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Necessary conditions for a reaction–diffusion system with delay to preserve positivity

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Abstract. We consider the reaction–diffusion system with delay

$$\begin{cases} \dfrac{\partial u}{\partial t} = A(t,x)\Delta u - \displaystyle\sum_{i=1}^{k} \gamma_i(t,x)\partial_{x_i} u + f(t,u_t), \quad x \in \Omega; \\ B(u)|_{\partial\Omega} = 0. \end{cases}$$

We show that this system with delay preserves positivity if and only if its diffusion matrix A and convection matrix γ_i are diagonal with non-negative elements and nonlinear delay term f satisfies the normal tangential condition.

Keywords: positivity, monotonicity, reaction–diffusion equation with delay.

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1 Introduction

Consider the following initial-boundary value problem (IBVP) of reaction–diffusion equations with delay

$$\begin{cases} \dfrac{\partial u}{\partial t} = A(t,x)\Delta u - \displaystyle\sum_{i=1}^{k} \gamma_i(t,x)\partial_{x_i} u + f(t,u_t), \quad x \in \Omega; \\ B(u)|_{\partial\Omega} = 0; \\ u_{t_0}(\theta,x) = \varphi(\theta,x), \quad \varphi \in C([-\tau,0]\times\overline{\Omega}, R^n) \end{cases} \tag{1.1}$$

where we assume:

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(A.1). $A(t,x)$ and $\gamma_i(t,x)$ are $n \times n$ matrices, with each element in $C(R \times \overline{\Omega}, R)$ and $\Omega \subseteq R^k$ is an open bounded domain;

(A.2). u_t is defined by $u_t(\theta,x) = u(t+\theta,x)$ for any $t \geq t_0$ and $\theta \in [-\tau,0]$, where t_0 is the initial time, τ is a positive number and $u(t,x)$ is a solution of (1.1);

(A.3). f is a continuous and locally Lipschitz mapping from $R \times C([-\tau,0],R^n)$ to R^n;

(A.4). The boundary condition is given by $B(u)(t,x) = a(x)u(t,x) + b(x)\frac{\partial u}{\partial n}(t,x)$ for any $t > t_0$, where

$$a(x) = \mathrm{diag}(a_1(x),\dots,a_n(x)), \qquad b(x) = \mathrm{diag}(b_1(x),\dots,b_n(x))$$

with each element $a_i, b_i \in C(\overline{\Omega}, \overline{R^{n+}})$.

It is well known that solutions of IBVP (1.1) starting from nonnegative initial conditions remain nonnegative under the assumptions that the diffusion matrix is diagonal and the kinetics f satisfies a certain sub-tangential condition with respect to a cone of nonnegative functions. See, for example, results for ordinary delay differential equations (Smith [11] and Seifert [12]), for parabolic equations (Weinberger [13]), and for abstract functional differential equations including delayed reaction–diffusion equations (Martin and Smith [7,8], Ruess [9] and Summers [10]). Nonnegative properties are of course one of the fundamental behaviours of any dynamical model arising from biological systems if the state variables u represent the densities of the biological species involved. On the other hand, sufficient conditions for solutions to preserve nonnegative property can easily be extended to generate a certain monotonicity (order-preserving property) of the solutions which turns out to have significant implications for the global dynamics of the generated solution semiflows (Hirsch [2–6]).

It is natural to ask if these commonly used sufficient conditions are necessary as well, and to our best knowledge very little has been done in the literature except the work reported in Efendiev [1]. Here we confirm that these conditions are indeed necessary by constructing explicitly negative solutions with nonnegative initial conditions when these conditions are not met. This confirmation obviously provides a convenient first step to disprove any proposed mathematical model arising from population dynamics if the state variables are population densities. We also show how to use this necessary condition to identify *primitive* state variables, through a standard linear transformation, of any correct mathematical models when they fail to meet the necessary conditions.

2 Main results

We start with recalling a few notations and notions.

Definition 2.1. A set $K^+ \subseteq X$ is called a positive cone if

1. K^+ is closed;

2. $\alpha x \in C$ for all $x \in C$ and $\alpha \in R^+$;

3. $K^+ \bigcap (-K^+) = \{0\}$.

We will write $x \geq 0$ if $x \in K^+$.

In this paper, we will use various cones as follows.

1. If $X = R^n$, we choose $R^{n+} := \{x = (x_1, \dots, x_n) \in R^n, x_i \geq 0, 1 \leq i \leq n\}$ as a positive cone.

