Electronic Journal of Qualitative Theory of Differential Equations
2015, No. **16**, 1–14; doi: 10.14232/ejqtde.2015.1.16 http://www.math.u-szeged.hu/ejqtde/

Multiple solutions for Kirchhoff type problems involving super-linear and sub-linear terms

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Received 14 December 2015, appeared 24 March 2016
Communicated by Dimitri Mugnai

Abstract. In this paper, we consider the multiplicity of solutions for a class of Kirchhoff type problems with concave and convex nonlinearities on an unbounded domain. With the aid of Ekeland's variational principle, Jeanjean's monotone method and the Pohožaev identity we prove that the Kirchhoff problem has at least two solutions.

Keywords: Kirchhoff type problem, Pohožaev identity, variational method.

2010 Mathematics Subject Classification: 35A01, 35A15.

1 Introduction

This paper concerns the multiplicity of solutions for the following Kirchhoff type problem

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\triangle u+u=f(u)+g(x)|u|^{q-2}u,\qquad x\in\mathbb{R}^3, \tag{1.1}$$

where a,b are positive constants, $1<q<2$, $g(x)$ is a continuous function and f is a superlinear, subcritical nonlinearity.

Kirchhoff type problems were proposed by Kirchhoff in 1883 [14] as an extension of the classical D'Alembert's wave equation for free vibration of elastic strings. Kirchhoff's model takes into account the changes in the length of the string produced by transverse vibrations. It is related to the stationary analogue of the equation

$$\begin{cases} u_{tt}-(a+b\int_\Omega|\nabla u|^2dx)\triangle u=h(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where u denotes the displacement, $h(x,u)$ the external force and b the initial tension while a is related to the intrinsic properties of the string (such as Young's modulus). Such problems are often viewed as nonlocal because the presence of the integral term $\int_\Omega|\nabla u|^2dx$ which implies that the problem (1.2) is no longer a pointwise identity. This phenomenon causes some mathematical difficulties making the study of such problems particularly interesting.

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Besides, a similar nonlocal problem also appears in other fields such as physical and biological systems, where u describes a process that depends on its average, for example, the population density.

The case of Kirchhoff problems where the nonlinear term is super-triple or super-linear has been investigated in the last decades by many authors, for example [4–10, 13, 17–20] and references therein. Here, we are interested in the case of Kirchhoff problems where the nonlinearity includes super-linear and sub-linear terms. Recently, Chen and Li [3] considered the following nonhomogeneous Kirchhoff equation

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\triangle u+V(x)u=f(u)+g(x)\quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $a,b>0$, $N\leq 3$. Under the conditions $g(x)\in L^2(\mathbb{R}^N)$ and

- (V) $V(x)\in C(\mathbb{R}^N,\mathbb{R})$, $\inf_{\mathbb{R}^N}V(x)\geq C_1>0$ and for each $M>0$, meas$(\{x\in\mathbb{R}^N : V(x)\leq M\})<\infty$, where C_1 is a positive constant and "meas" denotes the abbreviation of Lebesgue measure in $\mathbb{R}^N$;
- (f_1) $\frac{f(s)}{s}\to 0$ as $s\to 0$;
- (f_2') $f\in C(\mathbb{R}^N,\mathbb{R})$ and for some $2<p<2^*=\frac{2N}{N-2}$, $C_2>0$, $|f(s)|\leq C_2(1+|s|^{p-1})$;
- (f_3') there exists $\overline{\mu}>4$ such that $sf(s)\geq\overline{\mu}F(s):=\overline{\mu}\int_0^s f(z)dz$;
- (f_4') $\inf_{x\in\mathbb{R}^N,|s|=1}F(x,s)>0$,

they proved that (1.3) has at least two solutions when $\|g\|_{L^2(\mathbb{R}^N)}$ is small.

Jiang, Wang and Zhou [12] studied the following nonhomogeneous Schrödinger–Maxwell system

$$\begin{cases}-\triangle u+u+\lambda\phi(x)u=|u|^{p-2}u+g(x) & \text{in } \mathbb{R}^3,\\ -\triangle\phi=u^2 & \text{in } \mathbb{R}^3,\end{cases} \tag{1.4}$$

where $\lambda>0$, $p\in(2,6)$ and $0\leq g(x)=g(|x|)\in L^2(\mathbb{R}^3)$. By using a cut-off functional to obtain a bounded Palais–Smale sequence ((PS) sequence in short), they proved that there is a constant $C_p>0$ such that (1.4) has at least two solutions for $p\in(2,6)$ provided that $\|g\|_{L^2}\leq C_p$; however, for $p\in(2,3)$ they needed to assume in addition that $\lambda>0$ is small.

Li, Li and Shi [15] considered the following Kirchhoff type problem

$$\left(a+\lambda\int_{\mathbb{R}^N}(|\nabla u|^2+bu^2)dx\right)(-\triangle u+bu)=f(u)\quad \text{in } \mathbb{R}^N, \tag{1.5}$$

where $N\geq 3$, $a,b>0$ and the parameter $\lambda\geq 0$. Under the conditions (f_1) and

- $(\overline{f})$ $f\in C(\mathbb{R}^N,\mathbb{R})$ and for some $2<p<2^*=\frac{2N}{N-2}$, $C>0$, $|f(s)|\leq C(|s|+|s|^{p-1})$;
- (f_2) $\frac{f(s)}{s}\to+\infty$ as $s\to+\infty$,

they proved that there exists $\lambda_0>0$ such that for any $\lambda\in[0,\lambda_0)$, (1.5) has at least one positive solution. Moreover, they pointed out that it is not clear whether (1.5) has a solution for large $\lambda>0$.

