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Periodic Solutions of p-Laplacian Systems with a Nonlinear Convection Term

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Abstract In this work, we study the existence of periodic solutions for the evolution of p-Laplacian system and we show that these periodic solutions belong to $L^{\infty}(\omega, W^{1,\infty}(\Omega))$ and give a bound of $\|\nabla u_i(t)\|_{\infty}$ under certain geometric conditions on $\partial\Omega$.

Keywords: Periodic solutions; nonlinear parabolic systems; p-Laplacian; existence of solutions; gradients estimates.

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1. Introduction

We consider the following nonlinear system $(\mathcal{S})$ of the form

$$(\mathcal{S})\begin{cases} \frac{\partial u_1}{\partial t} - \triangle_{p_1} u_1 + a_1(u_1).\nabla u_1 = f_1(t) u_1^{\alpha_1} + h_1(x, u_2) & \text{in } Q, \\ \frac{\partial u_2}{\partial t} - \triangle_{p_2} u_2 + a_2(u_2).\nabla u_2 = f_2(t) u_2^{\alpha_2} + h_2(x, u_1) & \text{in } Q, \\ u_1(x,t) = u_2(x,t) = 0 & \text{on } S, \\ ((u_1(x, t+\omega), (u_2(x, t+\omega)) = (u_1(x,t), u_2(x,t)) & in \;\; Q, \end{cases}$$

where $\triangle_p u = div(|\nabla u|^{p-2} \nabla u)$ is the so-called $p-$Laplacian operator, Ω is a bounded convex subset in $\mathbb{R}^N$, $Q := \Omega \times (0,\omega), S := \partial\Omega \times (0,\omega), \partial\Omega$ is the boundary of Ω and $\omega > 0$. Precise conditions on a_i, f_i and h_i will be given later.

The system of form $(\mathcal{S})$ is a class of degenerate parabolic systems and appears in the theory of non-Newtonian fluids perturbed by nonlinear terms and forced by rather irregular period in time excitations, see $[1, 5]$. Periodic parabolic single equations have been the subject of numbers of extentive works see $[3, 5, 11, 12, 14, 15, 16]$. In particular, Lui [11], has considered the following single equation:

$$\frac{\partial u}{\partial t} - \triangle_p u + b(u).\nabla u = f(t) u^{\alpha} + h(x,t),$$

with Dirichlet boundary condition and has obtained the existence and gradient estimates of a periodic solution.The basic tools that have been used are a priori estimates, Leray-Shauder fixed point theorem, topological degree theory and

Moser method. We first consider the parabolic regularisation of $(\mathcal{S})$ and we employ Moser's technique as in [4] and we make some devices as in [7, 13] to obtain the existence of periodic solutions and we derive estimates of $\nabla u_i(t)$.

2. Main results

For convenience, hereafter let $E = C_\omega(\overline{Q})$, the set of all functions in $C(\overline{\Omega}\times\mathbb{R})$ which are periodic in t with period w. $\|.\|_q$ and $\|.\|_{s,q}$ denote $L^q = L^q(\Omega)$ and $W^{s,q} = W^{s,q}(\Omega)$ norms, respectively, $1 \le q \le \infty$. Since , $\Omega \subset \mathbb{R}^N$ is a bounded convex domain, we take the equivalent norm in $W_0^{1,q}(\Omega)$ to be

$$\|\nabla u\|_q = \left(\sum_{j=1}^{N}\int_\Omega \left|\frac{\partial u}{\partial x_j}\right|^q\right)^{\frac{1}{q}} \; for \;\; any \;\; u \in W_0^{1,q}(\Omega).$$

In the sequel, the same symbol c will be used to indicate some positive constants, possibly different from each other, appearing in the various hypotheses and computations and depending only on data. When we need to fix the precise value of one constant, we shall use a notation like $M_i, i = 1, 2.....,$ instead.

We make the following assumptions:

(H1) $\begin{cases} a_i(s) = (a_{i1}, a_{i2}, ..., a_{IN}) \text{ is an } \mathbb{R}^N - \text{valued function on } \mathbb{R}, \\ |a_i(s)| \le k_i\,|s|^{\beta_i}, \text{ for some } 0 \le \beta_i < p_i - 2, (i = 1, 2). \end{cases}$

(H2) $\{\; f_i(t) \in L^{\infty}(0, \omega)$ is periodic in t $(i = 1, 2)$, with period ω.

