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Exact boundary behavior for the solutions to a class of infinity Laplace equations

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Abstract. In this paper, by Karamata regular variation theory and the method of lower and upper solutions, we give an exact boundary behavior for the unique solution near the boundary to the singular Dirichlet problem $-\Delta_\infty u = b(x)g(u)$, $u > 0$, $x \in \Omega$, $u|_{\partial\Omega} = 0$, where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $g \in C^1((0,\infty),(0,\infty))$, g is decreasing on $(0,\infty)$ and the function $b \in C(\bar{\Omega})$ which is positive in Ω. We find a new structure condition on g which plays a crucial role in the boundary behavior of the solutions.

Keywords: infinity-Laplacian, singular Dirichlet problem, the exact asymptotic behavior, lower and upper solutions.

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1 Introduction and the main results

Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^N$ $(N \geq 2)$. In this paper, we consider the exact asymptotic behavior near the boundary to the following singular Dirichlet problem

$$-\Delta_\infty u = b(x)g(u), \qquad u > 0,\ x \in \Omega, \qquad u|_{\partial\Omega} = 0, \tag{1.1}$$

where the operator Δ_∞ is the ∞-Laplacian, and it is defined as

$$\Delta_\infty u := \langle D^2 u Du, Du\rangle = \sum_{i,j=1}^{N} D_i u D_{ij} u D_j u, \tag{1.2}$$

b satisfies

$(\mathbf{b_1})$ $b \in C(\bar{\Omega})$ is positive in Ω,

and g satisfies

$(\mathbf{g_1})$ $g \in C^1((0,\infty),(0,\infty))$, $\lim_{s\to 0^+} g(s) = \infty$ and g is decreasing on $(0,\infty)$.

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This operator (1.2) is called the infinity Laplacian, which was first introduced in the work of Aronsson [2] in connection with the geometric problem of finding the so-called absolutely minimizing functions in Ω. As a result of the high degeneracy of the ∞-Laplacian, the associated Dirichlet problems may not have classical solutions. Therefore solutions are understood in the viscosity sense, a concept introduced by Crandall, Lions [13] and Crandall, Evans, Lions [12], and to be defined in Section 2. By using the viscosity solutions, Jensen [21] proved the existence and uniqueness of the viscosity solutions to the Dirichlet problem to the infinity harmonic equation. Later, Lu and Wang [23] obtained a uniqueness theorem for the Dirichlet problem to the infinity harmonic equation in the perspective of PDE. The infinity Laplace equation in turn is a very topical differential operator that appears in many contexts and has been extensively studied, see, for instance, [3, 5, 6, 11, 24–26, 29, 32, 33, 40] and the references therein.

Next, let us review the following singular elliptic boundary value problem involving the classical Laplace operator Δ, i.e.

$$-\Delta u = b(x)g(u), \qquad u > 0,\ x \in \Omega, \qquad u|_{\partial\Omega} = 0. \tag{1.3}$$

Problem (1.3) arises in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, as well as in the theory of heat conduction in electrical materials and has been discussed and extended by many authors in many contexts, for instance, the existence, uniqueness, regularity and boundary behavior of solutions, see, [1, 4, 14–20, 22, 28, 30, 34, 35, 41–43, 45, 46] and the references therein.

The pioneering work of problem (1.3) is Crandall, Rabinowitz, Tartar [14] and Fulks, Maybee [15]. For $b \equiv 1$ in Ω and g satisfying (g_1), [14] and [15] derived that problem (1.3) has a unique solution $u \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$. Moreover, in [14], the following result was established: if $\psi_1 \in C[0, \delta_0] \cap C^2(0, \delta_0]$ is the local solution to the problem

$$-\psi_1''(t) = g(\psi_1(t)), \qquad \psi_1(t) > 0, \qquad 0 < t < \delta_0, \qquad \psi_1(0) = 0, \tag{1.4}$$

then there exist positive constants c_1 and c_2 such that

$$c_1\psi_1(d(x)) \le u(x) \le c_2\psi_1(d(x)) \quad \text{near } \partial\Omega.$$

In particular, when $g(u) = u^{-\gamma}$, $\gamma > 1$, u has the property

$$c_1(d(x))^{2/(1+\gamma)} \le u(x) \le c_2(d(x))^{2/(1+\gamma)} \quad \text{near } \partial\Omega. \tag{1.5}$$

By constructing global subsolutions and supersolutions, Lazer and McKenna [22] showed that (1.5) continued to hold on $\bar{\Omega}$. Then, $u \in H_0^1(\Omega)$ if and only if $\gamma < 3$. This is a basic characteristic of problem (1.3).

It is very worthwhile to point out that Cîrstea and Rădulescu [8–10] introduced the Karamata regular variation theory which is a basic tool in stochastic process to study the boundary behavior and uniqueness of solutions to boundary blow-up elliptic problems and obtained a series of rich and significant information about the boundary behavior of solutions. For further insight on the boundary blow-up elliptic problems, please refer to [36, 37, 44] and the references therein.

Later, by means of Karamata regular variation theory, Zhang et al. [42, 43, 45, 46] proved the first or second boundary expansion of solutions to problem (1.3). The author et al. [28, 30] further proved the second boundary expansion of solutions to problem (1.3).