Motivated by these papers [3, 8, 12, 15, 17], we consider the Kirchhoff problem (1.1) with super-linear and sub-linear terms on the whole space $\mathbb{R}^3$. By the fact that the nonlocal term

$(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2$ is homogeneous of degree 4 and that the nonlinearity is a combination of super-linearity and sub-linearity, we are unable to use the method in [3] to obtain a bounded (PS) sequence. Here, we overcome the difficulties with the aid of Jeanjean's monotone method and the Pohožaev identity.

Theorem 1.1. *Assume that in the problem (1.1), $f(u) = |u|^{p-2}u$ with $2 < p < 6$ and $g(x)$ is a nonnegative function with the following property:*

(g_1) $0 \leq g(x) = g(|x|) \neq 0$ *and* $g(x) \in C^1(\mathbb{R}^3) \cap L^{q^*}(\mathbb{R}^3)$, *where* $q^* = \frac{2}{2-q}$;

(g_2) $\langle \nabla g(x), x\rangle \in L^{q^*}(\mathbb{R}^3)$.

There exists $\sigma > 0$ such that if $|g|_{q^*} \in (0, \sigma)$, the problem (1.1) has two positive solutions, one of which has a positive energy and the other a negative energy.

Theorem 1.2. *Assume that in the problem (1.1), $f \in C(\mathbb{R}^3, \mathbb{R})$ satisfies the conditions $(f_1), (f_2)$ and*

(f_3) *there exists $C_1, C_2 > 0$ such that $|\frac{f(s)}{s^5}| < C_1$ if $|s| \geq C_2$;*

(f_4) *there exists $\mu > 2$ such that $f(s)s \geq \mu F(s) > 0, \forall s \neq 0$, where $F(s) = \int_0^s f(z)dz$.*

Moreover, assume that $g(x)$ is a nonnegative function satisfying the conditions $(g_1), (g_2)$ and

(g_3) $g(x) - \langle \nabla g(x), x\rangle \in L^{\overline{q}}(\mathbb{R}^3)$, *where* $\overline{q} = \frac{\mu}{\mu - q} \in (2, 6)$.

There exists $\overline{\sigma} > 0$, which depends on f, such that if $|g|_{q^*} \in (0, \overline{\sigma})$, the problem (1.1) has two solutions, one of which has a positive energy and the other a negative energy. Moreover, if, in addition, $f(u)$ is odd, then the solutions are positive.

Remark 1.3. For example, $f(u) = |u|^{p-2}u$ $(2 < p < 6)$ satisfies the conditions (f_1)–(f_4) and $g(x) \in C_0^\infty(\mathbb{R}^3)$ satisfies the conditions (g_1)–(g_3).

Remark 1.4. In the previous papers, because of the nonlocal term $(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2$ with 4-degrees, the Kirchhoff problem (1.1) is usually considered under the condition (f_3'), which implies that $f(u)$ is super-triple. In other way, the nonlinear condition (f_3') demands $N \leq 3$, thus the corresponding Kirchhoff type problem is usually studied in $\mathbb{R}^N$ with $N \leq 3$. In the spirit of [3,8,12,15,17], we consider the Kirchhoff problem (1.1) under the condition (f_4), which implies the nonlinearity $f(u)$ is a super-linear term. This nonlinear condition (f_4) can allow the dimension $N \geq 3$. However, in this paper, for simplicity, we still consider the problem (1.1) in $\mathbb{R}^3$.

Remark 1.5. Consider the problem (1.1) with $q = 1$:

$$-(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx)\triangle u + u = f(u) + g(x), \qquad x \in \mathbb{R}^3, \tag{1.6}$$

where $0 \leq g(x) = g(|x|) \in L^2(\mathbb{R}^3)$ and f satisfies the conditions (f_1)–(f_4). By a similar method we can prove that there is a constant $\overline{C}_p > 0$ such that if $\|g\|_{L^2} \leq \overline{C}_p$, (1.6) has two solutions with different signs of the energies.

Remark 1.6. We can also consider the following Kirchhoff problem:

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\triangle u + V(x)u = f(u) + g(x)|u|^{q-2}u, \qquad x \in \mathbb{R}^3, \tag{1.7}$$

where $1 \leq q < 2$. Suppose that $V(x)$ satisfies the condition (V) and $V(x) + \langle \nabla V(x), x \rangle$ satisfies suitable condition; $g(x)$ is a continuous function and satisfies the conditions (g_1)–(g_2) or (g_1)–(g_3); f is a superlinear and subcritical nonlinearity, which satisfies the conditions (f_1)–(f_4). With a similar method we can obtain similar results for (1.7).

This paper is organized as follows: Section 2 is dedicated to the abstract framework and some preliminary results. Sections 3 and 4 are concerned with the proofs of Theorems 1.1 and 1.2, respectively.

