with [22, 26, 27]. For example, let us consider the simple time scale

$$\mathbb{T} = [0, 1] \cup [2, 3] \cup \dots = \bigcup_{k \in \mathbb{N} \cup \{0\}} [2k, 2k + 1].$$

Then the integral over  $\mathbb{T}$  can be written as

$$\begin{aligned} \int_{\mathbb{T}} f(t) \Delta t &= \int_0^1 f(t) \Delta t + \int_1^2 f(t) \Delta t + \int_2^3 f(t) \Delta t + \dots \\ &= \sum_{k=0}^{\infty} \int_{2k}^{2k+1} f(t) \Delta t + \sum_{k=0}^{\infty} \int_{2k+1}^{\sigma(2k+1)} f(t) \Delta t \\ &= \sum_{k=0}^{\infty} \int_{2k}^{2k+1} f(t) dt + \sum_{k=0}^{\infty} \mu(2k+1) f(2k+1) \end{aligned}$$

and similarly we obtain

$$\int_{\mathbb{T}} f^\sigma(t) \Delta t = \sum_{k=0}^{\infty} \int_{2k}^{2k+1} f(t) dt + \sum_{k=0}^{\infty} \mu(2k+1) f^\sigma(2k+1).$$

If we define the function  $f : \mathbb{T} \rightarrow \mathbb{R}$  as

$$f(t) = \frac{3k}{(k+1)^2} t^2 - \frac{2k(6k+1)}{(k+1)^2} t + \frac{12k^3 + 4k^2 + 1}{(k+1)^2} \quad \text{for } t \in [2k, 2k+1],$$

then  $f(2k) = \frac{1}{(k+1)^2}$ ,  $f(2k+1) = \frac{1}{k+1}$ , and  $\int_{2k}^{2k+1} f(t) dt = \frac{1}{(k+1)^2}$ . Therefore

$$\int_{\mathbb{T}} f^\sigma(t) \Delta t = \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} + \sum_{k=0}^{\infty} \frac{1}{(k+2)^2} = \frac{\pi^2}{3} - 1 < \infty,$$

while

$$\int_{\mathbb{T}} f(t) \Delta t = \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} + \sum_{k=0}^{\infty} \frac{1}{k+1} = \infty,$$

i.e., the integrals  $\int_a^{\infty} f(t) \Delta t$  and  $\int_a^{\infty} f^{\sigma}(t) \Delta t$  do not converge/diverge at the same time.

The contrapositive of Theorem 3.1 yields the following oscillation criterion for  $(E_{\lambda})$ .

**Corollary 3.3.** *Let condition (3.1) hold and assume that for every  $y(\cdot) \in \mathcal{L}_w^2$  there exists  $t_0 \in (a, \infty)_{\mathbb{T}}$  such that  $\int_{t_0}^{\infty} w^{\rho}(t) y^2(t) \Delta t < \infty$ . If equation  $(E_{\lambda})$  is in the limit circle case for some  $\lambda \in \mathbb{C}$ , then equation  $(E_{\lambda})$  is oscillatory for all  $\lambda \in \mathbb{R}$ .*

Several limit point criteria for equation  $(E_{\lambda})$  or its delta-nabla counterpart were established in [22, Section 4], [26, Section 4], and [24, Section 3]. In the following example we compare Theorem 3.1, [22, Theorem 4.1], and [26, Theorem 4.2] in the case  $\mathbb{T} = \mathbb{Z}$ . We note that the assumptions of [26, Theorem 4.2] are never satisfied if  $\mathbb{T} = \mathbb{R}$ .