2. If $X = C(\overline{\Omega}) := C(\overline{\Omega}, R)$ is equipped with the maximal norm, where Ω is a bounded domain in R^k, then we choose $C^+(\overline{\Omega}) = C(\overline{\Omega}, R^+)$ as a positive cone of X.

3. If $X = C([-\tau, 0] \times \overline{\Omega}, R^n)$ with the norm
$$\|u\|_{max} = \max_{i=1}^{n} \max_{\theta \in [-\tau,0], x \in \overline{\Omega}} \{u^i(\theta, x)\},$$
where Ω is a bounded domain in R^k and $\tau \geq 0$, then a positive cone is $C^+([-\tau, 0] \times \overline{\Omega}, R^{n+})$. Similarly, we define $C^+[-\tau, 0]$.

In what follows, we will say that a closed subset S in a chosen phase space X for IBVP (1.1) is *totally positively invariant set* if the solution $u_t \in S$ for all $t \geq t_0$ as long as $u_{t_0} \in S$ and the solution is defined at $t \geq t_0$. The *positivity property* of IBVP (1.1) refers to the property that the positive cone is a totally positively invariant set.

We can now state our main result.

Theorem 2.2. *The IBVP (1.1) satisfies the positivity property if and only if the following conditions hold:*

(i) $A(t, x) = \mathrm{diag}\{a_1(t, x), a_2(t, x), \dots, a_n(t, x)\}$ *with* $a_i(t, x) \geq 0$ *for all* $t \in R$ *and* $x \in \overline{\Omega}$, $i \in \{1, 2, \dots, n\}$;

(ii) $\gamma_l(t, x) = \mathrm{diag}\{\gamma_l^1(t, x), \gamma_l^2(t, x), \dots, \gamma_l^n(t, x)\}$, $l \in \{1, 2, \dots, k\}$;

(iii) for all $t \in R$ *and* $\psi \in C^+[-\tau, 0]$ *with* $\psi_i(0) = 0$, $f_i(t, \psi(\theta)) \geq 0$.

Proof. We only prove the necessity and refer to the aforementioned references for the proof of sufficiency. Assume that an initial data $u_{t_0} \in C^+([-\tau, 0] \times \overline{\Omega}, R^n)$ with $t_0 \in R$ is given so that the solution $u(t, x, u_{t_0}) \geq 0$ as long as it exists.

We can see that $u_{t_0}(0, \cdot) \in C^+(\overline{\Omega}, R^n) \subset L^2(\overline{\Omega}, R^n)$. Define the inner product of the function space $L^2(\overline{\Omega}, R^n)$ by
$$\langle \tilde{u}, \tilde{v} \rangle_{L^2(\overline{\Omega}, R^n)} = \sum_{i=1}^{n} \int_{\Omega} \tilde{u}_i \tilde{v}_i dx,$$
where u_i, v_i are the i-th component of the vector $\tilde{u}, \tilde{v}$. Then,
$$\langle u_{t_0}(0, \cdot), v(\cdot) \rangle_{L^2(\overline{\Omega}, R^n)} = \sum_{i=1}^{n} \int_{\Omega} u_{t_0}^i(0, x) v_i(x) dx,$$
for any vector $v \in L^2_+(\overline{\Omega}, R^n)$. Consequently, $\langle \cdot, v \rangle_{L^2(\overline{\Omega}, R^n)}$ is a positive linear functional of $L^2(\overline{\Omega}, R^n)$. Consider the action of this functional on the derivative of solution $u(t, x, u_{t_0})$, we have
$$\begin{aligned}\left\langle \frac{\partial u(t,\cdot,u_{t_0})}{\partial t}\Big|_{t=t_0}, v \right\rangle_{L^2} &= \lim_{t \to t_0^+} \left\langle \frac{u(t,\cdot,u_{t_0}) - u(t_0,\cdot,u_{t_0})}{t}, v \right\rangle_{L^2} \\ &= \lim_{t \to t_0^+} \left\langle \frac{u(t,\cdot,u_{t_0})}{t}, v \right\rangle_{L^2} - \left\langle \frac{u(t_0,\cdot,u_{t_0})}{t}, v \right\rangle_{L^2}.\end{aligned}$$
If $\left\langle \frac{u(t_0,\cdot,u_{t_0})}{t}, v \right\rangle_{L^2} = 0$, i.e, $\langle u_{t_0}(0, \cdot), v \rangle_{L^2(\overline{\Omega}, R^n)} = 0$, then
$$\left\langle \frac{\partial u(t,\cdot,u_{t_0})}{\partial t}\Big|_{t=t_0}, v \right\rangle_{L^2} = \lim_{t \to t_0^+} \left\langle \frac{u(t,\cdot,u_{t_0})}{t}, v \right\rangle_{L^2} \geq 0,$$