Throughout this paper, C or C_i is used in various places to denote distinct constants. $L^p(\mathbb{R}^N)$ denotes the usual Lebesgue space endowed with the standard norm

$$|u|_p = \left(\int_{\mathbb{R}^N} |u|^p dx \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$. When it causes no confusion, we still denote by $\{u_n\}$ a subsequence of the original sequence $\{u_n\}$.

2 Preliminary results

In this section, we will recall some preliminaries and establish the variational setting for our problem. Since g is radially symmetric, we consider the problem in the radial space $H_r^1(\mathbb{R}^3)$, whose compactness is very important to our proof. Let $E = H_r^1(\mathbb{R}^3)$ be the subspace of $H^1(\mathbb{R}^3)$ consisting of the radial functions and equipped with the norm

$$\|u\|^2 = \int_{\mathbb{R}^3} (a|\nabla u|^2 + u^2) dx,$$

which is equivalent to the usual one for $a > 0$.

The energy functional corresponding to (1.1) is

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^3} (a|\nabla u|^2 + u^2) dx + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 - \int_{\mathbb{R}^3} F(u) dx - \frac{1}{q} \int_{\mathbb{R}^3} g(x)|u|^q dx,$$

where $F(u) = \int_0^u f(s) ds$. It is well known that a weak solution of problem (1.1) is a critical point of the functional I. In the following, we are devoted to finding critical points of I.

First we give the following lemma.

Lemma 2.1 ([1, 22]). *The embedding $E \hookrightarrow L^p(\mathbb{R}^3)$ is continuous for $p \in [2, 2^*]$ and compact for $p \in (2, 2^*)$. Denote by S_p the best Sobolev constant for the embedding $E \hookrightarrow L^p(\mathbb{R}^3)$, which is given by*

$$S_p = \inf_{u \in E \setminus \{0\}} \frac{\|u\|^2}{|u|_p^2} > 0.$$

In particular,

$$|u|_p \leq S_p^{-\frac{1}{2}} \|u\|, \qquad \forall u \in E. \tag{2.1}$$

In what follows, we recall the following two lemmata, which play an important role in obtaining a bounded (PS) sequence of I.

Lemma 2.2 ([11]). *Let $(X, \|\cdot\|)$ be a Banach space and $J \subset \mathbb{R}^+$ be an interval. Consider the family of C^1 functionals on X of the form*

$$I_\lambda(u) = A(u) - \lambda B(u), \qquad \lambda \in J,$$

where $B(u) \geq 0$ and either $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$. Assume that there are two points $v_1, v_2 \in X$ such that

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\}, \qquad \lambda \in J,$$

where

$$\Gamma = \{\gamma \in C([0,1], X) \mid \gamma(0) = v_1,\ \gamma(1) = v_2\}.$$

Then, for almost every $\lambda \in J$, there is a sequence $\{v_n\} \subset X$ such that

(i) $\{v_n\}$ *is bounded;*

(ii) $I_\lambda(v_n) \to c_\lambda$;

(iii) $I'_\lambda(v_n) \to 0$ *in the dual X^{-1} of X.*

Furthermore, the map $\lambda \mapsto c_\lambda$ is continuous from the left and non-increasing.

Lemma 2.3 (Pohožaev identity [2,12,16]). *Let $u \in H^1(\mathbb{R}^3)$ be a weak solution to the problem (1.1), then we have the following Pohožaev identity:*

$$\begin{aligned} 0 =& \frac{1}{2}\int_{\mathbb{R}^3} a|\nabla u|^2 dx + \frac{3}{2}\int_{\mathbb{R}^3} u^2 dx + \frac{b}{2}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 - 3\int_{\mathbb{R}^3} F(u) dx - \frac{3}{q}\int_{\mathbb{R}^3} g(x)|u|^q dx \\ &- \frac{1}{q}\int_{\mathbb{R}^3} \langle \nabla g(x), x\rangle |u|^q dx =: P(u). \end{aligned} \tag{2.2}$$

3 Proof of Theorem 1.1

In this section, we are devoted to the proof of Theorem 1.1, so we suppose that the assumptions of Theorem 1.1 hold throughout this section. First, we prove some useful preliminary results.

Lemma 3.1. *There exists $\sigma > 0$ such that if $|g|_{q^*} \in (0, \sigma)$, then there exist $\alpha > 0$ and $\rho > 0$ such that*

$$I(u)|_{\|u\|=\alpha} \geq \rho > 0,$$

where

$$\sigma = qS_2^{\frac{q}{2}} C_{p,q} = qS_2^{\frac{q}{2}} \left(\frac{(2-q)pS_p^{\frac{p}{2}}}{2(p-q)}\right)^{\frac{2-q}{p-2}} \cdot \frac{p-2}{2(p-q)}.$$

Proof. By (g_1), the Hölder inequality and Lemma 2.1, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx - \frac{1}{q}\int_{\mathbb{R}^3} g(x)|u|^q dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{p}S_p^{-\frac{p}{2}}\|u\|^p - \frac{1}{q}|g|_{q^*} S_2^{-\frac{q}{2}}\|u\|^q \\ &= \|u\|^q \left(\frac{1}{2}\|u\|^{2-q} - \frac{1}{p}S_p^{-\frac{p}{2}}\|u\|^{p-q} - \frac{1}{q}|g|_{q^*} S_2^{-\frac{q}{2}}\right). \end{aligned} \tag{3.1}$$

