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Local invariant manifolds for delay differential equations with state space in $C^1((-\infty, 0], \mathbb{R}^n)$

Dedicated to Professor Tibor Krisztin on the occasion of his 60th birthday

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Abstract. Consider the delay differential equation $x'(t) = f(x_t)$ with the history $x_t : (-\infty, 0] \to \mathbb{R}^n$ of x at 'time' t defined by $x_t(s) = x(t+s)$. In order not to lose any possible entire solution of any example we work in the Fréchet space $C^1((-\infty, 0], \mathbb{R}^n)$, with the topology of uniform convergence of maps and their derivatives on compact sets. A previously obtained result, designed for the application to examples with unbounded state-dependent delay, says that for maps f which are slightly better than continuously differentiable the delay differential equation defines a continuous semiflow on a continuously differentiable submanifold $X \subset C^1$ of codimension n, with all time-t-maps continuously differentiable. Here *continuously differentiable* for maps in Fréchet spaces is understood in the sense of Michal and Bastiani. It implies that f is of *locally bounded delay* in a certain sense. Using this property – and a related further mild smoothness hypothesis on f – we construct stable, unstable, and center manifolds of the semiflow at stationary points, by means of transversality and embeddings.

Keywords: delay differential equation, state-dependent delay, unbounded delay, Fréchet space, local invariant manifold.

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1 Introduction

Let U be a set of maps $(-\infty, 0] \to \mathbb{R}^n$ and let a map $f : U \to \mathbb{R}^n$ be given. A solution of the delay differential equation

$$x'(t) = f(x_t) \tag{1.1}$$

is a map $x : (-\infty, 0] + I \to \mathbb{R}^n$, with $I \subset \mathbb{R}$ an interval of positive length, such that all its segments

$$x_t : (-\infty, 0] \ni s \mapsto x(t+s) \in \mathbb{R}^n, \quad t \in I,$$

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belong to U, x is differentiable on I, and satisfies (1.1) on I. In [18] we studied the initial value problem

$$x'(t) = f(x_t) \quad \text{for} \quad t > 0 \quad \text{and} \quad x_0 = \phi \in U \tag{1.2}$$

for a functional f on an open subset U of the Fréchet space C^1 of continuously differentiable maps $(-\infty, 0] \to \mathbb{R}^n$, with the topology of uniform convergence of maps and their derivatives on compact sets. Let us briefly recall the motivation for working in the Fréchet space C^1, and not in a smaller Banach space of continuously differentiable maps $(-\infty, 0] \to \mathbb{R}^n$: we did not want to exclude any possible continuously differentiable map satisfying (1.1) on some interval, neither by growth conditions at $-\infty$ nor by integrability conditions.

The main result of [18] says that if $f : C^1 \supset U \to \mathbb{R}^n$ is *continuously differentiable in the sense of Michal and Bastiani* (we come back to this below) and if its derivatives satisfy a mild extension property then the initial value problem (1.2) defines a continuous semiflow S on the continuously differentiable submanifold

$$X = \{\phi \in U : \phi'(0) = f(\phi)\}, \quad \operatorname{codim} X = n,$$

with all solution operators $S(t, \cdot) : \phi \mapsto x_t$ continuously differentiable. The extension property is that

(e) *each derivative $Df(\phi)$, $\phi \in U$, extends to a linear map $D_e f(\phi)$ on the Fréchet space C of continuous maps $(-\infty, 0] \to \mathbb{R}^n$, and the map*

$$U \times C \ni (\phi, \chi) \mapsto D_e f(\phi)\chi \in \mathbb{R}^n$$

is continuous.

The topology on C is given by uniform convergence on compact sets, of course. A first version of property (e) is the notion of being *almost Fréchet differentiable* due to [10].

A toy example covered by the result from [18] is the state-dependent delay equation

$$x'(t) = g(x(t - \Delta)), \quad \Delta = \delta(x(t)),$$

with $g : \mathbb{R} \to \mathbb{R}$ and $\delta : \mathbb{R} \to [0, \infty)$ continuously differentiable, not necessarily bounded.