**Example 3.4.** Let  $[a, \infty)_{\mathbb{T}} = [0, \infty)_{\mathbb{Z}}$ . Then equation  $(E_{\lambda})$  corresponds to the second order Sturm–Liouville difference equation

$$-\Delta[p_k \Delta y_k(\lambda)] + q_k y_{k+1}(\lambda) = \lambda w_k y_{k+1}(\lambda), \quad k \in [0, \infty)_{\mathbb{Z}}, \quad (\Delta E_{\lambda})$$

and Theorem 3.1 implies that  $(\Delta E_{\lambda})$  is in the limit point case if it is nonoscillatory for some  $\lambda \in \mathbb{R}$  and

$$\sum_{k=1}^{\infty} \frac{(w_k w_{k-1})^{1/4}}{|p_k|^{1/2}} = \sum_{k=0}^{\infty} \frac{(w_{k+1} w_k)^{1/4}}{|p_{k+1}|^{1/2}} = \infty, \quad (3.2)$$

see also [15, Theorem 5].

- (i) According to [22, Theorem 4.1], equation  $(\Delta E_{\lambda})$  is in the limit point case if it is nonoscillatory for some  $\lambda \in \mathbb{R}$ ,  $p_k < 0$ ,  $q_k > 0$ ,  $w_k \equiv 1$  on  $[0, \infty)_{\mathbb{Z}}$ , and

$$\sum_{k=0}^{\infty} \frac{1}{|p_k|} = \infty. \quad (3.3)$$

If  $\lim_{k \rightarrow \infty} \frac{1}{|p_k|} \neq 0$ , then conditions (3.2) and (3.3) hold simultaneously. On the other hand, if  $\lim_{k \rightarrow \infty} \frac{1}{|p_k|} = 0$ , then  $\frac{1}{|p_k|} < 1$  for all  $k$  large enough, in which case  $\frac{1}{|p_k|} \leq \frac{1}{|p_k|^{1/2}}$ . Hence condition (3.2) can be satisfied, while (3.3) fails, e.g. for  $p_k = -(k+1)^2$ . This shows that Theorem 3.1 yields a stronger criterion.

- (ii) By the criterion in [26, Theorem 4.2], see also [13, Theorem 10], equation  $(\Delta E_{\lambda})$  is in the limit point case if

$$\sum_{k=0}^{\infty} \frac{(w_{k+1} w_k)^{1/2}}{|p_{k+1}|} = \infty. \quad (3.4)$$

Observe that this criterion does not include any oscillatory/nonoscillatory behavior of  $(\Delta E_{\lambda})$  and does not depend on the value of  $q_k$ , i.e., if (3.4) is satisfied, then equation  $(\Delta E_{\lambda})$  is in the limit point case for any choice of  $q_k$ . Since conditions (3.3) and (3.4) coincide in the case  $w_k \equiv 1$ , it follows that [26, Theorem 4.2] yields a stronger criterion than [22, Theorem 4.1].

Similarly as in the previous part, if  $\lim_{k \rightarrow \infty} \frac{(w_{k+1} w_k)^{1/2}}{|p_{k+1}|} \neq 0$ , then conditions (3.4) and (3.2) hold simultaneously, i.e.,  $(\Delta E_{\lambda})$  is in the limit point case whether it is oscillatory or not.

The situation when  $\lim_{k \rightarrow \infty} \frac{(w_{k+1}w_k)^{1/2}}{|p_{k+1}|} = 0$  is more interesting. If condition (3.4) holds, then we obtain the same conclusion as before (again the limit point classification does not depend on  $q_k$ ). But it is also possible that the sum in (3.4) is convergent, while (3.2) is satisfied. For example, let  $p_k \equiv 1$ ,  $q_k \equiv 0$ , and  $w_k = \frac{1}{k^2+1}$ , i.e.,

$$-\Delta^2 y_k(\lambda) = \frac{\lambda}{k^2+1} y_{k+1}(\lambda). \quad (3.5)$$

Then direct calculations show that the sum in (3.4) is convergent, i.e., the assumptions of [26, Theorem 4.2] are not fulfilled, while the sum in (3.2) is divergent. Equation (3.5) with  $\lambda = 0$  has two linearly independent solutions  $y_k^{[1]}(0) \equiv 1$  and  $y_k^{[2]}(0) = k$ , which are obviously nonoscillatory. Therefore the assumptions of Theorem 3.1 are satisfied, which implies that the equation is in the limit point case. This fact can be verified directly, because the solution  $y^{[2]}(0)$  is not square summable with respect to  $w_k$ .