Set $l(t) = \frac{1}{2}t^{2-q} - \frac{1}{p}S_p^{-\frac{p}{2}}t^{p-q}$ for $t > 0$. Direct calculations yield that

$$\max_{t>0} l(t) = l(\alpha) = \left(\frac{(2-q)pS_p^{\frac{p}{2}}}{2(p-q)}\right)^{\frac{2-q}{p-2}} \cdot \frac{p-2}{2(p-q)} =: C_{p,q},$$

where

$$\alpha = \left(\frac{(2-q)pS_p^{\frac{p}{2}}}{2(p-q)}\right)^{\frac{1}{p-2}}.$$

Then it follows from (3.1) that, if $|g|_{q^*} < \sigma$, $I(u)|_{\|u\|=\alpha} \geq \rho > 0$, where $\sigma = qS_2^{q/2}C_{p,q}$ and $\rho = \alpha^q\big(l(\alpha) - \frac{1}{q}|g|_{q^*}S_2^{-q/2}\big) > 0$. □

Lemma 3.2. *If $\{u_n\} \subset E$ is a bounded (PS) sequence of I, then $\{u_n\}$ has a strongly convergent subsequence in E.*

Proof. By Lemma 2.1, going if necessary to a subsequence, we have

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } E,\\ &u_n \to u \quad \text{in } L^p(\mathbb{R}^3),\ p \in (2, 2^*). \end{aligned}$$

Note that

$$\begin{aligned} (I'(u_n) - I'(u), u_n - u) &= (I'(u_n), u_n - u) - (I'(u), u_n - u)\\ &= \left(a + b\int_{\mathbb{R}^3}|\nabla u_n|^2dx\right)\int_{\mathbb{R}^3}|\nabla(u_n - u)|^2dx + \int_{\mathbb{R}^3}|u_n - u|^2dx\\ &\quad - b\left(\int_{\mathbb{R}^3}|\nabla u|^2dx - \int_{\mathbb{R}^3}|\nabla u_n|^2dx\right)\int_{\mathbb{R}^3}\nabla u\nabla(u_n - u)dx\\ &\quad - \int_{\mathbb{R}^3}g(x)(|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u)dx\\ &\quad - \int_{\mathbb{R}^3}(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)dx, \end{aligned}$$

then

$$\begin{aligned} \|u_n - u\|^2 \leq\ & (I'(u_n) - I'(u), u_n - u)\\ & + b\left(\int_{\mathbb{R}^3}|\nabla u|^2dx - \int_{\mathbb{R}^3}|\nabla u_n|^2dx\right)\int_{\mathbb{R}^3}\nabla u\nabla(u_n - u)dx\\ & + \int_{\mathbb{R}^3}g(x)(|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u)dx\\ & + \int_{\mathbb{R}^3}(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)dx. \end{aligned}$$

From the boundedness of $\{u_n\}$ in E and Lemma 2.1, $\{u_n\}$ is bounded in $L^p(\mathbb{R}^3)$, $p \in [2, 6)$. By twice using the Hölder inequality we obtain

$$\begin{aligned} &\left|\int_{\mathbb{R}^3}g(x)(|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u)dx\right|\\ &\quad \leq \left(\int_{\mathbb{R}^3}|g|^{q^*}dx\right)^{\frac{1}{q^*}}\left(\int_{\mathbb{R}^3}\big||u_n|^{q-2}u_n - |u|^{q-2}u\big|^{\frac{p}{q}}|u_n - u|^{\frac{p}{q}}dx\right)^{\frac{q}{p}}\\ &\quad \leq C|g|_{q^*}\left(|u_n|_p^{q-1} + |u|_p^{q-1}\right)|u_n - u|_p \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

where C is a positive constant. Similarly, we have

$$\left| \int_{\mathbb{R}^3} (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)dx \right| \to 0 \quad \text{as } n \to \infty,$$

Combining with

$$b \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx - \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right) \int_{\mathbb{R}^3} \nabla u \nabla (u_n - u) dx \to 0 \quad \text{as } n \to \infty,$$

and

$$(I'(u_n) - I'(u), u_n - u) \to 0 \quad \text{as } n \to \infty,$$

we have $\|u_n - u\| \to 0$ as $n \to \infty$. This completes the proof. □

Lemma 3.3. *There exists $u_1 \in E$ such that*

$$I(u_1) = \inf\{I(u) : u \in \overline{B}_\alpha\} < 0,$$

where $\overline{B}_\alpha = \{u \in E : \|u\| \leq \alpha\}$ and α is given in Lemma 3.1.

Proof. We choose a function $v \in E$ such that $g(x)v(x) \neq 0$, then for $t > 0$ small enough, we have

$$I(tv) \leq \frac{a}{2} t^2 \int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{b}{4} t^4 \left(\int_{\mathbb{R}^3} |\nabla v|^2 dx \right)^2 + \frac{1}{2} t^2 \int_{\mathbb{R}^3} |v|^2 dx - \frac{1}{q} t^q \int_{\mathbb{R}^3} g(x)|v|^q dx < 0.$$

This shows that $c_1 := \inf\{I(u) : u \in \overline{B}_\alpha\} < 0$. By Ekeland's variational principle [21], there exists $\{u_n\} \subset \overline{B}_\alpha$ which is a bounded (PS) sequence of I. Then, by Lemma 3.2, there exists $u_1 \in E$ such that $u_n \to u_1$ as $n \to \infty$ in E. Hence $I(u_1) = c_1 < 0$ and $I'(u_1) = 0$. □