(H3) $\begin{cases} h_i(x,t) \in C_\omega(\overline{Q}) \cap L^{\infty}(0,\omega; W_0^{1,\infty}(\Omega)), \; h_i(x, 0) = 0 \\ h_i(x, t) > 0 \text{ , for all } (x, t) \in \Omega \times \mathbb{R}, \; (i = 1, 2). \end{cases}$

(H4) $\{0 \le \alpha_i < p_i - 1$ and $N > p_i \ge 2$.

We make the following definition of solutions.

Definition 2.1 A function $u = (u_1, u_2)$ is called a solution of system $(\mathcal{S})$ if

$$u_i \in L^{p_i}(0, \omega; W_0^{1,p_i}(\Omega)) \cap C_\omega(\overline{Q})$$

and u satisfies

$$\int\int_Q \{-u_1\varphi_{1t} + |\nabla u_1|^{p_1-2}\nabla u_1.\nabla\varphi_1 - A_1(u_1).\nabla\varphi_1 - f_1(t)u_1^{\alpha_1}\varphi_1 - h_1(x, u_2)\varphi_1\}\,dxdt = 0,$$

$$\int\int_Q \{-u_2\varphi_{2t} + |\nabla u_2|^{p_2-2}\nabla u_2.\nabla\varphi_2 - A_2(u_2).\nabla\varphi_2 - f_2(t)u_2^{\alpha_2}\varphi_2 - h_2(x, u_1)\varphi_2\}\,dxdt = 0,$$

for any $(\varphi_1, \varphi_2) \in C_0^1(0, \omega; C_0^1(\Omega))$ with $\varphi_i(x, 0) = \varphi_i(x, \omega)$, where $A_i(u) = \int_0^u a_i(s)ds$ is set.

Theorem 1.1.

Let (H1) to (H4) be satisfied. Then there exists a solution (u_1, u_2) of problem $(\mathcal{S})$ such that for $i = 1, 2$, we have

$$u_i(t) \in L^\infty(0, \omega; W_0^{1,p_i}(\Omega)) \cap C_\omega(\overline{Q}),$$

$$\frac{\partial u_i}{\partial t} \in L^2(Q).$$

Theorem 1.2. Let (H1) to (H4) be satisfied. Then there exists a solution (u_1, u_2) of problem $(\mathcal{S})$ such that for $i = 1, 2$, we have

$$u_i(t) \in L^\infty(0, \omega; W_0^{1,\infty}(\Omega)),$$

$$\sup_t \|\nabla u_i(t)\|_\infty \leq M_i < \infty.$$

For the proof of the theorems we use the following elementary lemmas.

Lemma 2.1. (Gagliardo-Nirenberg, cf[4])

let $\beta \geq 0, N > p \geq 1, \beta + 1 \leq q$, and $1 \leq r \leq q \leq (\beta + 1)Np/(N - p)$, then for v such that $|v|^\beta v \in W^{1,p}(\Omega)$,

$$\|v\|_q \leq C^{\frac{1}{\beta+1}} \|v\|_r^{1-\theta} \left\| |v|^\beta v \right\|_{1,p}^{\theta/(\beta+1)},$$

with $\theta = (\beta+1)(r^{-1} - q^{-1}) / \left\{N^{-1} - p^{-1} + (\beta + 1)r^{-1}\right\}$, where C is a constant independent of q, r, β, and θ.

Lemma 2.2. (cf [13])

Let $y(t) \in C^1(\mathbb{R})$ be a nonnegative ω periodic function satisfying the differential inequality

$$y'(t) + Ay^{\alpha+1}(t) \leq By(t) + C, \qquad t \in \mathbb{R},$$

with some $\alpha, A > 0, B \geq 0$, and $C \geq 0$. Then

$$y(t) \leq \max\left\{1, (A^{-1}(B + C))^{\frac{1}{\alpha}}\right\}.$$

3. Proofs of Theorem 1.1 and Theorem 1.2.

In this section we derive a priori estimates of solutions (u_1, u_2) of problem $(\mathcal{S})$. We first replace the p-Laplacian by the regularized one

$$\triangle_p^\varepsilon u = div(\left(|\nabla u|^2 + \varepsilon\right)^{\frac{p-2}{2}} \nabla u).$$

To prove theorem 1.1, we consider the following.