Let us recall results on semiflows on submanifolds of *Banach* spaces which will be used in the sequel. The Banach spaces are

$$C_d^1 = C^1([-d, 0], \mathbb{R}^n), \quad d > 0,$$

with the norm given by

$$|\phi|_{d,1} = \max_{-d \le s \le 0} |\phi(s)| + \max_{-d \le s \le 0} |\phi'(s)|,$$

and

$$B_a^1 = \left\{ \phi \in C^1 : \lim_{s \to -\infty} \phi(s) e^{as} = 0, \ \lim_{s \to -\infty} \phi'(s) e^{as} = 0 \right\}, \quad a > 0,$$

with the norm given by

$$|\phi|_{a,1} = \sup_{s \le 0} |\phi(s)| e^{as} + \sup_{s \le 0} |\phi'(s)| e^{as}.$$

In [6, 15, 16] the initial value problem (1.2) was studied for f defined on an open subset of C_d^1, and the results apply to differential equations with *bounded* state-dependent delay. In [17] the initial value problem (1.2) was studied for f defined on an open subset of B_a^1, which covers differential equations with unbounded delay. The hypotheses are that f is continuously differentiable and that the extension property holds, with an additional requirement in the second case: for a map $f : B^1_a \supset U \to \mathbb{R}^n$ in (1.1) it is also assumed in [17] that f represents *locally bounded delay* in the sense that

(lbd) *for every* $\phi \in U$ *there are a neighbourhood* $N(\phi) \subset U$ *and some* $d > 0$ *so that* $f(\psi) = f(\chi)$ *for all* ψ, χ *in* $N(\phi)$ *with* $\psi(s) = \chi(s)$ *on* $[-d, 0]$.

It may seem surprising that in case of maps $f : U \to \mathbb{R}^n$ on open sets in Fréchet spaces property (lbd) is linked to smoothness. In fact, [18, Proposition 1.1] says that property (lbd) *follows* from *continuous differentiability in the sense of Michal and Bastiani* [1,12]. The latter notion means for a continuous map

$$g : U \to G, \quad U \subset F \text{ open}, \quad F \text{ and } G \text{ Fréchet spaces},$$

that all directional derivatives

$$Dg(u)v = \lim_{0 \neq h \to 0} \frac{1}{h}(g(u + hv) - g(u)) \in G, \quad u \in U, v \in F,$$

exist and that the map

$$U \times F \ni (u, v) \mapsto Dg(u)v \in G$$

is continuous.

This notion of continuous differentiability avoids choosing a topology on the vector space $L_c(F, G)$ of linear continuous maps $F \to G$. In case F and G are Banach spaces it is obviously weaker than continuous differentiability in the sense of Fréchet (that is, there are derivatives $Dg(u) \in L_c(F, G)$, $u \in U$, in the sense of Fréchet and $U \ni u \mapsto Dg(u) \in L_c(F, G)$ is continuous with respect to the usual norm topology on $L_c(F, G)$).

In the sequel the labels (MB) and (F) are used in order to distinguish between both notions of continuous differentiability wherever confusion might arise.

In the present paper we find local invariant manifolds at a stationary point $\bar{\phi} \in X \subset C^1$ of the semiflow S, for f continuously differentiable (MB), with property (e), and satisfying a further mild smoothness assumption (d) which requires that a map induced by f via property (lbd) is continuously differentiable (F). In order to state this precisely consider the restriction map $R_{d,1} : C^1 \ni \phi \mapsto \phi|_{[-d,0]} \in C^1_d$ and the prolongation map $P_{d,1} : C^1_d \to C^1$ given by $(P_{d,1}\phi)(s) = \phi(s)$ for $-d \leq s \leq 0$ and $(P_{d,1}\phi)(s) = \phi(-d) + (s + d)\phi'(-d)$ for $s < -d$. Both maps are linear and continuous. Choose a neighbourhood $N = N(\bar{\phi}) \subset U$ of $\bar{\phi}$ and $d > 0$ according to property (lbd). Set $\bar{\phi}_d = R_{d,1}\bar{\phi}$ and notice that