where we used that the solution $u(t,\cdot,u_{t_0}) \geq 0$ due to necessary. It then follows from equation (1.1) that

$$\left\langle \frac{\partial u(t,\cdot,u_{t_0})}{\partial t}\Big|_{t=t_0}, v\right\rangle_{L^2} = \langle A(t_0,\cdot)\Delta u_{t_0}(0,\cdot) - \gamma(t_0,\cdot)\nabla u_{t_0}(0,\cdot) + f(t_0,u_{t_0}(\theta,\cdot)), v\rangle_{L^2} \geq 0. \quad (2.1)$$

We now choose initial data $u_{t_0}(\theta,\cdot) = (0,\ldots,u^i_{t_0}(\theta,\cdot),\ldots,0)$ and the vector $v = (0,\ldots,v^j(\cdot),\ldots,0)$ with $j \neq i$ and $u^i_{t_0}(\theta,\cdot), v^j(\cdot) \geq 0$, then

$$\langle u_{t_0}(0,\cdot), v\rangle_{L^2(\overline{\Omega},R^n)} = \int_{\overline{\Omega}} u^i_{t_0}(0,\cdot)\cdot 0dx + \int_{\overline{\Omega}} 0\cdot v^j(x)dx = 0.$$

From equation (2.1), we obtain

$$\left\langle a_{ji}(t_0,\cdot)\Delta u^i_{t_0}(0,\cdot) - \sum_{l=1}^{n} \gamma^{ji}_l(t_0,\cdot)\partial_l u^i_{t_0}(0,\cdot) + f_j(t_0,0,\ldots,0,u^i_{t_0}(\theta,\cdot),0,\ldots,0),\ v^j\right\rangle_{L^2} \geq 0, \quad (2.2)$$

for any $v^j \geq 0$. Since v^j is an arbitrary non-negative function, it follows from (2.2) that the following pointwise inequality

$$a_{ji}(t_0,x)\Delta u^i_{t_0}(0,x) - \sum_{l=1}^{n} \gamma^{ji}_l(t_0,x)\partial_l u^i_{t_0}(0,x) + f_j(t_0,0,\ldots,0,u^i_{t_0}(\theta,x),0,\ldots,0) \geq 0$$

holds for almost all $x \in \overline{\Omega}$. By the continuity of the left hand of the inequality above, we know $a_{ji}(t_0,x)\Delta u^i_{t_0}(0,x) - \sum_{l=1}^{n} \gamma^{ji}_l(t_0,x)\partial_l u^i_{t_0}(0,x) + f_j(t_0,0,\ldots,0,u^i_{t_0}(\theta,x),0,\ldots,0) \geq 0$ is true for all $x \in \overline{\Omega}$.

In order to obtain the condition for a_{ji}, we need to choose a family of special positive functions $u^i_{t_0}(\cdot,\cdot,\varepsilon) \in C^+([-\tau,0]\times\overline{\Omega},R)$ to take off the term $\sum_{l=1}^{n} \gamma^{ji}_l(t_0,x_0)\partial_l u^i_{t_0}(0,x_0)$ at some point $x_0 \in \Omega$. We may choose the functions $u^i_{t_0}(\theta,x,\varepsilon)$ such that:

1. they attain their maximum at $\theta = 0$ and $x_0 \in \Omega$;
2. their second derivative of them can achieve an given $\theta = 0$ and x_0 as ε varies;
3. $B(u^i_{t_0}(\theta,x,\varepsilon))|_{\partial\Omega} = 0$.