In order to apply Lemma 2.2 to get another solution, we introduce the following approximation problem:

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \triangle u + u = \lambda |u|^{p-2}u + g(x)|u|^{q-2}u, \qquad \lambda \in [\tfrac{1}{2}, 1]. \tag{3.2}$$

Define $I_\lambda : E \to \mathbb{R}$ by

$$I_\lambda(u) = A(u) - \lambda B(u), \qquad \lambda \in [\tfrac{1}{2}, 1],$$

where $B(u) = \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx$ and

$$A(u) = \frac{1}{2} \int_{\mathbb{R}^3} (a|\nabla u|^2 + u^2) dx + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 - \frac{1}{q} \int_{\mathbb{R}^3} g(x)|u|^q dx.$$

Then $\{I_\lambda\}_{\lambda \in J}$ is a family of C^1-functionals on E associated with the problem (3.2), where $J = [\frac{1}{2}, 1]$. Obviously, we have $B(u) \geq 0, \forall u \in E$, and

$$A(u) \geq \frac{1}{2} \|u\|^2 - \frac{1}{q} |g|_{q^*} S_2^{-\frac{q}{2}} \|u\|^q \to \infty \quad \text{as } \|u\| \to \infty.$$

In the following lemma, we show that the family of functionals $\{I_\lambda\}$ satisfies the assumptions of Lemma 2.2.

Lemma 3.4. *If $|g|_{q^*} < \sigma$, then for any $\lambda \in J$ the following conclusions hold.*

(i) *There exist $\eta, \xi > 0$ and $e \in E$ with $\|e\| > \xi$ such that*

$$I_\lambda(u) \geq \eta > 0 \quad with \quad \|u\| = \xi \quad and \quad I_\lambda(e) < 0.$$

(ii)

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(0), I_\lambda(e)\}, \qquad \lambda \in J,$$

where $\Gamma = \{\gamma \in C([0,1], E) \mid \gamma(0) = 0, \ \gamma(1) = e\}$.

Proof. (i) Since $I_\lambda \geq I_1$ for all $u \in E$ and $\lambda \in [\frac{1}{2}, 1]$, by Lemma 3.1 there exist $\eta > 0$ and $\xi > 0$, which are independent of $\lambda \in [\frac{1}{2}, 1]$, such that $I_\lambda(u) \geq \eta > 0$ with $\|u\| = \xi$.

Let $v \in E \backslash \{0\}$ and set $v_t(x) = tv(t^{-2}x)$ for $t > 0$, then we have

$$I_\lambda(v_t) \leq \frac{a}{2} t^4 \int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{b}{4} t^8 \left(\int_{\mathbb{R}^3} |\nabla v|^2 dx \right)^2 + \frac{1}{2} t^8 \int_{\mathbb{R}^3} |v|^2 dx - \frac{1}{p} t^{p+6} \lambda \int_{\mathbb{R}^3} |v|^p dx.$$

Noting that $p \in (2, 6)$, there exists $t_0 > 0$ large enough, which is independent of $\lambda \in [\frac{1}{2}, 1]$, such that $I_\lambda(v_{t_0}) < 0$ for all $\lambda \in [\frac{1}{2}, 1]$. Thus, by taking $e = v_{t_0}(x)$, (i) holds.

(ii) By (i) and the definition of c_λ,

$$c_\lambda \geq c_1 \geq \eta > 0 \quad \text{for all } \lambda \in [\tfrac{1}{2}, 1].$$

Since $I_\lambda(0) = 0, I_\lambda(e) < 0$ for all $\lambda \in [\frac{1}{2}, 1]$, (ii) holds. □

Then, thanks to Lemmata 2.2, 3.2 and 3.4, there exists $\{(\lambda_n, u_n)\} \subset [\frac{1}{2}, 1] \times E$ such that $\lambda_n \to 1$ as $n \to \infty$ and

$$0 < \eta \leq I_{\lambda_n}(u_n) = c_{\lambda_n} \leq c_{\frac{1}{2}}, \qquad I'_{\lambda_n}(u_n) = 0 \quad \text{for all } n \geq 1. \tag{3.3}$$

In view of Lemma 3.2, if the sequence $\{u_n\} \subset E$ given above is bounded, there exists $u_2 \neq 0$ such that $I'(u_2) = 0$. In particular, u_2 is a non-trivial positive solution of the problem (1.1).

To complete the proof of Theorem 1.1, we just require that $\{u_n\} \subset E$ is bounded. Let

$$A_n = \int_{\mathbb{R}^3} |\nabla u_n|^2 dx, \qquad B_n = \int_{\mathbb{R}^3} |u_n|^2 dx, \qquad C_n = \lambda_n \int_{\mathbb{R}^3} |u_n|^p dx$$

and

$$D_n = \int_{\mathbb{R}^3} g(x) |u_n|^q dx, \qquad E_n = \int_{\mathbb{R}^3} \langle \nabla g(x), x \rangle |u_n|^q dx.$$

From (3.3) and Lemma 2.3, we have

$$\begin{cases} \frac{1}{2}(aA_n + B_n) + \frac{b}{4}(A_n)^2 - \frac{1}{p} C_n - \frac{1}{q} D_n = c_{\lambda_n}, \\ aA_n + B_n + b(A_n)^2 - C_n - D_n = 0, \\ \frac{a}{2} A_n + \frac{3}{2} B_n + \frac{b}{2}(A_n)^2 - \frac{3}{p} C_n - \frac{3}{q} D_n - \frac{1}{q} E_n = 0. \end{cases} \tag{3.4}$$

Lemma 3.5. *$\{u_n\}$ is bounded in E.*

Proof. We prove the lemma by the following two steps.
Step 1. $\{|u_n|_2\}$ is bounded.