$$P_{d,1}\bar{\phi}_d = \bar{\phi} \in N$$

(because $\bar{\phi}$ is constant, see the preliminaries at the end of this introduction). By continuity there exist neighbourhoods V of $\bar{\phi}_d$ in C^1_d with $P_{d,1}V \subset N$, and due to the chain rule the composition $f \circ (P_{d,1}|_V)$ is continuously differentiable (MB), with

$$D(f \circ (P_{d,1}|_V))(\phi)\chi = Df(P_{d,1}\phi)P_{d,1}\chi.$$

We assume that

(d) *there is an open neighbourhood* U_d *of* $\bar{\phi}_d$ *in* C^1_d *with* $P_{d,1}U_d \subset N$ *so that* $f_d = f \circ (P_{d,1}|_{U_d})$ *is continuously differentiable (F).*

Combining (e) and (d) we shall see in Proposition 2.2 below that the map f_d has an extension property analogous to (e). Then results from [15,16] apply and show that the equation

$$x'(t) = f_d(x_t) \tag{1.3}$$

(with segments $x_t : [-d, 0] \ni s \mapsto x(t+s) \in \mathbb{R}^n$) defines a semiflow S_d of continuously differentiable solution operators $S_d(t, \cdot)$ on domains in the continuously differentiable submanifold

$$X_d = \{\phi \in U_d : \phi'(0) = f_d(\phi)\}, \quad \operatorname{codim} X_d = n,$$

of the Banach space C^1_d. The restriction $\bar{\phi}_d$ is a stationary point of S_d. From [6] we get local stable, center, and unstable manifolds of S_d at $\bar{\phi}_d \in X_d \subset C^1_d$.

We construct each local invariant manifold of S at $\bar{\phi} \in X \subset C^1$ in a different way. For the local stable manifold of S at $\bar{\phi} \in X \subset C^1$ we need the local stable manifold of S_d at $\bar{\phi}_d \in X_d \subset C^1_d$, and make use of a local transversality result in Fréchet spaces which is derived in the Appendix (Section 7). The local unstable manifold of S at $\bar{\phi} \in X \subset C^1$ results from embedding the local unstable manifold obtained in [17], which sits in a Banach space B^1_a, $a > 0$. The construction of a local center manifold of S at $\bar{\phi} \in X \subset C^1$ begins as in Krisztin's Lyapunov–Perron type approach to a local center manifold of S_d at $\bar{\phi}_d \in X_d \subset C^1_d$ from [6,8], and deviates at a certain point.

Section 3 below provides the tangent spaces of the local invariant manifolds of S at $\bar{\phi} \in X \subset C^1$. Using the decomposition of the Banach space $Y_d = T_{\bar{\phi}_d} X_d \subset C^1_d$ into stable, center and unstable spaces of the linearized solution operators

$$T_{d,t} : Y_d \ni \eta \mapsto D_2 S_d(t, \bar{\phi}_d)\eta \in Y_d, \quad t \geq 0,$$

from [6] we construct linear stable, center and unstable spaces in the tangent space

$$Y = T_{\bar{\phi}} X = \{\chi \in C^1 : \chi'(0) = Df(\bar{\phi})\chi\} \subset C^1,$$

for the linearized solution operators

$$T_t : Y \ni \chi \mapsto D_2 S(t, \bar{\phi})\chi \in Y, \quad t \geq 0.$$

This is done without recourse to spectral properties of the operators T_t.

Returning to the hypotheses on $f : C^1 \supset U \to \mathbb{R}^n$ it may be of interest to note that we could have started from another arrangement, in order to obtain the desired local invariant manifolds in C^1. As the objectives are local in nature it is possible to begin with property (lbd) of f where $N(\bar{\phi}) = U$. Next one can assume that an induced map like f_d on a neighbourhood of $\bar{\phi}_d$ in C^1_d is continuously differentiable (F) and that an analogue of the extension property (e) holds for the induced map. It would then follow that a restriction of f to a neighbourhood of $\bar{\phi}$ in C^1 is continuously differentiable (MB) and has property (e), which means that we are back at the set of hypotheses which we prefer and actually use in this paper. – Arguing this way one finds in particular that for the toy example, where

$$f(\phi) = g(\phi(-\delta(\phi(0)))) \quad \text{for all } \phi \in C^1 \quad (\text{with } n = 1),$$

all our hypotheses are satisfied.