By theorem 1 in [3], we can choose $u_{i\varepsilon}^0$ such that :

$$\frac{\partial u_{i\varepsilon}^0}{\partial t} - \triangle_{p_1}^\varepsilon u_{i\varepsilon}^0) + a_i(u_{i\varepsilon}^0).\nabla u_{i\varepsilon}^0 = f_i(t)(u_{i\varepsilon}^n)^{\alpha_i} \quad \text{in } Q,$$

$$u_{i\varepsilon}^0 = 0 \qquad \text{on } S,$$

$$u_{i\varepsilon}^0(x,\omega) = u_{i\varepsilon}^0(x,t) \qquad \text{in } Q,$$

and we construct two sequences of functions $(u_{1\varepsilon}^n)$ and $(u_{2\varepsilon}^n)$, such that :

$$\frac{\partial u_{1\varepsilon}^n}{\partial t} - \triangle_{p_1}^{\varepsilon} u_{1\varepsilon}^n + a_1(u_{1\varepsilon}^n).\nabla u_{1\varepsilon}^n = f_1(t)(u_{1\varepsilon}^n)^{\alpha_1} + h_1(x, u_{2\varepsilon}^{n-1}) \quad \text{in } Q\ , \qquad (1.1)$$
$$u_{1\varepsilon}^n = 0 \qquad \text{on } S, \qquad (1.2)$$
$$u_{1\varepsilon}^n(x,\omega) = u_{1\varepsilon}^n(x,t) \qquad \text{in } \ Q. \qquad (1.3)$$

$$\frac{\partial u_{2\varepsilon}^n}{\partial t} - \triangle_{p_2}^{\varepsilon} u_{2\varepsilon}^n + a_2(u_{2\varepsilon}^n).\nabla u_{2\varepsilon}^n = f_2(t)(u_{2\varepsilon}^n)^{\alpha_2} + h_2(x, u_{1\varepsilon}^{n-1}) \quad \text{in } Q \qquad (1.6)$$
$$u_{2\varepsilon}^n = 0 \qquad \text{on } S, \qquad (1.5)$$
$$u_{2\varepsilon}^n(x,\omega) = u_{2\varepsilon}^n(x,t) \qquad \text{in } \ Q. \qquad (1.3)$$

It is clear that for each $n = 1, 2, 3, ...$, the above systems consist of two nondegenerated and uncoupled initial boundary-value problems. By theorem 1, [3] for fixed n and ε, the problem (1.1)-(1.3) and (1.4)-(1.6) has a solution $(u_{1\varepsilon}^n, u_{2\varepsilon}^n)$.

We need lemma 2.3 and lemma 2.4 below to complete the proof of theorem 1.1.

Lemma 2.3. There exists a positive constant M_i independent of ε and n, such that

$$\|u_{i\varepsilon}^n(t)\|_\infty \leq M_i. \qquad (1.7)$$

Proof. For $n = 0$, (1.7) is proved in [11], so suppose (1.7) for $(n-1)$.

Multiplying (1.1) by $|u_{1\varepsilon}^n|^k u_{1\varepsilon}^n$, k integer, and integrating over Ω to obtain :

$$\frac{1}{k+2}\int_\Omega |u_{1\varepsilon}^n|^{k+2} dx + \varepsilon(k+1)(\frac{p_1}{k+p_1})^{p_1} \left\| \nabla (u_{1\varepsilon}^n)^{(k+p_1)/p_1} \right\|_{p_1}^{p_1}$$
$$+ \int_\Omega a_1(u_{1\varepsilon}^n).\nabla u_{1\varepsilon}^n |u_{1\varepsilon}^n|^k u_{1\varepsilon}^n dx =$$
$$\int_\Omega f_1(t)(u_{1\varepsilon}^n)^{\alpha_1+1} |u_{1\varepsilon}^n|^k dx + \int_\Omega h_1(x, u_{2\varepsilon}^{n-1}) u_{1\varepsilon}^n |u_{1\varepsilon}^n|^k dx. \qquad (1.8)$$