Ben Othman, Maagli, Masmoudi, Zribi [4] and Gontara, Maagli, Masmoudi, Turki [19] introduced a large class of functions $b(x)$ which belong to the Kato class $K(\Omega)$ and proved the boundary behavior of solutions for problem (1.1) when g is normalized regularly varying at zero with index $-\gamma$ ($\gamma > 0$). Later, Zhang et al. [45] extend the previous results on the boundary behavior of the solution u of problem (1.1) to the case where the weight functions $b(x)$ belong to the Kato class $K(\Omega)$ or $b(x)$ lie into a class of functions Λ that was introduced by Cîrstea and Rădulescu in [8–10] for non-decreasing functions and by Mohammed in [31] for nonincreasing functions as the set of positive monotonic functions $C^1(0,\delta_0) \cap L^1(0,\delta_0)$ ($\delta_0 > 0$) which satisfy

$$\lim_{t\to 0^+} \frac{d}{dt}\left(\frac{K(t)}{k(t)}\right) =: C_k \in [0,\infty), \qquad K(t) = \int_0^t k(s)ds. \tag{1.6}$$

Recently, N. Zeddini et al. [41] gave a common proof for theorems in [45] and extended these results.

Now let us return to problem (1.1).

When Ω is a bounded domain that satisfies both the uniform interior and uniform exterior sphere conditions and $b \equiv 1$ in Ω, Bhattacharya and Mohammed [5] established that: let g satisfy (g_1) and u be a solution of (1.1), then there are positive constants a and c, with $0 < a < c$ such that

$$\psi_a^{-1}(\sqrt{2}d(x)) \leq u(x) \leq \psi_c^{-1}(\sqrt{2}d(x))$$

where $d(x)$ is the distance of x from $\partial\Omega$, and

$$\psi_a(t) = \int_0^t \frac{1}{G_a(s)^{\frac{1}{4}}}ds, \qquad G_a(t) = \int_t^a g(s)ds, \qquad 0 < t < a.$$

Recently, the author [29] extended the result in [5] to the weight function b which belong to the set Λ. Theorem 1.1 in [29] established the following result: let g satisfy (g_1) and

$(\mathbf{g}_2')$ there exists $\gamma > 1$ such that

$$\lim_{s\to 0^+} \frac{g'(s)s}{g(s)} =: -\gamma;$$

b satisfy (b_1) and

$(\mathbf{b}_2')$ there exist some $k \in \Lambda$ and a positive constant $b_0 \in \mathbb{R}$ such that

$$\lim_{d(x)\to 0} \frac{b(x)}{k^4(d(x))} = b_0.$$

If $C_k(\gamma+3) > 4$, then for the unique solution u of problem (1.1), it holds that

$$\lim_{d(x)\to 0} \frac{u(x)}{\phi(K^{\frac{4}{3}}(d(x)))} = \xi_0, \tag{1.7}$$

where ϕ is uniquely determined by

$$\int_0^{\phi(t)} \frac{ds}{(g(s))^{\frac{1}{3}}} = t, \qquad t > 0 \tag{1.8}$$

and

$$\xi_0 = \left(\frac{27b_0(3+\gamma)}{64((3+\gamma)C_k - 4)}\right)^{\frac{1}{3+\gamma}}. \tag{1.9}$$

For convenience, we introduce the following class of functions.

Let Λ_1 denote the set of all Karamata functions $\hat{L}$, which are **normalized** slowly varying at zero defined on $(0,a)$ for some $a>0$ by

$$\hat{L}(t)=c_0\exp\left(\int_s^{a_1}\frac{y(\nu)}{\nu}d\nu\right),\qquad s\in(0,a_1),$$

for some $a_1\in(0,a)$, where $c_0>0$ and the function $y\in C((0,a_1])$ with $\lim_{s\to 0^+}y(s)=0$.

Inspired by the above works, in this paper, by Karamata regular variation theory and the method of lower and upper solutions, we investigate the new boundary asymptotic behavior of solutions to problem (1.1) when the weight function b lies into Λ_1 and the nonlinear term g satisfies the following structure condition

($\mathbf{g_2}$) there exists $C_g>0$ such that

$$\lim_{s\to 0}\frac{1}{3g^{\frac{2}{3}}(s)}g'(s)\int_0^s g^{-1/3}(\nu)d\nu=-C_g,$$

i.e.

$$\lim_{s\to 0}(g^{\frac{1}{3}}(s))'\int_0^s g^{-1/3}(\nu)d\nu=-C_g.$$

A complete characterization of g in (g_2) is provided in Lemma 3.2.

Note that in this paper we extend the previous results in all two directions. We extend $g(u)$ to a more general class of functions which include the condition (g_2') and $b(x)$ belongs to another class of functions Λ_1.

Our main results are summarized as follows.

Theorem 1.1. *Let g satisfy (g_1)–(g_2), b satisfy (b_1) and*

($\mathbf{b_2}$) *There exists a positive constant $b_0\in\mathbb{R}$ such that*

$$\lim_{d(x)\to 0}\frac{b(x)}{a(d(x))}=b_0,$$

where

$$a(t)=t^{-\lambda}L(t),\quad L\in\Lambda_1,\quad \lambda\le 4\quad\text{and}\quad \int_0^\eta s^{\frac{1-\lambda}{3}}L(s)ds<\infty\quad\text{for some }\eta>0. \tag{1.10}$$

If $C_g<1$ and $4C_g+\lambda(1-C_g)>1$, for the unique solution u of problem (1.1), it holds that

$$\lim_{d(x)\to 0}\frac{u(x)}{\phi\big(h(d(x))\big)}=\xi_0, \tag{1.11}$$

where ϕ is uniquely determined by (1.8),

$$h(t)=\int_0^t s^{\frac{1-\lambda}{3}}L^{\frac{1}{3}}(s)ds, \tag{1.12}$$

and

$$\xi_0=\left(\frac{3b_0}{(4-\lambda)C_g+(\lambda-1)}\right)^{\frac{1-C_g}{3}}. \tag{1.13}$$

Remark 1.2 (Existence and uniqueness [5, Corollary 6.3.]). Let $g : (0,\infty) \to (0,\infty)$ be non-increasing and $b \in C(\Omega)$ be a positive function such that $\sup_{x\in\Omega} b(x) < \infty$. The singular boundary value problem (1.1) admits a unique solution.