Let us mention other recent work on invariant manifolds for equations with unbounded delay: in [11] Matsunaga et al. obtain local center manifolds for integral equations with unbounded state- and time-invariant delay.

Preliminaries, notation. Banach spaces also are Fréchet spaces, that is, locally convex topological vector spaces which are complete and metrizable. For each $k \in \mathbb{N}_0$ the topology on the Fréchet space C^k of k times continuously differentiable maps $(-\infty, 0] \to \mathbb{R}^n$ is given by the seminorms $|\cdot|_{k,j}$, $j \in \mathbb{N}$, with

$$|\phi|_{k,j} = \sum_{\kappa=1}^{k} \max_{-j \leq s \leq 0} |\phi^{(\kappa)}(s)|,$$

with the sets

$$V_{k,j}=\left\{\phi\in C^k:|\phi|_{k,j}<\frac{1}{j}\right\}$$

forming a neighbourhood base at the origin. In C^k we have $\phi_m\to\phi$ as $m\to\infty$ if and only if for every $j\in\mathbb{N}$, $|\phi_m-\phi|_{k,j}\to 0$ as $m\to\infty$.

Continuously differentiable submanifolds of Fréchet spaces and continuously differentiable maps on such submanifolds are defined using continuous differentiability (MB). The reference for results on calculus in Fréchet spaces based on continuous differentiability (MB) which are freely used in the sequel is [5]. See also the survey [14]. For basic facts about topological vector spaces, see [13].

In the sequel also the vector space $C^\infty=\cap_{k=0}^\infty C^k$ occurs, but without a topology on it.

It is convenient to denote the unique maximal solution to the initial value problem

$$x'(t)=f(x_t)\quad\text{for } t>0,\quad x_0=\phi\in X,$$

by x^ϕ.

Stationary points of the semiflow S are constant. (Proof of this: suppose $S(t,\phi)=\phi$ for all $t\geq 0$. The solution x of (1.1) on $[0,\infty)$ with $x_0=\phi$ satisfies $x(t)=x_t(0)=S(t,\phi)(0)=\phi(0)$ for all $t\geq 0$. For all $s<0$ we have $x(s)=\phi(s)=S(-s,\phi)(s)=x_{-s}(s)=x(0)=\phi(0)$.)

For reals $a<b$ and $k\in\mathbb{N}_0$ let $C^k([a,b],\mathbb{R}^n)$ denote the Banach space of k times continuously differentiable maps $[a,b]\to\mathbb{R}^n$, with the norm given by

$$|\phi|_{[a,b],k}=\sum_{\kappa=1}^{k}\max_{a\leq s\leq b}|\phi^{(\kappa)}(s)|.$$

In case $a=-d<b=0$ we abbreviate $C^k_d=C^k([-d,0],\mathbb{R}^n)$ and $|\cdot|_{d,k}=|\cdot|_{[-d,0],k}$.

It is easy to see that the linear restriction maps

$$R_{d,k}:C^k\to C^k_d,\quad d>0\quad\text{and}\quad k\in\mathbb{N}_0,$$

and the linear prolongation maps

$$P_{d,k}:C^k_d\to C^k,\quad d>0\quad\text{and}\quad k\in\mathbb{N}_0,$$

given by $(P_{d,k}\phi)(s)=\phi(s)$ for $-d\leq s\leq 0$ and

$$(P_{d,k}\phi)(s)=\sum_{\kappa=0}^{k}\frac{\phi^{(\kappa)}(-d)}{\kappa!}(s+d)^\kappa\quad\text{for } s<-d$$

are continuous, and for all $d>0$ and $k\in\mathbb{N}_0$,

$$R_{d,k}\circ P_{d,k}=\mathrm{id}_{C^k_d}.$$

Solutions of equations

$$x'(t)=g(x_t),\quad\text{with}\quad g:C^1_d\supset U\to\mathbb{R}^n\quad\text{or}\quad g:B^1_a\supset U\to\mathbb{R}^n,$$

on some interval $I\subset\mathbb{R}$ are defined as in case of (1.1): with $J=[-d,0]$ or $J=(-\infty,0]$, respectively, they are continuously differentiable maps $x:J+I\to\mathbb{R}^n$ so that $x_t\in U$ for all $t\in I$ and the differential equation holds for all $t\in I$. Notice that x_t may denote a map on $[-d,0]$ or on $(-\infty,0]$, depending on the context.