- (iii) Although the criterion of Theorem 3.1 does not include explicitly  $q_k$ , these coefficients play a significant role in contrast to [26, Theorem 4.2]. Let us slightly modify equation (3.5) to the form

$$-\Delta^2 y_k(\lambda) - 2y_{k+1}(\lambda) = \frac{\lambda}{k^2+1} y_{k+1}(\lambda), \quad (3.6)$$

i.e., with  $q_k \equiv -2$ . Observe that the coefficients of (3.5) and (3.6) satisfy (3.2), but equation (3.5) is in the limit point case, while (3.6) is in the limit circle case. Indeed, equation (3.6) has for  $\lambda = 0$  two linearly independent solutions  $y_k^{[1]}(0) = \sin(k\pi/2)$  and  $y_k^{[2]}(0) = \cos(k\pi/2)$ , which are square summable with respect to  $w_k$ , i.e., it is in the limit circle case for all  $\lambda \in \mathbb{C}$  by Theorem 2.9. Note that this conclusion does not contradict the result of Theorem 3.1, because equation (3.6) is oscillatory for  $\lambda = 0$ . In fact, Corollary 3.3 implies that equation (3.6) is oscillatory for all  $\lambda \in \mathbb{R}$ . Similarly we can show that, e.g., the equation

$$-\Delta[(-1)^k \Delta y_k(\lambda)] = \frac{\lambda}{k^2+1} y_{k+1}(\lambda)$$

is oscillatory for all  $\lambda \in \mathbb{R}$ , compare with [3, Theorem 4.51].

As already mentioned, whenever the principal solution  $\tilde{y}(\cdot, \lambda)$  of equation  $(E_\lambda^{\mathbb{R}})$  exists, it is unique up to a nonzero constant multiple. The same is true also for a square integrable solution (being the Weyl solution) of equation  $(E_\lambda)$ , which is in the limit point case. In the final part of this paper we establish an intimate connection between these two solutions.

Let  $\alpha \in [0, \pi)$  be fixed and  $\nu \in \mathbb{R}$  be such that equation  $(E_\nu)$  is nonoscillatory. Then there exists  $t_0 \in [a, \infty)_{\mathbb{T}}$  such that the quotient  $\varphi(t, \nu)/\psi(t, \nu)$  is well defined for all  $t \in [t_0, \infty)_{\mathbb{T}}$ . Moreover, from the fact  $W[\psi(t, \nu), \varphi(t, \nu)] \equiv 1$  and the quotient rule on time scales, see [3, Theorem 1.20], we get

$$\left( \frac{\varphi(t, \nu)}{\psi(t, \nu)} \right)^\Delta = \frac{W[\psi(t, \nu), \varphi(t, \nu)]}{p(t) \psi^\sigma(t, \lambda) \psi(t, \lambda)} = -\frac{1}{p(t) \psi^\sigma(t, \lambda) \psi(t, \lambda)},$$

which upon integrating both sides from  $t_0$  to  $t \in [t_0, \infty)_{\mathbb{T}}$  yields

$$-\frac{\varphi(t, \nu)}{\psi(t, \nu)} = -\frac{\varphi(t_0, \nu)}{\psi(t_0, \nu)} + \int_{t_0}^t \frac{1}{p(\tau) \psi^\sigma(\tau, \lambda) \psi(\tau, \lambda)} \Delta\tau. \quad (3.7)$$

From the positivity of the denominator in the integral on the right-hand side of (3.7) it follows that the limit of the left-hand side exists (finite or infinite) as  $t \rightarrow \infty$ , see also (2.16) and (2.20). In the following theorem we show that if this limit is finite, then the Weyl solution is well defined and it is the principal solution. Moreover, these solutions belong to  $\mathcal{L}_w^2$ , whenever equation  $(E_\nu)$  possesses any square integrable solution. This statement reduces to [5, Theorem 2.13] in the case  $\mathbb{T} = \mathbb{R}$ , otherwise it is new.