We note that

$$\int_\Omega a_1(u_{1\varepsilon}^n).\nabla u_{1\varepsilon}^n |u_{1\varepsilon}^n|^k u_{1\varepsilon}^n dx = \int_\Omega \sum_{j=1}^N a_{1j}(u_{1\varepsilon}^n) |u_{1\varepsilon}^n|^k u_{1\varepsilon}^n \frac{\partial u_{1\varepsilon}^n}{\partial x_j} dx$$
$$= \sum_{j=1}^N \int_\Omega \left(\int_0^{u_{1\varepsilon}^n} a_{1j}(s) |s|^k s ds \right)_{x_j} dx$$

$$= \sum_{j=1}^{N} \int_{\partial\Omega} \left(\int_0^{u_{1\varepsilon}^n} a_{1j}(s) |s|^k sds \right) \cos(n, x_j)ds = 0, \tag{1.9}$$

$$\int_\Omega f_1(t)(u_{1\varepsilon}^n)^{\alpha_1+1} |u_{1\varepsilon}^n|^k dx + \int_\Omega h_1(x, u_{2\varepsilon}^{n-1}) u_{1\varepsilon}^n |u_{1\varepsilon}^n|^k dx$$

$$\leq C \left[f(t) \|u_{1\varepsilon}^n\|_{k+\alpha_1+1}^{k+\alpha_1+1} + \|h\|_{k+2} \|u_{1\varepsilon}^n\|_{k+2}^{k+1} \right]. \tag{1.10}$$

If $1 \leq \alpha_1 < p_i - 1$, by Hölder's inequality and Lemma 2.1, we have

$$\|u_{1\varepsilon}^n\|_{k+\alpha_1+1}^{k+\alpha_1+1} \leq C \|u_{1\varepsilon}^n\|_{k+2}^{\theta_1} \|u_{1\varepsilon}^n\|_q^{\theta_2}$$

$$\leq C \|u_{1\varepsilon}^n\|_{k+2}^{\theta_1} \left\| \nabla (u_{1\varepsilon}^n)^{(k+p_1)/p_1} \right\|_{p_i}^{\theta_2 p_1/(k+p_1)}$$

$$\leq \frac{\varepsilon}{2M}(k+1)(\frac{p_1}{k+p_1})^{p_1} \left\| \nabla (u_{1\varepsilon}^n)^{(k+p_1)/p_1} \right\|_{p_1}^{p_1}$$

$$+ C \left(\|u_{1\varepsilon}^n\|_{k+2}^{\theta_1} \right)^{r/\theta_1} (k+2)^\sigma, \tag{1.11}$$

where we set
$q = (k+p_1)N/(N-p_1), \theta_1 = (k+2)\left[q - (p_1 + \alpha_1 + 1)\right]/(q-k-2)$
$, \theta_2 = q(\alpha_1 - 1)/(q-k-2), r < k+2$ and $\sigma > 0$ is a constant independent of k.

If $0 < \alpha_1 < 1$, by Hölder's inequality and Young's inequality, we have

$$\|u_{1\varepsilon}^n\|_{k+\alpha_1+1}^{k+\alpha_1+1} \leq \|u_{1\varepsilon}^n\|_{k+2}^{k+2} + C \|u_{1\varepsilon}^n\|_{k+2}^{k+1}. \tag{1.12}$$

If $\alpha_1 = 0$, by Hölder's inequality , we have

$$\|u_{1\varepsilon}^n\|_{k+\alpha_1+1}^{k+\alpha_1+1} \leq \max\left(1, |\Omega|^{\frac{1}{2}}\right) \|u_{1\varepsilon}^n\|_{k+2}^{k+1}. \tag{1.13}$$

It follows from (1.8) - (1.13) that

$$\frac{d}{dt} \|u_{1\varepsilon}^n\|_{k+2}^{k+2} + c_1(k+2)^{-(p_1-2)} \left\| \nabla (u_{1\varepsilon}^n)^{(k+p_1)/p_1} \right\|_{p_1}^{p_1}$$

$$\leq c' \left[(k+2)^{\sigma+1} \|u_{1\varepsilon}^n\|_{k+2}^{k+2} + 1 \right]. \tag{1.14}$$