The following statement on "globally bounded delay" for continuous linear maps corresponds to a special case of [18, Proposition 1.1].

Proposition 1.1. *For every continuous linear map $L : C^0 \to B$, B a Banach space, there exists $r > 0$ with $L\phi = 0$ for all $\phi \in C^0$ with $\phi(s) = 0$ on $[-r, 0]$.*

Proof. Otherwise there are sequences $r_m \to \infty$ and $(\phi_m)_1^\infty$ in C^0 with $\phi_m(s) = 0$ on $[-r_m, 0]$ and $0 \neq L\phi_m$ for all $m \in \mathbb{N}$. For $c_m = |L\phi_m| > 0$ we get $\frac{1}{c_m}\phi_m \to 0$ because for each $j \in \mathbb{N}$ and for all integers m with $r_m \geq j$, $|\frac{1}{c_m}\phi_m|_{0,j} = 0$. By continuity,

$$L\left(\frac{1}{c_m}\phi_m\right) \to 0 \quad \text{as } m \to \infty,$$

contradicting

$$\left|L\left(\frac{1}{c_m}\phi_m\right)\right| = 1 \quad \text{for all } m \in \mathbb{N}.$$

□

For results on strongly continuous semigroups given by solutions of linear autonomous retarded functional differential equations

$$x'(t) = \Lambda x_t$$

with $\Lambda : C^0_d \to \mathbb{R}^n$ linear and continuous, see [2,4].

2 On locally bounded delay, the extension property, and prolongation and restriction

This section contains proofs of a few facts which were used already in Section 1, and further relations between the functionals f and f_d and between the semiflows S and S_d. Recall that f is continuously differentiable (MB) and has property (lbd), with $N = N(\bar{\phi})$ and $d > 0$.

Proposition 2.1. *For every $\phi \in N$ we have*

$$Df(\phi)\psi = 0 \quad \text{for all } \psi \in C^1 \text{ with } \psi(s) = 0 \text{ on } [-d, 0],$$

and

$$D_e f(\phi)\chi = 0 \quad \text{for all } \chi \in C^0 \text{ with } \chi(s) = 0 \text{ on } [-d, 0].$$

Proof. Let $\phi \in N$ and $\psi \in C^1$ with $\psi(s) = 0$ on $[-d, 0]$ be given. For $h \neq 0$ sufficiently small, $\phi + h\psi \in N$ (due to continuity of multiplication with scalars), hence $f(\phi + h\psi) = f(\phi)$, and thereby,

$$Df(\phi)\psi = \lim_{0 \neq h \to 0} \frac{1}{h}(f(\phi + h\psi) - f(\phi)) = \lim_{0 \neq h \to 0} \frac{1}{h}(f(\phi) - f(\phi)) = 0.$$

Let $\phi \in N$ and $\chi \in C^0$ with $\chi(s) = 0$ on $[-d, 0]$ be given. Choose a sequence of points $\chi_m \in C^1$ with $\chi_m(s) = 0$ on $[-d, 0]$ which converges to χ in the topology of C^0. (For example, let $m \geq d$ and find $\hat{\chi}_m \in C^1([-m, 0], \mathbb{R}^n)$ with

$$|\hat{\chi}_m(s) - \chi(s)| < \frac{1}{m} \quad \text{on } [-m, 0] \quad \text{and} \quad \hat{\chi}_m(s) = 0 \quad \text{on } [-d, 0].$$