By contradiction, we assume that $|u_n|_2 \to +\infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{|u_n|_2}$ and

$$X_n = a\int_{\mathbb{R}^3} |\nabla v_n|^2 dx, \qquad Y_n = b|u_n|_2^2\left(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx\right)^2, \qquad Z_n = \lambda_n |u_n|_2^{p-2}\int_{\mathbb{R}^3} |v_n|^p dx.$$

Using (g_1) and (g_2), and multiplying (3.4) by $\frac{1}{|u_n|_2^2}$, we see that

$$\begin{cases} \frac{1}{2}X_n + \frac{1}{4}Y_n - \frac{1}{p}Z_n = -\frac{1}{2} + o_n(1), \\ X_n + Y_n - Z_n = -1 + o_n(1), \\ \frac{1}{2}X_n + \frac{1}{2}Y_n - \frac{3}{p}Z_n = -\frac{3}{2} + o_n(1), \end{cases} \tag{3.5}$$

where $o_n(1) \to 0$ as $n \to \infty$. For $p \in (2, 6)$, solving (3.5), we have

$$X_n = \frac{2-p}{6-p} + o_n(1).$$

This is a contradiction for n large enough, since $X_n \geq 0$ for all $n \in \mathbb{N}$. Thus, $\{|u_n|_2\}$ is bounded.
Step 2. $\{|\nabla u_n|_2\}$ is bounded.

Similarly to the proof of Step 1, arguing by contradiction, if $|\nabla u_n|_2 \to \infty$ as $n \to \infty$. Let $w_n = \frac{u_n}{|\nabla u_n|_2}$, $M_n = b|\nabla u_n|_2^2(\int_{\mathbb{R}^3} |\nabla w_n|^2 dx)^2$ and $N_n = \lambda_n|\nabla u_n|_2^{p-2}\int_{\mathbb{R}^3} |w_n|^p dx$. Using (g_1), (g_2) and Step 1, and multiplying (3.4) by $\frac{1}{|\nabla u_n|_2^2}$, we see that

$$\begin{cases} \frac{1}{4}M_n - \frac{1}{p}N_n = -\frac{a}{2} + o_n(1), \\ M_n - N_n = -a + o_n(1), \\ \frac{1}{2}M_n - \frac{3}{p}N_n = -\frac{a}{2} + o_n(1). \end{cases} \tag{3.6}$$

From the first two equations of (3.6),we have

$$M_n = \frac{2a(p-2)}{4-p} + o_n(1), \qquad N_n = \frac{ap}{4-p} + o_n(1).$$

This together with the third equation of (3.6) implies that $p = 6 + o_n(1)$. So, if $p \neq 6$, (3.6) is impossible to hold. Thus, $\{|\nabla u_n|_2\}$ is bounded. □

Now we are in a position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.3, we find a solution u_1 of the equation (1.1) with negative energy. By Lemma 3.4, due to the Mountain Pass Theorem [21], we get a critical point u_2 of I corresponding to positive energy. Because u_1 and u_2 have different energies, it follows that $u_1 \neq u_2$. Moreover, by strong maximum principle, u_1 and u_2 are positive. Thus, we obtain two positive solutions u_1 and u_2, one of which corresponds to positive energy and another one negative energy. □

4 Proof of Theorem 1.2

At first, we assume that the assumptions of Theorem 1.2 always hold in this section. Since f is a nonhomogeneous nonlinearity, the method of Lemma 3.5 is not available. However, by the condition (g_3), we can still obtain a bounded (PS) sequence. Before proving Theorem 1.2, we give some useful preliminary results.

Lemma 4.1. *There exists $\overline{\sigma} > 0$, which depends on f, such that if $|g|_{q^*} \in (0, \overline{\sigma})$, then there exist $\overline{\alpha} > 0$ and $\overline{\rho} > 0$ such that*

$$I(u)|_{\|u\|=\overline{\alpha}} \geq \overline{\rho} > 0.$$

Proof. From (f_1) and (f_3), for all $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$|f(u)| \leq \epsilon|u| + C_\epsilon|u|^5. \tag{4.1}$$

Then, by Lemma 2.1, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^2 - \int_{\mathbb{R}^3}F(u)dx - \frac{1}{q}\int_{\mathbb{R}^3}g(x)|u|^q dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\epsilon}{2}|u|_2^2 - \frac{C_\epsilon}{6}|u|_6^6 - \frac{1}{q}|g|_{q^*}S_2^{-\frac{q}{2}}\|u\|^q \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\epsilon}{2}S_2^{-1}\|u\|^2 - \frac{C_\epsilon}{6}S_6^{-3}\|u\|^6 - \frac{1}{q}|g|_{q^*}S_2^{-\frac{q}{2}}\|u\|^q \\ &\geq \|u\|^q\left(\frac{1-S_2^{-1}\epsilon}{2}\|u\|^{2-q} - \frac{C_\epsilon}{6}S_6^{-3}\|u\|^{6-q} - \frac{1}{q}|g|_{q^*}S_2^{-\frac{q}{2}}\right). \end{aligned} \tag{4.2}$$