Setting
$k_1 = p_1, k_l = p_1 k_{l-1} - p_1 + 2, \delta_l = \frac{(k_l+p_1-2)}{\theta_l} - k_l \quad (> p_1 - 2)$,
$\theta_l = \frac{N((p_1-1)k_l - p_1 + 2)}{k_l((p_1-1)N+p_1)}$. By lemma 2.1, we have

$$\|u_{1\varepsilon}^n\|_{k_l} \leq c \|u_{1\varepsilon}^n\|_{k_{l-1}}^{1-\theta_l} \left\| \nabla (u_{1\varepsilon}^n)^{(k_l+p_1)/p_1} \right\|_{p_1}^{\theta_l p_1/(k_l+p_1-2)}. \tag{1.15}$$

Set $k = k_l$ in (1.14) and by (1.15), we have

$$\frac{d}{dt}\left\|u_{1\varepsilon}^{n}\right\|_{k_l}^{k_l}+c_1c^{-(k_l+p_1-2)\theta_l^{-1}}k_l^{-p_1+2}.\left\|u_{1\varepsilon}^{n}(t)\right\|_{k_{l-1}}^{(k_l+p_1-2)(\theta_l-1)/\theta_l}\left\|u_{1\varepsilon}^{n}(t)\right\|_{k_l}^{(k_l+p_1-2)/\theta_l}$$

$$\leq c\left[(k_l)^{\sigma+1}\left\|u_{1\varepsilon}^{n}(t)\right\|_{k_l}^{k_l}+1\right]. \tag{1.16}$$

Therefore,

$$\frac{d}{dt}\left\|u_{1\varepsilon}^{n}\right\|_{k_l}+c_1c^{-(k_l+p_1-2)\theta_l^{-1}}k_l^{-p_1+1}\ \left\|u_{1\varepsilon}^{n}(t)\right\|_{k_l}^{(p_1-1-\delta_l)}\left\|u_{1\varepsilon}^{n}(t)\right\|_{k_l}^{\delta_l+1}$$

$$\leq c\left[(k_l)^{\sigma}\left\|u_{1\varepsilon}^{n}\right\|_{k_l}+1\right]. \tag{1.17}$$

Let $\lambda_l=\sup_t\left\|u_{1\varepsilon}^{n}\right\|_{k_l}$, by lemma 2.2, we obtain

$$\lambda_l\leq\max(1,R),\qquad R=\left\{c'c^{(k_l+p_1-2)\theta_l^{-1}}k_l^{p_1-1+\sigma}.\lambda_{l-1}^{(p_1-1-\delta_l)}\right\}^{\frac{1}{\delta l}}.$$

By [7], we have $\{\lambda_l\}$ bounded and we set without loss of generality that $R>1$, which implies

$$\sup_t\left\|u_{1\varepsilon}^{n}\right\|_{\infty}\leq c_0. \tag{1.18}$$

The same holds also for $u_{2\varepsilon}^{n}$.■

Lemma 2.4. $u_{i\varepsilon}^{n}$ satisfies the following:

$$\int_0^{\omega}\left\|\nabla u_{i\varepsilon}^{n}\right\|_{p_i}^{p_i}dt\leq c_i<\infty, \tag{1.19}$$

$$\int_0^{\omega}\left\|\frac{\partial u_{i\varepsilon}^{n}}{\partial t}\right\|_2^2 dt\leq c_i<\infty. \tag{1.20}$$

Proof of lemma 2.4. Multiplying (1.1) by $u_{1\varepsilon}^{n}$, we get :

$$\int_0^{\omega}\int_{\Omega}u_{1\varepsilon}^{n}\frac{\partial u_{1\varepsilon}^{n}}{\partial t}dxdt+\int_0^{\omega}\int_{\Omega}\left|\nabla u_{1\varepsilon}^{n}\right|^{p_1}dxdt+\int_0^{\omega}\int_{\Omega}a_1(u_{1\varepsilon}^{n}).\nabla u_{1\varepsilon}^{n}u_{1\varepsilon}^{n}dxdt$$

$$=\int_0^{\omega}\int_{\Omega}f(t)(u_{1\varepsilon}^{n})^{\alpha_1+1}dxdt+\int_0^{\omega}\int_{\Omega}h_1(t,u_{2\varepsilon}^{n-1})dxdt. \tag{1.21}$$