Remark 1.3. By the following Proposition 2.7, one can see that when $\lambda < 4$, h in (1.12) satisfies

$$h(t) \cong \frac{3}{4-\lambda} t^{\frac{4-\lambda}{3}} L^{\frac{1}{3}}(t).$$

Remark 1.4. Some basic examples of the functions which satisfy (g_2) are

($\mathbf{i_1}$) When $g(s) = s^{-\gamma}$, $\gamma > 0$, $C_g = \frac{\gamma}{\gamma+3}$,

$$\phi(t) = \big(((\gamma+3)t)/3\big)^{\frac{3}{3+\gamma}}, \qquad \forall\, t > 0.$$

($\mathbf{i_2}$) When $g(s) = s^{-\gamma} e^{(-\ln s)^\beta}$, $\gamma > 0$, $\beta < 1$, $\beta \neq 0$, $s \in (0, s_0]$, $s_0 \in (0,1)$, $C_g = \frac{\gamma}{\gamma+3}$.

($\mathbf{i_3}$) When $g(s) = \beta^{-3} s^{3(1+\beta)} e^{3s^{-\beta}}$, $\beta > 0$, $s \in (0, \left(\frac{\beta}{1+\beta}\right)^{\frac{1}{\beta}}]$, $C_g = 1$.

($\mathbf{i_4}$) When $g(s) = \beta^{-3} s^{3(1+\beta)} e^{-3s^{-\beta}} e^{e^{3s^{-\beta}}}$, $\beta > 0$, $s \in (0, s_0]$, $s_0 \in (0,1)$, $C_g = 1$.

The outline of this paper is as follows. In Sections 2–3, we give some preparation that will be used in the next section. The proof of Theorem 1.1 will be given in Section 4.

2 Preparation

Our approach relies on Karamata regular variation theory established by Karamata in 1930 which is a basic tool in the theory of stochastic process (see [7, 27, 39] and the references therein). In this section, we first give a brief account of the definition and properties of regularly varying functions involved in our paper (see [7,27,39]).

Definition 2.1. A positive measurable function f defined on $[a,\infty)$, for some $a > 0$, is called **regularly varying at infinity** with index ρ, written as $f \in RV_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,

$$\lim_{s\to\infty} \frac{f(\xi s)}{f(s)} = \xi^\rho. \tag{2.1}$$

In particular, when $\rho = 0$, f is called **slowly varying at infinity**.

Clearly, if $f \in RV_\rho$, then $L(s) := f(s)/s^\rho$ is slowly varying at infinity.

Definition 2.2. A positive measurable function f defined on $[a,\infty)$, for some $a > 0$, is called **rapidly varying at infinity** if for each $\rho > 1$

$$\lim_{s\to\infty} \frac{f(s)}{s^\rho} = \infty. \tag{2.2}$$

We also see that a positive measurable function g defined on $(0,a)$ for some $a > 0$, is **regularly varying at zero** with index σ (written as $g \in RVZ_\sigma$) if $t \to g(1/t)$ belongs to $RV_{-\sigma}$. Similarly, g is called **rapidly varying at zero** if $t \to g(1/t)$ is rapidly varying at infinity.

Proposition 2.3 (Uniform convergence theorem). *If $f \in RV_\rho$, then* (2.1) *holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$. Moreover, if $\rho < 0$, then uniform convergence holds on intervals of the form (a_1, ∞) with $a_1 > 0$; if $\rho > 0$, then uniform convergence holds on intervals $(0, a_1]$ provided f is bounded on $(0, a_1]$ for all $a_1 > 0$.*

Proposition 2.4 (Representation theorem). *A function L is slowly varying at infinity if and only if it may be written in the form*

$$L(s) = \varphi(s) \exp\left(\int_{a_1}^{s} \frac{y(\tau)}{\tau} d\tau\right), \qquad s \geq a_1, \tag{2.3}$$

for some $a_1 \geq a$, where the functions φ and y are measurable and for $s \to \infty$, $y(s) \to 0$ and $\varphi(s) \to c_0$, with $c_0 > 0$.

We say that

$$\hat{L}(s) = c_0 \exp\left(\int_{a_1}^{s} \frac{y(\tau)}{\tau} d\tau\right), \qquad s \geq a_1, \tag{2.4}$$

is **normalized** slowly varying at infinity and

$$f(s) = c_0 s^\rho \hat{L}(s), \qquad s \geq a_1, \tag{2.5}$$

is **normalized** regularly varying at infinity with index ρ (and written as $f \in NRV_\rho$).

Similarly, g is called **normalized** regularly varying at zero with index ρ, written as $g \in NRVZ_\rho$ if $t \to g(1/t)$ belongs to $NRV_{-\rho}$.