**Theorem 3.5.** *Let us assume that equation  $(E_\lambda)$  is in the limit point case,  $\nu \in \mathbb{R}$  is such that equation  $(E_\nu)$  is nonoscillatory, and  $\alpha \in [0, \pi)$  is such that  $\psi(\cdot, \nu)$  is a nonprincipal solution. Then the definition of the Weyl solution  $\mathcal{X}(\cdot, \nu)$  given in (2.14) can be extended to  $(\mathbb{C} \setminus \mathbb{R}) \cup \{\nu\}$  and  $\mathcal{X}(\cdot, \nu)$  is the principal solution of  $(E_\nu)$ . Moreover, if equation  $(E_\nu)$  possesses a square integrable solution, i.e., it holds  $n(\nu) = 1$ , then it is the Weyl solution, i.e.,  $\mathcal{X}(t, \nu) \in \mathcal{L}_w^2$ .*

*Proof.* Since  $\psi(\cdot, \nu)$  is a nonprincipal solution, the integral on the right-hand side of (3.7) is convergent on  $[t_0, \infty)_\mathbb{T}$  by (2.22). Hence one easily conclude from (2.16) and (3.7) that the function  $m_+(\lambda)$  is well defined for all  $\lambda \in (\mathbb{C} \setminus \mathbb{R}) \cup \{\nu\}$  and the Weyl solution  $\mathcal{X}(\cdot, \lambda)$  does as well. If  $m_+(\nu) = 0$ , then  $\varphi(\cdot, \nu)$  is the principal solution of  $(E_\nu)$  and at the same time we get from (2.14) that  $\mathcal{X}(t, \nu) = \varphi(t, \nu)$  for all  $t \in [a, \infty)_\mathbb{T}$ , i.e.,  $\mathcal{X}(t, \nu)$  is also the principal solution of  $(E_\nu)$ . If  $m_+(\nu) = L \neq 0$  and  $y(\cdot, \nu)$  is any solution of  $(E_\nu)$ , which is linearly independent with  $\mathcal{X}(\cdot, \nu)$ , i.e.,  $y(t, \nu) = c_1 \varphi(t, \nu) + c_2 \psi(t, \nu)$  on  $[a, \infty)_\mathbb{T}$  for some  $c_1, c_2 \in \mathbb{R}$  with  $c_2/c_1 \neq L$ , then  $\lim_{t \rightarrow \infty} \mathcal{X}(t, \nu)/y(t, \nu) = 0$ , i.e.,  $\mathcal{X}(t, \nu)$  is (again) the principal solution of  $(E_\nu)$ . Finally, let us assume that  $n(\nu) = 1$  and that  $\mathcal{X}(\cdot, \nu) \notin \mathcal{L}_w^2$ . Then it follows from the previous part that the square integrable solution  $\tilde{y}(\cdot, \nu)$  has to be a nonprincipal solution, i.e.,

$$\lim_{t \rightarrow \infty} \frac{\mathcal{X}(t, \nu)}{\tilde{y}(t, \nu)} = 0.$$

It means that  $|\mathcal{X}(t, \nu)/\tilde{y}(t, \nu)| \leq M$ , i.e.,  $|\mathcal{X}(t, \nu)| \leq M|\tilde{y}(t, \nu)|$  for some  $M > 0$  and all  $t \in [t_1, \infty)_\mathbb{T}$ , where  $t_1 \in [a, \infty)_\mathbb{T}$  is such that  $\tilde{y}(t, \nu) \neq 0$  on  $[t_1, \infty)_\mathbb{T}$ . The existence of such  $t_1$  is guaranteed by the nonoscillatory behavior of  $(E_\nu)$ . But the latter inequality together with the the square integrability of  $\tilde{y}(\cdot, \nu)$  yields

$$\int_{t_1}^{\infty} w(t) \mathcal{X}^{\sigma^2}(t, \nu) \Delta t \leq M^2 \int_{t_1}^{\infty} w(t) \tilde{y}^{\sigma^2}(t, \nu) \Delta t < \infty,$$

i.e., we have also  $\mathcal{X}(\cdot, \nu) \in \mathcal{L}_w^2$ , which contradicts the limit point hypothesis.  $\square$