Now we begin to construct the family of functions. Firstly, let $w^i_{t_0}(\theta,x,\varepsilon) = e^{\frac{-1}{\varepsilon}(x^1-x^1_0)^2+\theta}$, where $\varepsilon \in R$. By calculation,

$$\nabla w^i_{t_0}(\theta,x,\varepsilon) = \left(\frac{-2(x^1-x^1_0)}{\varepsilon}e^{\frac{-1}{\varepsilon}(x^1-x^1_0)^2+\theta}, 0,\ldots,0\right)$$

and

$$\Delta w^i_{t_0}(\theta,x,\varepsilon) = \left(-\frac{2}{\varepsilon}e^{-\frac{1}{\varepsilon}(x^1-x^1_0)^2+\theta} + \frac{4}{\varepsilon^2}(x^1-x^1_0)^2 e^{-\frac{1}{\varepsilon}(x^1-x^1_0)^2+\theta}, 0,\ldots,0\right).$$

Consequently, $\nabla w^i_{t_0}(0,x_0,\varepsilon) = (0,\ldots,0)$ and $\Delta w^i_{t_0}(0,x_0,\varepsilon) = (-\frac{2}{\varepsilon},0,\ldots,0)$. Since Ω is an open bounded domain in R^k, $\partial\Omega$ is a compact subset of R^k. Then we can define $d_{x_0} = \min_{x\in\partial\Omega}\sum_{i=1}^{k}(x^i - x^i_0)^2$ for any $x_0 \in \Omega$. It is easy to see $d_{x_0} > 0$. Next, we construct a non-negative cut-off function $g(x) \in C^\infty(\overline{\Omega})$ such that $g(x) \equiv 1$ for any $x \in B_{x_0}(\frac{d_{x_0}}{3})$ and $g(x) \equiv 0$ for any $x \notin B_{x_0}(\frac{2d_{x_0}}{3})$. Let

$$g_1(t) = \begin{cases} \exp\left\{\dfrac{1}{\left(t-\frac{d^2_{x_0}}{9}\right)\left(t-\frac{4d^2_{x_0}}{9}\right)}\right\}, & t \in \left(\frac{d^2_{x_0}}{9}, \frac{4d^2_{x_0}}{9}\right) \\ 0, & t \notin \left(\frac{d^2_{x_0}}{9}, \frac{4d^2_{x_0}}{9}\right) \end{cases}$$

and $g_2(t) = \frac{\int_t^{+\infty} g_1(s)ds}{\int_{-\infty}^{+\infty} g_1(s)ds}$. We can see that

$$g_2(t) = \begin{cases} 1, & t \le \frac{d_{x_0}^2}{9} \\ 0, & t \ge \frac{4d_{x_0}^2}{9}. \end{cases}$$

Then $g(x) = g_2(|x - x_0|^2)$ is the cut-off function we need. Finally, we get the family of functions $u_{t_0}^i(\theta, x, \varepsilon) = g(x) w_{t_0}^i(\theta, x, \varepsilon)$.

Then we have the following inequality

$$\begin{aligned} &a_{ji}(t_0, x_0)\Delta u_{t_0}^i(0, x_0) - \sum_{l=1}^{n} \gamma_l^{ji}(t_0, x_0)\partial_l u_{t_0}^i(0, x_0) + f_j(t_0, 0, \dots, 0, u_{t_0}^i(\theta, x_0), 0, \dots, 0) \\ &\quad = -\frac{2}{\varepsilon} a_{ji}(t_0, x_0) + f_j(t_0, 0, \dots, 0, e^{\theta}, 0, \dots, 0) \ge 0. \end{aligned}$$

As ε can be chosen arbitrarily small for any given $t_0 \in R$ and $x_0 \in \Omega$ and $a_{ji} \in C(R \times \overline{\Omega}, R)$, equation (2.2) implies that $a_{ji}(t, x) = 0$, $j \neq i$ must be satisfied.

Next, we consider the term $\gamma_l^{ji}(t_0, x)$ for $j \neq i$. Let $u_{t_0}^i(\theta, x) = g(x) e^{-\frac{1}{\varepsilon}(x^l - x_0^l) + \theta}$, then $\nabla u_{t_0}^i(0, x_0) = (0, \dots, 0, -\frac{1}{\varepsilon}, 0, \dots, 0)$. Hence,

$$\begin{aligned} &a_{ji}(t_0, x_0)\Delta u_{t_0}^i(0, x_0) - \sum_{l=1}^{n} \gamma_l^{ji}(t_0, x_0)\partial_l u_{t_0}^i(0, x_0) + f_j(t_0, 0, \dots, 0, u_{t_0}^i(\theta, x_0), 0, \dots, 0) \\ &\quad = -\frac{1}{\varepsilon} \gamma_l^{ji}(t_0, x_0) + f_j(t_0, 0, \dots, 0, e^{\theta}, 0, \dots, 0) \ge 0. \end{aligned}$$

Since $\varepsilon \in R$ is arbitrary for any given $t_0 \in R$ and $x_0 \in \Omega$ and the continuity of γ_l^{ji} in the set $R \times \overline{\Omega}$, it is clear that $\gamma_l^{ji}(t, x) = 0$ for any $i \neq j$.