A function $f \in RV_\rho$ belongs to NRV_ρ if and only if

$$f \in C^1[a_1, \infty) \text{ for some } a_1 > 0 \text{ and } \lim_{s \to \infty} \frac{s f'(s)}{f(s)} = \rho. \tag{2.6}$$

Proposition 2.5. *If functions L, L_1 are slowly varying at zero, then*

(**i**) *L^ρ (for every $\rho \in \mathbb{R}$), $c_1 L + c_2 L_1$ ($c_1 \geq 0$, $c_2 \geq 0$ with $c_1 + c_2 > 0$), $L \circ L_1$ (if $L_1(t) \to 0$ as $t \to 0^+$), are also slowly varying at zero;*

(**ii**) *for every $\rho > 0$ and $t \to 0^+$,*

$$t^\rho L(t) \to 0, \;\; t^{-\rho} L(t) \to \infty;$$

(**iii**) *for $\rho \in \mathbb{R}$ and $t \to 0^+$, $\ln(L(t))/\ln t \to 0$ and $\ln(t^\rho L(t))/\ln t \to \rho$.*

Proposition 2.6.

(**i**) *If $g_1 \in RVZ_{\rho_1}$, $g_2 \in RVZ_{\rho_2}$ with $\lim_{t \to 0^+} g_2(t) = 0$, then $g_1 \circ g_2 \in RVZ_{\rho_1 \rho_2}$.*

(**ii**) *If $g \in RVZ_\rho$, then $g^\alpha \in RVZ_{\rho\alpha}$ for every $\alpha \in \mathbb{R}$.*

Proposition 2.7 (Asymptotic behavior). *If a function L is slowly varying at zero, then for $a > 0$ and $t \to 0^+$,*

(**i**) *$\int_0^t s^\rho L(s) ds \cong (\rho + 1)^{-1}\, t^{1+\rho}\, L(t)$, for $\rho > -1$;*

(**ii**) *$\int_t^a s^\rho L(s) ds \cong (-\rho - 1)^{-1}\, t^{1+\rho}\, L(t)$, for $\rho < -1$.*

Proposition 2.8. *If a function L be defined on $(0, \eta]$, is slowly varying at zero. Then we have*

$$\lim_{t\to 0^+} \frac{L(t)}{\int_t^\eta \frac{L(s)}{s} ds} = 0. \tag{2.7}$$

If further $\int_0^\eta \frac{L(s)}{s} ds$ converges, then we have

$$\lim_{t\to 0^+} \frac{L(t)}{\int_0^t \frac{L(s)}{s} ds} = 0. \tag{2.8}$$

Proposition 2.9 ([46, Proposition 2.6])**.** *Let $Z \in C^1(0, \eta]$ be positive and $\lim_{t\to 0^+} \frac{sZ'(s)}{Z(s)} = +\infty$. Then Z is rapidly varying to zero at zero.*

Proposition 2.10 ([46, Proposition 2.7])**.** *Let $Z \in C^1(0, \eta)$ be positive and $\lim_{t\to 0^+} \frac{sZ'(s)}{Z(s)} = -\infty$. Then Z is rapidly varying to infinity at zero.*

Next, we recall here the precise definition of viscosity solutions for problem (1.1).

Definition 2.11. A function $\underline{u} \in C(\Omega)$ is a viscosity subsolution of the PDE $\Delta_\infty u = -b(x)g(u)$ in Ω if for every $\varphi \in C^2(\Omega)$, with the property that $\underline{u} - \varphi$ has a local maximum at some $x_0 \in \Omega$, then

$$\Delta_\infty \varphi(x_0) \geq -b(x_0)g(\underline{u}(x_0)).$$

Definition 2.12. A function $\bar{u} \in C(\Omega)$ is a viscosity supersolution of the PDE $\Delta_\infty u = -b(x)g(u)$ in Ω if for every $\varphi \in C^2(\Omega)$, with the property that $\bar{u} - \varphi$ has a local minimum at some $x_0 \in \Omega$, then

$$\Delta_\infty \varphi(x_0) \leq -b(x_0)g(\bar{u}(x_0)).$$

Definition 2.13. A function $u \in C(\Omega)$ is a viscosity solution of the PDE $\Delta_\infty u = -b(x)g(u)$ in Ω if it is both a subsolution and a supersolution.

3 Some auxiliary results

In this section, we collect some useful results that will be used in the proof of the theorem.

Lemma 3.1. *Let*

$$a(t) = t^{-\lambda} L(t)$$

and

$$h(t) = \int_0^t s^{\frac{1-\lambda}{3}} (L(s))^{\frac{1}{3}} ds,$$

where $t \in (0, \delta_0)$, $\lambda \leq 4$, $\int_0^\eta s^{\frac{1-\lambda}{3}} (L(s))^{\frac{1}{3}} ds < \infty$ for some $\eta > 0$ and $L \in \Lambda_1$. Then

(i) $\lim_{t\to 0^+} \frac{(h'(t))^4}{h(t)a(t)} = \frac{4-\lambda}{3}$ *and* $\lim_{t\to 0^+} \frac{th'(t)}{h(t)} = \frac{4-\lambda}{3}$;

(ii) $\lim_{t\to 0^+} \frac{th''(t)}{h'(t)} = \frac{1-\lambda}{3}$;

(iii) $\lim_{t\to 0^+} \frac{(h'(t))^2 h''(t)}{a(t)} = \frac{1-\lambda}{3}$.