**Remark 3.6.**

- (i) Let us point out that the assumption of Theorem 3.5 concerning the choice  $\alpha$  is not truly restrictive. If  $\alpha \in [0, \pi)$  is such that  $\psi(\cdot, \nu)$  is the principal solution, then the integral on the right-side of (3.7) is divergent over  $[t_0, \infty)_\mathbb{T}$  by (2.21), i.e., the limit on the right-hide side of (2.16) is infinite. But in that case it suffices to switch the roles of  $\varphi(\cdot, \nu)$  and  $\psi(\cdot, \nu)$ , i.e., to replace  $\alpha$  by  $\tilde{\alpha} := \alpha \pm \pi/2 \in [0, \pi)$ , which yields the pair of the fundamental solutions  $\tilde{\varphi}(t, \nu) := \mp \psi(t, \nu)$  and  $\tilde{\psi}(t, \nu) := \pm \varphi(t, \nu)$ . Then  $\tilde{\psi}(t, \nu)$  is a nonprincipal solution, i.e., the particular assumption of Theorem 3.5 is satisfied for  $\tilde{\alpha}$ , in which case we get  $m_+(\nu) = 0$ .
- (ii) The first part of Theorem 3.5 is even true for any  $\lambda \leq \nu$  such that  $\varphi(\cdot, \lambda)$  is a nonprincipal solution, because equation  $(E_\lambda)$  remains nonoscillatory by the Sturmian theory. Moreover, from [5, Theorem 2.13] one can infer that in the case  $\mathbb{T} = \mathbb{R}$  there exists  $\tilde{\nu} \in \mathbb{R}$  such

that also the square integrability of the Weyl (or principal) solution holds for all  $\lambda < \tilde{\nu}$ . The existence of such number  $\tilde{\nu}$  follows from the characterization of the spectrum of an operator associated with equation  $(E_\lambda)$ . We conjecture that the same conclusion can be established also for equation  $(E_\lambda)$  on any time scale  $\mathbb{T}$ , compare Examples 2.11 and 3.7. The solution of this problem is currently under development.

Finally, the following example illustrates Theorem 3.5 in the case  $\mathbb{T} = \mathbb{Z}$ .

**Example 3.7.** Let  $[a, \infty)_{\mathbb{T}} = [0, \infty)_{\mathbb{Z}}$  and consider the particular case of  $(\Delta E_\lambda)$  given as

$$-\Delta^2 y_k(\lambda) - 2y_{k+1}(\lambda) = \lambda y_{k+1}(\lambda), \quad \text{i.e.,} \quad y_{k+2}(\lambda) + \lambda y_{k+1}(\lambda) + y_k(\lambda) = 0. \quad (3.8)$$

From the characteristic equation  $\nu^2 + \lambda\nu + 1 = 0$  we conclude that two linearly independent solutions  $y_k^{[1]}(\lambda)$  and  $y_k^{[2]}(\lambda)$  of equation (3.8) can be expressed for  $\lambda \in \mathbb{R}$  as follows

$$\begin{aligned} \lambda = -2 : & \quad y_k^{[1]}(\lambda) \equiv 1, & \quad y_k^{[2]}(\lambda) = k, \\ \lambda = 2 : & \quad y_k^{[1]}(\lambda) = (-1)^k, & \quad y_k^{[2]}(\lambda) = k(-1)^k, \\ \lambda \in (-2, 2) : & \quad y_k^{[1]}(\lambda) = \cos(k\omega), & \quad y_k^{[2]}(\lambda) = \sin(k\omega), \\ \lambda \in \mathbb{R} \setminus [-2, 2] : & \quad y_k^{[1]}(\lambda) = \gamma_+^k, & \quad y_k^{[2]}(\lambda) = \gamma_-^k, \end{aligned}$$