Now, we verify the sign of $a_{ii}(t, x)$. If $a_{ii}(t_0, x_0) < 0$ at some time t_0 and point $x_0 \in \Omega$, let $u_{t_0}(\theta, x) = (0, \dots, u_{t_0}^i(\theta, x), \dots, 0)$, where

$$u_{t_0}^i(\theta, x) = g(x)\left(e^{\frac{(x^1 - x_0^1)^2}{\varepsilon} - \theta} - 1\right) \ge 0$$

with $\varepsilon > 0$ for $\theta \in [-\tau, 0]$. Then we have $\frac{\partial u(t_0, x_0)}{\partial t} = a_{ii}(t_0, x_0)\frac{2}{\varepsilon} + f(t_0, 0, \dots, e^{-\theta}, 0, \dots, 0)$. It is easy to see that $\frac{\partial u(t_0, x_0)}{\partial t} < 0$ if ε is small enough. Notice $u(t_0, x_0) = u_{t_0}(0, x_0) = 0$, then there exists a positive number $\delta > 0$ such that $u(t, x_0) < 0$ for any $t \in [t_0, t_0 + \delta]$, a contradiction. So, by the continuity of $a_{ii}(t, x)$, $a_{ii}(t, x) \ge 0$ for any $t \in R$, $x \in \overline{\Omega}$.

Finally, we show $f_i(t, \psi(\theta)) \ge 0$ for any $\psi(\theta) \in C^+[-\tau, 0]$ with $\psi_i(0) = 0$ and any time t. Indeed, taking $A(t, x) = \mathrm{diag}(a_1(t, x), a_2(t, x), \dots, a_n(t, x))$ and $\gamma_l(t, x) = \mathrm{diag}(\gamma_l^1(t, x), \gamma_l^2(t, x), \dots, \gamma_l^n(t, x))$, $l = 1, 2, \dots, k$ into account, for pair $u_{t_0} = (u_{t_0}^1, u_{t_0}^2, \dots, u_{t_0}^i, \dots, u_{t_0}^n)$ satisfying $u_{t_0}(\theta, \cdot) \equiv \psi(\theta)$, and $v = (0, \dots, 0, v^i, 0, \dots, 0)$ with $v^i \ge 0$, from (2.1) we obtain that $f_i(t_0, u_{t_0}^1, \dots, u_{t_0}^i, \dots, u_{t_0}^n) \ge 0$, i.e., for any $t \in R$ and $\psi(\theta) \in C^+[-\tau, 0]$ with $\psi_i(0) = 0$, $f_i(t, \psi(\theta)) \ge 0$ for any $t \in R$. □

Remark 2.3. The case where $\tau = 0$, the diffusion and convection matrix of (1.1) and mapping f of (1.1) are all independent on time t, we get a reaction–diffusion equation. Theorem 2.2 is obtained in [1] when we further assume that the matrices γ_l, A are $(n \times n)$-matrices with constant coefficients.

Remark 2.4. If the boundary value condition of (1.1) is that $B(u)|_{\partial\Omega} = g(x)$, where $g \in C(\partial\Omega)$, then the necessary and sufficient condition satisfies to keep positiveness is that same as the one in Theorem 2.2 in addition to the following condition: $g(x) \geq 0$ for $x \in \partial\Omega$. To prove this, in the argument for Theorem 2.2, we need to use the cut-off function to find the special initial data u_{t_0} such that $B(u_{t_0})(0,\cdot)|_{\partial\Omega} = g(x)$ and $u_{t_0}(0,\cdot)|_{B_{x_0}(2\varepsilon)\setminus B_{x_0}(\varepsilon)} \equiv 0$, where the open ball $B_{x_0}(2\varepsilon)$ is a proper subset of Ω and $\varepsilon > 0$.