Proof. (i) Since $h'(t) = t^{\frac{1-\lambda}{3}}(L(t))^{\frac{1}{3}}$, then

$$\frac{(h'(t))^4}{h(t)a(t)} = \frac{t^{\frac{4-4\lambda}{3}} L^{\frac{4}{3}}(t)}{t^{-\lambda}L(t)\int_0^t s^{\frac{1-\lambda}{3}}(L(s))^{\frac{1}{3}}ds} = \frac{t^{\frac{4-\lambda}{3}} L^{\frac{1}{3}}(t)}{\int_0^t s^{\frac{1-\lambda}{3}}(L(s))^{\frac{1}{3}}ds}$$

and

$$\frac{th'(t)}{h(t)} = \frac{t^{\frac{4-\lambda}{3}} L^{\frac{1}{3}}(t)}{\int_0^t s^{\frac{1-\lambda}{3}}(L(s))^{\frac{1}{3}}ds}.$$

Hence, when $\lambda < 4$, by Proposition 2.7, we get $\lim_{t\to 0^+} \frac{(h'(t))^4}{h(t)a(t)} = \lim_{t\to 0^+} \frac{th'(t)}{h(t)} = \frac{4-\lambda}{3}$; when $\lambda = 4$, by Proposition 2.8, we get $\lim_{t\to 0^+} \frac{(h'(t))^4}{h(t)a(t)} = \lim_{t\to 0^+} \frac{th'(t)}{h(t)} = 0$.
(ii) By a direct computation, we get

$$h''(t) = \frac{1-\lambda}{3} t^{-\frac{2+\lambda}{3}}(L(t))^{\frac{1}{3}} + \frac{1}{3} t^{\frac{1-\lambda}{3}}(L(t))^{-\frac{2}{3}}L'(t)$$

and

$$\frac{th''(t)}{h'(t)} = \frac{1}{3}\frac{tL'(t)}{L(t)} + \frac{1-\lambda}{3}.$$

It follows by $L \in \Lambda_1$ that $\lim_{t\to 0^+} \frac{tL'(t)}{L(t)} = 0$. Hence,

$$\lim_{t\to 0^+} \frac{th''(t)}{h'(t)} = \frac{1-\lambda}{3}.$$

(iii) Since

$$\frac{(h'(t))^2 h''(t)}{a(t)} = \frac{th''(t)}{h'(t)}\frac{(h'(t))^3}{ta(t)} = \frac{th''(t)}{h'(t)},$$

by (ii), we get

$$\lim_{t\to 0^+} \frac{(h'(t))^{p-2} h''(t)}{a(t)} = \frac{1-\lambda}{3}.$$

□

Lemma 3.2. *Let g satisfy* (g$_1$)–(g$_2$).

(i) *If g satisfies* (g$_2$), *then $C_g \le 1$;*

(ii) (g$_2$) *holds for $C_g \in (0,1)$ if and only if $g \in NRV_{-\gamma}$; with $\gamma > 0$. In this case $\gamma = 3C_g/(1-C_g)$;*

(iii) (g$_2$) *holds for $C_g = 0$ if and only if g is normalized slowly varying at zero;*

(iv) *if* (g$_2$) *holds with $C_g = 1$, then g is rapidly varying to infinity at zero;*

(v) *if*

$$\lim_{s\to 0^+} \frac{g''(s)g(s)}{(g'(s))^2} = 1, \tag{3.1}$$

then g satisfies (g$_2$) *with $C_g = 1$.*

Proof. Since g satisfies (g_1) and is strictly decreasing on $(0, S_0)$, we see that

$$0 < \int_0^s \frac{1}{g^{1/3}(\nu)} d\nu < \frac{s}{g^{1/3}(s)}, \qquad \forall s \in (0, S_0),$$

i.e.,

$$0 < g^{1/3}(s) \int_0^s \frac{1}{g^{1/3}(\nu)} d\nu < s, \qquad \forall s \in (0, S_0), \tag{3.2}$$

and

$$\lim_{s \to 0} g^{1/3}(s) \int_0^s \frac{1}{g^{1/3}(\nu)} d\nu = 0. \tag{3.3}$$

(i) Let

$$I(s) = -\frac{1}{3g^{\frac{2}{3}}(s)} g'(s) \int_0^s g^{-1/3}(\nu) d\nu, \qquad \forall s \in (0, s_0).$$

Integrate $I(t)$ from 0 to s and integrate by parts, we obtain by (3.3) that

$$\int_0^s I(t) dt = -g^{1/3}(s) \int_0^s \frac{1}{g^{1/3}(\nu)} d\nu + s, \qquad \forall s \in (0, s_0),$$

i.e.

$$0 < \frac{g^{1/3}(s)}{s} \int_0^s \frac{1}{g^{1/3}(\nu)} d\nu = 1 - \frac{\int_0^s I(t) dt}{s}, \qquad \forall s \in (0, s_0).$$

It follows from L'Hospital's rule that

$$0 \leq \lim_{s \to 0^+} \frac{g^{1/3}(s)}{s} \int_0^s \frac{1}{g^{1/3}(\nu)} d\nu = 1 - \lim_{s \to 0^+} I(s) = 1 - C_g. \tag{3.4}$$

So (i) holds.

(ii) When (g_2) holds with $C_g \in (0, 1)$, it follows by (3.4) that

$$\lim_{s \to 0^+} \frac{g(s)}{sg'(s)} = \lim_{s \to 0^+} \frac{g^{1/3}(s) \int_0^s \frac{1}{g^{1/3}(\nu)} d\nu}{sg'(s) \int_0^s \frac{1}{g^{1/3}(\nu)} d\nu g^{\frac{1}{3}-1}(s)} = -\frac{1 - C_g}{3C_g}, \tag{3.5}$$

i.e., $g \in NRV_{-3C_g/(1-C_g)}$.