We fix C_ϵ with $\epsilon = \frac{S_2}{2}$. Then set $\overline{l}(t) = \frac{1}{4}t^{2-q} - \frac{C_\epsilon}{6}S_6^{-3}t^{6-q}$ for $t > 0$. By direct calculations, it yields

$$\max_{t>0}\overline{l}(t) = \overline{l}(\overline{\alpha}) = \left(\frac{3S_6^3(2-q)}{2C_\epsilon(6-q)}\right)^{\frac{2-q}{4}} \cdot \frac{1}{6-q} =: \overline{C}_q,$$

where

$$\overline{\alpha} = \left(\frac{3S_6^3(2-q)}{2C_\epsilon(6-q)}\right)^{\frac{1}{4}}.$$

Taking $\overline{\sigma} = qS_2^{q/2}\overline{C}_q$, then it follows from (4.2) that, if $|g|_{q^*} < \overline{\sigma}$, $I(u)|_{\|u\|=\overline{\alpha}} \geq \overline{\rho} > 0$, where $\overline{\rho} = \overline{\alpha}^q\big(l(\overline{\alpha}) - \frac{1}{q}|g|_{q^*}S_2^{-q/2}\big) > 0$. Note that C_ϵ depends on f, so does $\overline{\sigma}$. $\square$

Lemma 4.2. *If $\{u_n\} \subset E$ is a bounded (PS) sequence of I, then $\{u_n\}$ has a strongly convergent subsequence in E.*

Proof. Going if necessary to a subsequence, we have $u_n \rightharpoonup u$ in E. Inequality (4.1) and Lemma 3.2 of [18] imply that $\int_{\mathbb{R}^3}F(u_n)dx = \int_{\mathbb{R}^3}F(u_n - u)dx + \int_{\mathbb{R}^3}F(u)dx + o(1)$, then, similarly to Lemma 3.2, we can prove the result. $\square$

Lemma 4.3. *There exists $\overline{u}_1 \in E$ such that*

$$I(\overline{u}_1) = \inf\{I(u) : u \in \overline{B}_{\overline{\alpha}}\} < 0,$$

where $\overline{B}_{\overline{\alpha}} = \{u \in E : \|u\| \leq \overline{\alpha}\}$ and $\overline{\alpha}$ is given in Lemma 4.1.

Proof. Since the proof is similar to Lemma 3.3, we omit its details here. □

In the same way as the previous section, we introduce the following approximation problem:

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\triangle u+u=\lambda f(u)+g(x)|u|^{q-2}u, \qquad \lambda\in[\tfrac{1}{2},1]. \tag{4.3}$$

Define $I_\lambda : E\to\mathbb{R}$ by

$$I_\lambda(u)=A(u)-\lambda B(u), \qquad \lambda\in[\tfrac{1}{2},1]$$

where $B(u)=\int_{\mathbb{R}^3}F(u)dx$ and

$$A(u)=\frac{1}{2}\int_{\mathbb{R}^3}(a|\nabla u|^2+u^2)dx+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2-\frac{1}{q}\int_{\mathbb{R}^3}g(x)|u|^qdx.$$

Then $\{I_\lambda\}_{\lambda\in J}$ is a family of C^1-functionals on E corresponding to (4.3), where $J=[\frac{1}{2},1]$. It is easy to see that $B(u)\geq 0$, $\forall u\in E$ and

$$A(u)\geq\frac{1}{2}\|u\|^2-\frac{1}{q}|g|_{q^*}S_2^{-\frac{q}{2}}\|u\|^q\to\infty \quad \text{as } \|u\|\to\infty.$$

Similarly to Lemma 3.4, in the following lemma we want to show that $\{I_\lambda\}$ satisfies the assumptions of Lemma 2.2.

Lemma 4.4. *If $|g|_{q^*}<\overline{\sigma}$, then for any $\lambda\in J$, the following conclusions hold.*

(i) There exist $\overline{\eta},\overline{\xi}>0$ and $e\in E$ with $\|e\|>\overline{\xi}$ such that

$$I_\lambda(u)\geq\overline{\eta}>0 \quad \textit{with} \quad \|u\|=\overline{\xi} \quad \textit{and} \quad I_\lambda(e)<0.$$

(ii)

$$c_\lambda:=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I_\lambda(\gamma(t))>\max\{I_\lambda(0),I_\lambda(e)\}, \qquad \lambda\in J,$$

where $\Gamma=\{\gamma\in C([0,1],E)\mid\gamma(0)=0,\gamma(1)=e\}$.