Remark 2.5. For equations (1.1) with non-homogeneous boundary conditions in Remark 2.4, we assume that the diffusion matrix $A(t,x)$ can be diagonalized, that means there exists a reversible matrix $P(t,x)$ such that $P^{-1}AP = J$ for any $t \in R$, $x \in \overline{\Omega}$, where J is a diagonal matrix. Then the necessary and sufficient conditions for the set $PC^+([-\tau,0] \times \overline{\Omega}, R^n) = \{\phi \in C([-\tau,0] \times \overline{\Omega}, R^n) \mid \phi(\theta,x) \in P\overline{R^{n+}}$, for any $\theta \in [-\tau,0], x \in \overline{\Omega}\}$ totally positively invariant are that :

(1′) each element of J is equal to or greater than 0;

(2′) $P^{-1}\gamma_l P = \mathrm{diag}\{\gamma_l^1(t,x), \gamma_l^2(t,x), \dots, \gamma_l^n(t,x)\}$, $l \in \{1,2,\dots,k\}$;

(3′) for any $t \in R$ and $\psi \in C^+[-\tau,0]$ with $\psi_i(0) = 0$, the mapping $F_i(t,\psi) = P^{-1}f_i(t,P\psi) \geq 0$.

Therefore, we conclude that Pu rather than u should be the "prime" variable.

Remark 2.6. Assume that $u_1(t,x), u_2(t,x)$ are solutions of equations (1.1) satisfying that $u_1(t_0+\theta,x) \geq u_2(t_0+\theta,x)$ for any $x \in \Omega$ and $\theta \in [-\tau,0]$. Let $w(t,x) = u_1(t,x) - u_2(t,x)$, it then follows from (1.1) that

$$\begin{cases} \dfrac{\partial w}{\partial t} = A(t,x)\Delta w - \displaystyle\sum_{i=1}^{k} \gamma_i(t,x)\partial_{x_i} w + f(t,u_{1t}) - f(t,u_{2t}), \quad x \in \Omega, \\ B(w)|_{\partial\Omega} = 0. \end{cases}$$

If mapping f is smooth enough, then $f(t,u_{1t}) - f(t,u_{2t}) = \int_0^1 Df(t, su_{1t} + (1-s)u_{2t})ds \cdot w$.

Consider the following system

$$\begin{cases} \dfrac{\partial v}{\partial t} = A(t,x)\Delta v - \displaystyle\sum_{i=1}^{k} \gamma_i(t,x)\partial_{x_i} v + \int_0^1 Df(t, su_{1t} + (1-s)u_{2t})ds \cdot v, \\ B(v)|_{\partial\Omega} = 0, \\ v_0 = \varphi \in C([-\tau,0] \times \overline{\Omega}, R^n). \end{cases} \tag{2.3}$$

Then equations (2.3) preserving positivity if only if

(a). $A = \mathrm{diag}\{a_1(t,x), a_2(t,x), \dots, a_n(t,x)\}$ with $a_i(t,x) \geq 0$ for any $t \in R$ and $x \in \overline{\Omega}$, $i \in \{1,2,\dots,n\}$;

(b). $\gamma_l = \mathrm{diag}\{\gamma_1^l(t,x), \gamma_2^l(t,x), \dots, \gamma_n^l(t,x)\}$, $l \in \{1,2,\dots,k\}$;

(c). for any $t \in R$ and $\psi \in C^+[-\tau,0]$ with $\psi_i(0) = 0$, $\int_0^1 Df_i(t, su_{1t} + (1-s)u_{2t})ds \cdot \psi \geq 0$.

Since $w(t,x) = u_1(t,x) - u_2(t,x)$ is a special solution of (2.3) satisfying $w(t_0+\theta,x) \geq 0$ for any $\theta \in [-\tau,0]$, $u_1(t,x) \geq u_2(t,x)$ for any $t \geq t_0$ if (a), (b), (c) holds. In fact, if we just require that the special solution $w(t,x)$ remains non-negative for $t > 0$, the condition (c) can be replaced by (c′) below:

(c′) for any $t \in R$ and $u_{1t} \geq u_{2t}$ with $u_{1t}(0) = u_{2t}(0)$, $\int_0^1 Df_i(t, su_{1t} + (1-s)u_{2t})ds \cdot w = f_i(t,u_{1t}) - f_i(t,u_{2t}) \geq 0$. Therefore, we are naturally led to (c″), for any $t \in R$ and $\phi \geq \psi$ with $\phi(0) = \psi(0)$, $f_i(t,\phi) - f_i(t,\psi) \geq 0$.

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