Conversely, when $g \in NRV_{-\gamma}$ with $\gamma > 0$, i.e., $\lim_{s \to 0^+} \frac{sg'(s)}{g(s)} = -\gamma$ and there exist positive constant η and $\hat{L} \in \Lambda_1$ such that $g(s) = c_0 s^{-\gamma} \hat{L}(s)$, $s \in (0, \eta]$. It follows by (2.8) and Proposition 2.7 (i) that

$$\begin{aligned}
-\lim_{s \to 0^+} \frac{1}{3g^{\frac{2}{3}}(s)} g'(s) \int_0^s g^{-1/3}(\nu) d\nu &= -\frac{1}{3} \lim_{s \to 0^+} \frac{sg'(s)}{g(s)} \lim_{s \to 0^+} \frac{g^{1/3}(s)}{s} \int_0^s g^{-1/3}(\nu) d\nu \\
&= \frac{\gamma}{3} \lim_{s \to 0^+} s^{-\frac{\gamma}{3}-1} (\hat{L}(s))^{\frac{1}{3}} \int_0^s \nu^{\frac{\gamma}{3}} (\hat{L}(\nu))^{-\frac{1}{3}} d\nu \\
&= \frac{\gamma}{3 + \gamma} = C_g.
\end{aligned}$$

(iii) By $C_g = 0$ and the proof of (ii), one can see that

$$\begin{aligned}
\lim_{s \to 0^+} \frac{sg'(s)}{g(s)} &= \lim_{s \to 0^+} \frac{sg'(s) \int_0^s \frac{1}{g^{1/3}(\nu)} d\nu g^{\frac{1}{3}-1}(s)}{g^{1/3}(s) \int_0^s \frac{1}{g^{1/3}(\nu)} d\nu} \\
&= 3 \left(\lim_{s \to 0^+} \frac{g^{1/3}(s)}{s} \int_0^s \frac{1}{g^{1/3}(\nu)} d\nu \right)^{-1} \lim_{s \to 0^+} \frac{1}{3g^{1-\frac{1}{3}}(s)} g'(s) \int_0^s g^{-1/3}(\nu) d\nu \\
&= 0,
\end{aligned}$$

i.e., g is normalized slowly varying at zero.

Conversely, when g is normalized slowly varying at zero, i.e., $\lim_{s\to 0^+}\frac{sg'(s)}{g(s)}=0$, it follows by (3.4) that

$$\lim_{s\to 0^+}\frac{1}{3g^{1-\frac{1}{t}}(s)}g'(s)\int_0^s g^{-1/3}(\nu)d\nu=\lim_{s\to 0^+}\frac{1}{3}\frac{sg'(s)}{g(s)}\frac{g^{1/3}(s)}{s}\int_0^s\frac{1}{g^{1/3}(\nu)}d\nu=0.$$

(iv) By $C_g=1$ and the proof of (ii), we see that $\lim_{s\to 0^+}\frac{g(s)}{sg'(s)}=0$, i.e., $\lim_{s\to 0^+}\frac{sg'(s)}{g(s)}=-\infty$, we see by Proposition 2.10 that g is rapidly varying to infinity at zero.
(v) By (3.1) and L'Hospital's rule, we obtain that

$$\lim_{s\to 0}\frac{g(s)}{sg'(s)}=\lim_{s\to 0}\frac{\frac{g(s)}{g'(s)}}{s}=\lim_{s\to 0}\frac{d}{ds}\left(\frac{g(s)}{g'(s)}\right)=1-\lim_{s\to 0}\frac{g(s)g''(s)}{(g'(s))^2}=0. \tag{3.6}$$

Hence, by (g_1) and (3.6), we get that

$$\lim_{s\to 0}\frac{g^{\frac{2}{3}}(s)}{g'(s)}=\lim_{s\to 0}\frac{g(s)}{sg'(s)}\frac{s}{g^{\frac{1}{3}}(s)}=\lim_{s\to 0}\frac{g(s)}{sg'(s)}\lim_{s\to 0}\frac{s}{g^{\frac{1}{3}}(s)}=0. \tag{3.7}$$

It follows by the L'Hospital's rule and (3.7) that

$$\begin{aligned}&\lim_{s\to 0}\frac{1}{3g^{\frac{2}{3}}(s)}g'(s)\int_0^s g^{-1/3}(\nu)d\nu\\ &=\lim_{s\to 0}\frac{1}{3}\frac{\int_0^s g^{-1/3}(\nu)d\nu}{\frac{g^{\frac{2}{3}}(s)}{g'(s)}}=\lim_{s\to 0}\frac{1}{3}\frac{1}{\frac{2}{3}-\frac{g''(s)g(s)}{(g'(s))^2}}\\ &=-1,\end{aligned}$$

i.e. $C_g=1$. □

Lemma 3.3. *Let g satisfy* (g_1)–(g_2) *and ϕ be the solution to the problem*

$$\int_0^{\phi(t)}\frac{ds}{(g(s))^{\frac{1}{3}}}=t,\qquad \forall\ t>0.$$

Then

(i) $\phi'(t)=\big(g(\phi(t))\big)^{\frac{1}{3}},\ \phi(t)>0,\ t>0,\ \phi(0)=0$ *and* $\phi''(t)=\frac{1}{3}\big(g(\phi(t))\big)^{-\frac{1}{3}}g'(\phi(t)),\ t>0;$

(ii) $\phi\in NRVZ_{1-C_g}$ *and* $\phi'\in NRVZ_{-C_g}$;

(iii) $\lim_{t\to 0^+}\frac{t}{\phi(\xi h(t))}=0$ *uniformly for* $\xi\in[c_1,c_2]$ *with* $0<c_1<c_2$, *where h is given as in* (1.12).

Proof. By the definition of ϕ and a direct calculation, we show that (i) holds.