By the periodicity, Hölder's inequality, Poincare's inequality and the L$^{\infty}$ norm bounded of $u_{i\varepsilon}^{n}$, we have

$$\begin{aligned}\int_0^{\omega}\left\|\nabla u_{1\varepsilon}^{n}\right\|_{p_1}^{p_1}dt &\leq \int_0^{\omega}\left\|f(t)\right\|_{r^*}\left\|u_{1\varepsilon}^{n}\right\|_{p_1}^{\alpha_1+1}dt+\int_0^{\omega}\left\|h(.,t)\right\|_{p_1}\left\|u_{1\varepsilon}^{n}\right\|_{p_1}^{p_1-1}dt,\\ &\leq \left(\int_0^{\omega}\left\|f(t)\right\|_{r^*}^{r^*}dt\right)^{1/r^*}\left(\int_0^{\omega}\left\|u_{1\varepsilon}^{n}\right\|_{p_1}^{p_1}dt\right)^{(\alpha_1+1)/p_1}\\ &\quad+\left(\int_0^{\omega}\left\|h(.,t)\right\|_{p_1}^{p_1}dt\right)^{1/p_1}\left(\int_0^{\omega}\left\|u_{1\varepsilon}^{n}\right\|_{p_1}^{p_1}dt\right)^{(p_1-1)/p_1}\end{aligned}$$

$$c\left[\left(\int_0^\omega \|\nabla u_{1\varepsilon}^n\|_{p_1}^{p_1}\,dt\right)^{(\alpha_1+1)/p_1}+\left(\int_0^\omega \|\nabla u_{1\varepsilon}^n\|_{p_1}^{p_1}\,dt\right)^{(p_1-1)/p_1}\right], \tag{1.22}$$

in which $r^* = p_1/(p_1-1-\alpha_1)$. Thus, we have

$$\int_0^\omega \|\nabla u_1^n\|_{p_1}^{p_1}\,dt \le c < \infty, \tag{1.23}$$

Multiplying (1.1) by $\frac{\partial u_{1\varepsilon}^n}{\partial t}$ and integrating over $[0,\omega]\times\Omega$, we get :

$$\int_0^\omega \left\|\frac{\partial u_{1\varepsilon}^n}{\partial t}(t)\right\|_2^2 dt + \int_0^\omega\int_\Omega a_1(u_{1\varepsilon}^n).\nabla u_{1\varepsilon}^n \frac{\partial u_{1\varepsilon}^n}{\partial t}dxdt$$

$$\le \int_0^\omega f(t)dt\int_\Omega (u_{1\varepsilon}^n)^{\alpha_1}\frac{\partial u_{1\varepsilon}^n}{\partial t}dx + \int_0^\omega\int_\Omega \left|h_1(t,u_{2\varepsilon}^{n-1})\frac{\partial u_{1\varepsilon}^n}{\partial t}\right|dxdt \tag{1.24}$$

Hence, by (H1), (1.24) and Young's inequality, we have

$$\int_0^\omega \left\|\frac{\partial u_{1\varepsilon}^n}{\partial t}\right\|_2^2 dt \le c < \infty. \tag{1.25}$$

Whence lemma 2.4. Passing to the limit in $\varepsilon \to 0$ and $n \to +\infty$ is almost the same in [9].