where  $\omega := \arg\left(\frac{-\lambda + i\sqrt{4-\lambda^2}}{2}\right)$  and  $\gamma_\pm := \frac{-\lambda \pm \sqrt{\lambda^2 - 4}}{2}$ . Thus one observes that equation (3.8) is nonoscillatory only for  $\lambda \leq -2$ , while  $n(\lambda) = 1$  for  $\lambda \in \mathbb{R} \setminus [-2, 2]$  and  $n(\lambda) = 0$  for  $\lambda \in [-2, 2]$ . Moreover, the principal solution corresponds to  $y^{[1]}(-2)$  and to  $y^{[2]}(\lambda)$  for  $\lambda < -2$ . Now let  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ . Then the solutions of the characteristic equation can be written as

$$\gamma_\pm = \frac{-\lambda \pm \sqrt{\lambda^2 - 4}}{2} = -\frac{\lambda}{2} \pm \frac{1}{2} \left( \sqrt{\frac{c}{2}} + \frac{\sqrt{2ab}}{c} \right),$$

where  $a := \operatorname{Re} \lambda$ ,  $b := \operatorname{Im} \lambda$ , and  $c := a^2 - b^2 - 4 + \sqrt{(a^2 - b^2 - 4)^2 + 4a^2b^2}$ . Note that we have  $\gamma_+\gamma_- = 1$  and  $\gamma_+ \neq \overline{\gamma_-}$ , because  $b \neq 0$ . In addition,  $|\gamma_+| < |\gamma_-|$  if  $a > 0$  or  $a = 0$  and  $b > 0$ , while  $|\gamma_+| > |\gamma_-|$  if  $a < 0$  or  $a = 0$  and  $b < 0$ . If  $\alpha = \pi/2$ , the two linearly independent solutions  $\varphi(\lambda)$  and  $\psi(\lambda)$  determined by the initial conditions in (2.15) are

$$\varphi_k(\lambda) = [\gamma_+^k(\gamma_- - 1) + \gamma_-^k(1 - \gamma_+)] / (\gamma_- - \gamma_+) \quad \text{and} \quad \psi_k(\lambda) = (\gamma_-^k - \gamma_+^k) / (\gamma_- - \gamma_+).$$

Hence, by (2.16), we get the function  $m_+(\lambda) : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$  as

$$m_+(\lambda) = \begin{cases} \frac{1 - \gamma_-}{\gamma_-} & \text{if } a > 0 \text{ or } a = 0 \text{ and } b > 0, \\ \frac{1 - \gamma_+}{\gamma_+} & \text{if } a < 0 \text{ or } a = 0 \text{ and } b < 0, \end{cases}$$

which is true even for  $\lambda \in \mathbb{C} \setminus (-2, 2)$ , because the square root is well defined on  $\mathbb{C} \setminus (-\infty, 0)$ . Consequently, the Weyl solution is for any  $\lambda \in \mathbb{C} \setminus (-2, 2)$  given by

$$\mathcal{X}(t, \lambda) = \begin{cases} \gamma_+^k & \text{if } a > 0 \text{ or } a = 0 \text{ and } b > 0, \\ \gamma_-^k & \text{if } a < 0 \text{ or } a = 0 \text{ and } b < 0. \end{cases}$$



This shows that  $\mathcal{X}_k(-2) = y_k^{[1]}(-2)$  and  $\mathcal{X}_k(\lambda) = y_k^{[2]}(\lambda)$  for  $\lambda < -2$ , i.e., the Weyl solution coincides with the principal solution for all  $\lambda \in (-\infty, -2]$  and it is square summable for  $\lambda \in (-\infty, -2)$  as stated in Theorem 3.5, see also Remark 3.6(ii).

Let us point out yet another interesting fact. Although  $\lim_{k \rightarrow \infty} y_k^{[1]}(\lambda)/y_k^{[2]}(\lambda) = 0$  also in the case  $\lambda > 2$ , the solution  $y_k^{[1]}(\lambda)$  is not principal, because equation (3.8) is oscillatory for all  $\lambda > -2$ . However the previous calculations (again) yield  $\mathcal{X}(t, \lambda) = y_k^{[1]}(\lambda) \in \mathcal{L}_w^2$  for any  $\lambda \in (2, \infty)$ .

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