(ii) It follows from (i), (3.5) and (g_2) that

$$\begin{aligned}\lim_{t\to 0^+}\frac{t\phi'(t)}{\phi(t)}&=\lim_{t\to 0^+}\frac{t(g(\phi(t)))^{\frac{1}{p-1}}}{\phi(t)}\\ &=\lim_{s\to 0}\frac{(g(s))^{\frac{1}{3}}\int_0^s\frac{d\nu}{(g(\nu))^{\frac{1}{3}}}}{s}=1-C_g,\end{aligned}$$

i.e., $\phi \in NRVZ_{1-C_g}$, and

$$\begin{aligned}\lim_{t\to 0^+}\frac{t\phi''(t)}{\phi'(t)} &= \frac{1}{3}\lim_{t\to 0^+}\frac{g'(\phi(t))(g(\phi(t)))^{\frac{1}{3}}\int_0^{\phi(t)}(g(\nu))^{-\frac{1}{3}}d\nu}{g(\phi(t))}\\ &= \frac{1}{3}\lim_{s\to 0^+}\frac{g'(s)(g(s))^{\frac{1}{3}}\int_0^{s}(g(\nu))^{-\frac{1}{3}}d\nu}{g(s)}\\ &= -C_g.\end{aligned}$$

(iii) By Lemma 3.1 (i), we see $h \in NRVZ_{\frac{4-\lambda}{3}}$. It follows by Proposition 2.4 that $\phi\circ h \in NRVZ_{\frac{(4-\lambda)(1-C_g)}{3}}$. Since $4C_g+\lambda(1-C_g)>1$, the result follows by Proposition 2.5 (ii).

□

4 Proof of the Theorem

In this section, we prove Theorem 1.1. First, we need the following result.

Lemma 4.1 (Comparison principle [5, Lemma 4.3])**.** *Suppose that $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is continuous, $f(x,t)$ is non-decreasing in t. Assume further that f has one sign (either positive or negative) in $\Omega\times\mathbb{R}$. If $u,v\in C(\bar\Omega)$ are such that*

$$\Delta_\infty u \ge f(x,u), \qquad \Delta_\infty v \le f(x,v) \quad \textit{and} \quad u\le v \quad \textit{on } \partial\Omega,$$

then $u\le v$ in Ω.

First fix $\varepsilon>0$. For any $\delta_0>0$, we define $\Omega_{\delta_0}=\{x\in\Omega: 0<d(x)<\delta_0\}$. Since Ω is C^2-smooth, choose $\delta_1\in(0,\delta_0)$ such that $d\in C^2(\Omega_{\delta_1})$ and $|\nabla d(x)|=1$, $\forall\, x\in\Omega_{\delta_1}$, and consequently $\Delta_\infty d=0$ in Ω_{δ_1} in the viscosity sense.

Proof of Theorem 1.1. Let $v\in C(\bar\Omega)$ be the unique solution of the problem

$$-\Delta_\infty v=1, \qquad v>0, \qquad x\in\Omega, \qquad v|_{\partial\Omega}=0. \tag{4.1}$$

By Theorem 7.7 in [5], we see that

$$c_1 d(x)\le v(x)\le c_2 d(x), \qquad \forall x\in\Omega \ \text{ near } \partial\Omega. \tag{4.2}$$

where c_1, c_2 are positive constants.

Now, we define

$$\bar u_\varepsilon=(\xi_0+\varepsilon)\phi\big(h(d(x))\big) \quad \text{for any } x\in\Omega_{\delta_1},$$

where h is given as in (1.12).

Let

$$\eta(t)=(\xi_0+\varepsilon)\phi\big(h(t)\big), \qquad t\in(0,\delta_1).$$

Note that h and ϕ are all increasing in their respective definition domains. Therefore, when δ_1 is small enough, η is increasing in $(0,\delta_1)$. Let ζ be the inverse of η. One can easily check that

$$\zeta'(t)=\frac{1}{\eta'(\zeta(t))}=\Big((\xi_0+\varepsilon)\phi'\big(h(\zeta(t))\big)h'(\zeta(t))\Big)^{-1} \tag{4.3}$$

and

$$\begin{aligned}\zeta''(t) = &-\big((\xi_0+\varepsilon)\phi'\big(h(\zeta(t))\big)h'(\zeta(t))\big)^{-3}\\ &\times\big((\xi_0+\varepsilon)\phi''\big(h(\zeta(t))\big)(h'(\zeta(t)))^2\\ &+(\xi_0+\varepsilon)\phi'\big(h(\zeta(t))\big)h''(\zeta(t))\big).\end{aligned} \tag{4.4}$$

Let $(x_0,\psi) \in \Omega_{\delta_1} \cap C^2(\Omega_{\delta_1})$ be a pair such that $\bar{u}_\varepsilon \geq \psi$ in a neighborhood N of x_0 and $\bar{u}_\varepsilon(x_0) = \psi(x_0)$ Then $\varphi = \zeta(\psi) \in C^2(\Omega_{\delta_1})$, and

$$d(x) \geq \varphi(x) \quad \text{in } N, \qquad d(x_0) = \varphi(x_0).$$

Since $\Delta_\infty d = 0$ in Ω_{δ_1}, we have $\Delta_\infty\varphi(x_0) \leq 0$. A simple computation shows that

$$\Delta_\infty\varphi = \zeta''(\psi)(\zeta'(\psi))^2|D\psi|^4 + (\zeta'(\psi))^3\Delta_\infty\psi.$$

It follows by $\Delta_\infty\varphi(x_0) \leq 0$ and $\zeta' > 0$ that

$$\Delta_\infty\psi(x_0) \leq -\zeta''(\psi(x_0))(\zeta'(\psi(x_0)))^{-1}|D\psi(x_0)|^4.$$