Proof of theorem 1.2

Multiplying (1.1) by $-div(|\nabla u_{1\varepsilon}^n|^{k-2}\nabla u_{1\varepsilon}^n)$ $(k > p_1)$, and integrating over Ω, we have

$$\frac{1}{k}\frac{d}{dt}\|\nabla u_{1\varepsilon}^n(t)\|_k^k + \int_\Omega \operatorname{div}\left[\left(|\nabla u_{1\varepsilon}^n|^2+\varepsilon\right)^{\frac{p_1-2}{2}}\nabla u_{1\varepsilon}^n)\right]\operatorname{div}(|\nabla u_{1\varepsilon}^n|^{k-2}\nabla u_{1\varepsilon}^n)dx$$

$$= \int_\Omega a_1(u_{1\varepsilon}^n).\nabla u_{1\varepsilon}^n \operatorname{div}(|\nabla u_{1\varepsilon}^n|^{k-2}\nabla u_{1\varepsilon}^n)dx - \int_\Omega f_1(t)(u_{1\varepsilon}^n)^{\alpha_1+1}\operatorname{div}(|\nabla u_{1\varepsilon}^n|^{k-2}\nabla u_{1\varepsilon}^n)dx$$

$$-\int_\Omega h_1(x,u_{2\varepsilon}^{n-1})\operatorname{div}(|\nabla u_{1\varepsilon}^n|^{k-2}\nabla u_{1\varepsilon}^n)dx. \tag{1.26}$$

By hypothesis and Young's inequality, we get

$$\int_\Omega a_1(u_{1\varepsilon}^n).\nabla u_{1\varepsilon}^n \operatorname{div}(|\nabla u_{1\varepsilon}^n|^{k-2}\nabla u_{1\varepsilon}^n)dx$$

$$\le \int_\Omega |\nabla u_{1\varepsilon}^n|^{k+p_1-4}\left|D^2u_{1\varepsilon}^n\right|^2dx + \int_\Omega k^2\,|a_1(u_{1\varepsilon}^n)|^2\,|\nabla u_{1\varepsilon}^n|^{k-p_1-2}dx$$

$$\le \int_\Omega |\nabla u_{1\varepsilon}^n|^{k+p_1-4}\left|D^2u_{1\varepsilon}^n\right|^2dx + ck^2(1+\|\nabla u_{1\varepsilon}^n(t)\|_k^k), \tag{1.27}$$

$$-\int_\Omega f_1(t)(u_{1\varepsilon}^n)^{\alpha_1+1}\operatorname{div}(|\nabla u_{1\varepsilon}^n|^{k-2}\nabla u_{1\varepsilon}^n)dx = -\int_\Omega \nabla f_1.\,|\nabla u_{1\varepsilon}^n|^{k-2}\nabla u_{1\varepsilon}^n)dx$$

$$\leq c\left(\|\nabla u_{1\varepsilon}^{n}(t)\|_{k}^{k-1}+\|\nabla u_{1\varepsilon}^{n}(t)\|_{k}^{k}\right), \tag{1.28}$$

$$-\int_{\Omega} h_1(x,u_{2\varepsilon}^{n-1})\,\mathrm{div}(|\nabla u_{1\varepsilon}^{n}|^{k-2}\,\nabla u_{1\varepsilon}^{n})dx$$

$$=-\int_{\Omega}\left(\nabla_x h_1(x,u_{2\varepsilon}^{n-1})+\frac{\partial h_1(x,s)}{\partial s}\nabla u_{2\varepsilon}^{n-1}\right).\,|\nabla u_{1\varepsilon}^{n}|^{k-2}\,\nabla u_{1\varepsilon}^{n})dx$$

$$\leq c\,\|\nabla u_{1\varepsilon}^{n}(t)\|_{k}^{k-1}. \tag{1.29}$$

By integration by parts, we have

$$\int_{\Omega}\mathrm{div}\left[\left(|\nabla u_{1\varepsilon}^{n}|^{2}+\varepsilon\right)^{\frac{p_i-2}{2}}\nabla u_{1\varepsilon}^{n})\right]\mathrm{div}(|\nabla u_{1\varepsilon}^{n}|^{k-2}\,\nabla u_{1\varepsilon}^{n})dx\geq\int_{\Omega}|\nabla u_{1\varepsilon}^{n}|^{k+p_1-4}\left|D^2u_{1\varepsilon}^{n}\right|^{2}dx$$

$$\geq\int_{\Omega}|\nabla u_{1\varepsilon}^{n}|^{k+p_1-4}\left|D^2u_{1\varepsilon}^{n}\right|^{2}dx+\frac{p_1-2}{4}\int_{\Omega}|\nabla u_{1\varepsilon}^{n}|^{k+p_1-6}\left|\nabla\left(|\nabla u_{1\varepsilon}^{n}|^{2}\right)\right|^{2}dx$$