Proof. (i) Since $I_\lambda\geq I_1$ for all $u\in E$ and $\lambda\in[\frac{1}{2},1]$, by Lemma 4.1 there exist $\overline{\eta},\overline{\xi}>0$, which are independent of $\lambda\in[\frac{1}{2},1]$, such that $I_\lambda(u)\geq\overline{\eta}>0$ with $\|u\|=\overline{\xi}$. Let $v\in C_0^\infty(\mathbb{R}^3)$ such that $0\leq v\leq 1, v(x)=1$ for $|x|\leq 1$, $v(x)=0$ for $|x|\geq 2$, $|\nabla v|\leq C$ and set $v_t(x)=tv(t^{-2}x)$ for $t>0$, then we have

$$\begin{aligned}I_\lambda(v_t)&\leq\frac{a}{2}t^4\int_{\mathbb{R}^3}|\nabla v|^2dx+\frac{b}{4}t^8\left(\int_{\mathbb{R}^3}|\nabla v|^2dx\right)^2+\frac{1}{2}t^8\int_{\mathbb{R}^3}|v|^2dx-t^6\lambda\int_{\mathbb{R}^3}F(tv)dx\\&=\frac{a}{2}t^4\int_{|x|\leq 2}|\nabla v|^2dx+\frac{b}{4}t^8\left(\int_{|x|\leq 2}|\nabla v|^2dx\right)^2+\frac{1}{2}t^8\int_{|x|\leq 2}|v|^2dx\\&\quad-t^8\lambda\int_{|x|\leq 2}\frac{F(tv)}{t^2v^2}v^2dx.\end{aligned}$$

From the condition (f_2), $F(tv)/(t^2v^2)\to\infty$ as $t\to\infty$, so there exists $t_0>0$ large enough, which is independent of $\lambda\in[\frac{1}{2},1]$, such that $I_\lambda(v_{t_0})<0$ for all $\lambda\in[\frac{1}{2},1]$. Thus, by taking $e=v_{t_0}(x)$, (i) holds.

(ii) By (i) and the definition of c_λ,

$$c_\lambda\geq c_1\geq\overline{\eta}>0, \quad \text{for all } \lambda\in[\tfrac{1}{2},1].$$

Since $I_\lambda(0)=0, I_\lambda(e)<0$ for all $\lambda\in[\frac{1}{2},1]$, (ii) holds. □

Then, thanks to Lemmata 2.2, 4.2 and 4.4, there exists $\{(\lambda_n, u_n)\} \subset [\frac{1}{2}, 1] \times E$ such that $\lambda_n \to 1$ as $n \to \infty$ and

$$0 < \overline{\eta} \le I_{\lambda_n}(u_n) = c_{\lambda_n} \le c_{\frac{1}{2}}, \qquad I'_{\lambda_n}(u_n) = 0 \quad \text{for all } n \ge 1. \tag{4.4}$$

In view of Lemma 4.2, if the sequence $\{u_n\} \subset E$ given above is bounded, there exists $\overline{u}_2 \neq 0$ such that $I'(\overline{u}_2) = 0$. In particular, $\overline{u}_2$ is a non-trivial solution of problem (1.1).

To complete the proof of Theorem 1.2, it is sufficient to prove that $\{u_n\}$ is bounded in E.

Lemma 4.5. *$\{u_n\}$ is bounded in E.*

Proof. From the condition (f_4), we can deduce that $F(u) \ge C|u|^\mu$ with $C = F(1)$. Let $\beta \in (\frac{1}{\mu}, \frac{1}{2})$, then by (2.1), (4.4) and Lemma 2.3, we have

$$\begin{aligned}
&4I_{\lambda_n}(u_n) - \beta I'_{\lambda_n}(u_n)u_n - P_{\lambda_n}(u_n) \\
&\quad = \left(\frac{3}{2} - \beta\right) \int_{\mathbb{R}^3} a|\nabla u_n|^2 + \left(\frac{1}{2} - \beta\right) \int_{\mathbb{R}^3} |u_n|^2 dx + \left(\frac{1}{2} - \beta\right) b \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2 \\
&\quad\quad + \lambda_n \int_{\mathbb{R}^3} (\beta f(u_n)u_n - F(u_n))dx - \int_{\mathbb{R}^3} \left[\left(\frac{1}{q} - \beta\right) g(x) - \frac{1}{q}\langle \nabla g(x), x\rangle\right] |u_n|^q dx \\
&\quad \ge \left(\frac{1}{2} - \beta\right) \|u_n\|^2 + \lambda_n(\mu\beta - 1)C|u_n|_\mu^\mu - \frac{1}{q}|u_n|_\mu^q |g(x) - \langle \nabla g(x), x\rangle|_{\overline{q}},
\end{aligned}$$

where $\overline{q} = \frac{\mu}{\mu - q} \in (2, 6)$. Since $q \in (1, 2), \mu > 2$, $\{u_n\}$ is bounded in E. □

Now we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. By Lemma 4.3, we find a solution $\overline{u}_1$ of the equation (1.1) with negative energy. By Lemma 4.4, due to the Mountain Pass Theorem [21], we get a critical point $\overline{u}_2$ of I, whose energy is positive. Thus, $\overline{u}_1$ and $\overline{u}_2$ are two different solutions with their energies having different signs. If, in addition, $f(u)$ is odd, the corresponding functional is even, then the solutions $\overline{u}_1$ and $\overline{u}_2$ are positive. □

Acknowledgements

The authors sincerely thank the reviewers for their valuable suggestions and useful comments that have led to the present improved version. This work was supported by Natural Science Foundation of China (Nos: 11371090, 11571140), Natural Science Foundation of Jiangsu Province (Nos: BK 20140525, BK 20150478) .

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