Moreover, since $|Dd(x)| = 1$ for $x \in \Omega_{\delta_1}$ and $d - \varphi$ attains a local maximum at x_0, it follows that

$$1 = |Dd(x_0)| = |\zeta'(\psi(x_0))D\psi(x_0)|.$$

Hence

$$\Delta_\infty\psi(x_0) \leq -\zeta''(\psi(x_0))(\zeta'(\psi(x_0)))^{-5}.$$

Combing with (4.3) and (4.4), we further obtain

$$\begin{aligned}\Delta_\infty\psi(x_0) \leq\ &((\xi_0+\varepsilon))^3\left(\phi'\big(h(\varphi(x_0))\big)\right)^3 a(\varphi(x_0))\\ &\times\left[\frac{\phi''\big(h(\varphi(x_0))\big)h(\varphi(x_0))}{\phi'\big(h(\varphi(x_0))\big)}\frac{(h'(\varphi(x_0)))^4}{h(\varphi(x_0))a(\varphi(x_0))} + \frac{h''(\varphi(x_0))(h'(\varphi(x_0)))^2}{a(\varphi(x_0))}\right].\end{aligned}$$

Hence,

$$\begin{aligned}&\Delta_\infty\psi(x_0) + b(x_0)g(\bar{u}_\varepsilon(x_0))\\ &\quad\leq ((\xi_0+\varepsilon))^3\left(\phi'\big(h(\varphi(x_0))\big)\right)^3 a(\varphi(x_0))\\ &\qquad\times\left[\frac{\phi''\big(h(\varphi(x_0))\big)h(\varphi(x_0))}{\phi'\big(h(\varphi(x_0))\big)}\frac{(h'(\varphi(x_0)))^4}{h(\varphi(x_0))a(\varphi(x_0))} + \frac{h''(\varphi(x_0))(h'(\varphi(x_0)))^2}{a(\varphi(x_0))}\right.\\ &\qquad\quad\left. + ((\xi_0+\varepsilon))^{-3}\frac{b(x_0)}{a(\varphi(x_0))}\frac{g(\bar{u}_\varepsilon(x_0))}{\left(\phi'\big(h(\varphi(x_0))\big)\right)^3}\right]\\ &\quad=: ((\xi_0+\varepsilon))^3\left(\phi'\big(h(d(x_0))\big)\right)^3 a(d(x_0))I(x_0).\end{aligned}$$

Notice that $h(d(x_0)) \to 0$ as $\delta_1 \to 0$ (and thereby x_0 tends to the boundary of Ω). Then, it follows from Lemmas 3.1 and 3.3 that

$$I(x_0) \to \frac{(\lambda-4)C_g + (1-\lambda)}{3} + b_0\,(\xi_0+\varepsilon)^{-3-\gamma} \qquad \text{as } \delta_1 \to 0.$$

By the choice of ξ_0, we have $I(x_0) < 0$ provided $\delta_{1\varepsilon} \in (0, \frac{\delta_1}{2})$ small enough. Thus

$$\Delta_\infty\psi(x_0) \leq -b(x_0)g(\bar{u}_\varepsilon(x_0)),$$

i.e., $\bar{u}_\varepsilon$ is a supersolution of equation (1.1) in $\Omega_{\delta_{1\varepsilon}}$.

In a similar way, we can show that

$$\underline{u}_\varepsilon = (\xi_0 - \varepsilon)\phi(h(d(x)))$$

is a subsolution of equation (1.1) in $\Omega_{\delta_{1\varepsilon}}$.

Let $u \in C(\Omega)$ be the unique solution to problem (1.1). We assert that there exists M large enough such that

$$u(x) \le Mv(x) + \bar{u}_\varepsilon(x), \qquad \underline{u}_\varepsilon(x) \le u(x) + Mv(x), \qquad x \in \Omega_{\delta_{1\varepsilon}}, \tag{4.5}$$

where v is the solution of problem (4.1).

In fact, we can choose M large enough such that

$$u(x) \le \bar{u}_\varepsilon(x) + Mv(x) \quad \text{and} \quad \underline{u}_\varepsilon(x) \le u(x) + Mv(x) \quad \text{on } \{x \in \Omega : d(x) = \delta_{1\varepsilon}\}.$$

We see by (g_1) that $\bar{u}_\varepsilon(x) + Mv(x)$ and $u(x) + Mv(x)$ are also supersolutions of equation (1.1) in $\Omega_{\delta_{1\varepsilon}}$. Since $u = \bar{u}_\varepsilon + Mv = u + Mv = \underline{u}_\varepsilon = 0$ on $\partial\Omega$, (4.5) follows by (g_1) and Lemma 4.1. Hence, for $x \in \Omega_{\delta_{1\varepsilon}}$

$$\xi_0 - \varepsilon - \frac{Mv(x)}{\phi(h(d(x)))} \le \frac{u(x)}{\phi(h(d(x)))}$$

and

$$\frac{u(x)}{\phi(h(d(x)))} \le \xi_0 + \varepsilon + \frac{Mv(x)}{\phi(h(d(x)))}.$$

Consequently, by (4.2) and Lemma 3.3 (iii),

$$\xi_0 - \varepsilon \le \liminf_{d(x)\to 0} \frac{u(x)}{\phi(h(d(x)))};$$

$$\limsup_{d(x)\to 0} \frac{u(x)}{\phi(h(d(x)))} \le \xi_0 + \varepsilon.$$

Thus, letting $\varepsilon \to 0$, we obtain (1.11).

Thus the proof is finished by letting $\varepsilon \to 0$. □

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