$$-c(N-1)\int_{\partial\Omega}|\nabla u_{1\varepsilon}^{n}|^{k+p_1-2}\,H(x)dS, \tag{1.30}$$

where we note that $|\nabla v|=\left|\frac{\partial v}{\partial n}\right|$ on $\partial\Omega$ and $H(x)=(N-1)^{-1}((\frac{\partial v}{\partial n})^{-1}(\triangle v-\frac{\partial^2 v}{\partial n^2})$ is the mean curvature on $\partial\Omega$. Using that Ω is convex so $\partial\Omega$ is of C^2 and $H(x)$ at $x\in\partial\Omega$ nonpositive with respect to the ouward normal. We have from (1.26)-(1.30) that

$$\frac{1}{k}\frac{d}{dt}\|\nabla u_{1\varepsilon}^{n}(t)\|_{k}^{k}+\frac{c}{k}\left\||\nabla u_{1\varepsilon}^{n}|^{\frac{k+p_1-2}{2}}\right\|_{1,2}^{2}\leq ck^2(1+\|\nabla u_{1\varepsilon}^{n}(t)\|_{k}^{k})+c\,\|\nabla u_{1\varepsilon}^{n}(t)\|_{k}^{k-1}$$

$$+\frac{c}{k}\int_{\partial\Omega}|\nabla u_{1\varepsilon}^{n}|^{k+p_1-2}\,dx. \tag{1.31}$$

By lemma 2.1 and Young's inequality, we have

$$\|\nabla u_{1\varepsilon}^{n}(t)\|_{k+p_1-2}^{k+p_1-2}\leq\frac{1}{2}\left\||\nabla u_{1\varepsilon}^{n}|^{\frac{k+p_1-2}{2}}\right\|_{1,2}^{2}+c\,\|\nabla u_{1\varepsilon}^{n}\|_{p_1}^{p_1-1}\,\|\nabla u_{1\varepsilon}^{n}\|_{k}^{k-1}. \tag{1.32}$$

Therefore, (1.31) can be rewritten as

$$\frac{d}{dt}\|\nabla u_{1\varepsilon}^{n}(t)\|_{k}^{k}+c\left\||\nabla u_{1\varepsilon}^{n}|^{\frac{k+p_1-2}{2}}\right\|_{1,2}^{2}\leq ck^3(1+\|\nabla u_{1\varepsilon}^{n}(t)\|_{k}^{k})+ck\,\|\nabla u_{1\varepsilon}^{n}\|_{k}^{k-1}. \tag{1.33}$$

Then, setting

$$k_1 = p_1 - 2, k_l = 2k_{l-1} - p_1 + 2,\ \theta_l = \frac{2N}{N+2}(1 - \frac{p_{l-1}}{p_l}), l = 2, 3, ...$$

Again by lemma 2.1, we have

$$\|\nabla u_{1\varepsilon}^n(t)\|_{p_l} \le c^{\frac{2}{p_l+p_1-2}} \|\nabla u_{1\varepsilon}^n\|_{p_{l-1}}^{1-\theta_l} \left\| |\nabla u_{1\varepsilon}^n|^{\frac{p_l+p_1-2}{2}} \right\|_{1,2}^{2\theta_l/(p_l+p_1-2)}. \tag{1.34}$$

Therefore, from (1.33) and (1.34), we obtain

$$\frac{d}{dt}\|\nabla u_{1\varepsilon}^n(t)\|_{p_l}^{p_l} + cc^{-2/\theta_l}\||\nabla u_{1\varepsilon}^n|\|_{1,2}^{(p_l+p_1-2)(\theta_l-1)/\theta_l}\|\nabla u_{1\varepsilon}^n(t)\|_{p_l}^{(p_l+p_1-2)/\theta_l})$$

$$\le ck_l^3(1 + \|\nabla u_{1\varepsilon}^n(t)\|_{k_l}^{k_l}) + ck_l\|\nabla u_{1\varepsilon}^n\|_{k_l}^{k_l-1}. \tag{1.35}$$

Applying Lemma 2.2, by the same argument as in lemma 2.3, we can obtain (1.20).

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