Extend $\hat{\chi}_m$ to $\chi_m \in C^1$ by $\chi_m(s) = \hat{\chi}_m(-m) + (\hat{\chi}_m)'(-m)(s + m)$ for $s < -m$. Conclude that for each $j \in \mathbb{N}$, $|\chi_m - \chi|_{0,j} \to 0$ as $m \to \infty$.) – We obtain

$$D_e f(\phi)\chi = \lim_{m \to \infty} D_e f(\phi)\chi_m = \lim_{m \to \infty} Df(\phi)\chi_m = 0,$$

where the last equation follows from the first part of the assertion, with $\chi_m(s) = 0$ on $[-d,0]$. □

With regard to the next result on the extension property of f_d observe that each directional derivative of f_d, at a point $\phi \in U_d$ in direction of $\chi \in C^1_d$, is given by $Df_d(\phi)\chi$ with the Fréchet derivative $Df_d(\phi) \in L_c(C^1_d,\mathbb{R}^n)$.

Proposition 2.2. *Each Fréchet derivative* $Df_d(\phi) \in L_c(C^1_d,\mathbb{R}^n)$, $\phi \in U_d$, *extends to a linear map* $D_ef_d(\phi) : C^0_d \to \mathbb{R}^n$ *and the map* $U_d \times C^0_d \ni (\phi,\chi) \mapsto D_ef_d(\phi)\chi \in \mathbb{R}^n$ *is continuous.*

Proof. 1. Let $\phi \in U_d$ be given. By the chain rule for continuous differentiability (MB) in combination with the remark preceding Proposition 2.2 the Fréchet derivative of f_d at ϕ is given by $Df_d(\phi) = Df(P_{d,1}\phi) \circ P_{d,1}$. Define $D_ef_d(\phi) : C^0_d \to \mathbb{R}^n$ by $D_ef_d(\phi)\chi = D_ef(P_{d,1}\phi)P_{d,0}\chi$. The map $D_ef_d(\phi)$ is linear. It also is a continuation of $Df_d(\phi)$ since for $\chi \in C^1_d$ we have

$$\begin{aligned} D_ef_d(\phi)\chi &= D_ef(P_{d,1}\phi)P_{d,0}\chi \\ &= D_ef(P_{d,1}\phi)P_{d,1}\chi \quad \text{(with Proposition 2.1 and } P_{d,0}\chi(s) = P_{d,1}\chi(s) \text{ on } [-d,0]) \\ &= Df(P_{d,1}\phi)P_{d,1}\chi = Df_d(\phi)\chi. \end{aligned}$$

2. The continuity of the map

$$U_d \times C^0_d \ni (\phi,\chi) \mapsto D_ef_d(\phi)\chi \in \mathbb{R}^n$$

follows from its definition in combination with property (e) of f and the continuity of $P_{d,1}$ and $P_{d,0}$. □

Proposition 2.3. $X_d = R_{d,1}(X \cap N \cap R^{-1}_{d,1}(U_d))$

Proof. 1. On $X_d \subset R_{d,1}(X \cap N \cap R^{-1}_{d,1}(U_d))$. For $\phi \in X_d \subset U_d$ we have $P_{d,1}\phi \in N$. Using this and $\phi = R_{d,1}P_{d,1}\phi$ we get $P_{d,1}\phi \in N \cap R^{-1}_{d,1}(U_d)$ and

$$\begin{aligned} (P_{d,1}\phi)'(0) &= \phi'(0) = f_d(\phi) \quad (\text{by } \phi \in X_d) \\ &= f(P_{d,1}\phi), \end{aligned}$$

which means $P_{d,1}\phi \in X$. It follows that $\phi = R_{d,1}P_{d,1}\phi$ is in $R_{d,1}(X \cap N \cap R^{-1}_{d,1}(U_d))$.

2. On $R_{d,1}(X \cap N \cap R^{-1}_{d,1}(U_d)) \subset X_d$. Consider $\phi = R_{d,1}\psi$ with $\psi \in X \cap N \cap R^{-1}_{d,1}(U_d)$. Then $\phi = R_{d,1}\psi \in U_d \subset P^{-1}_{d,1}(N)$, $P_{d,1}R_{d,1}\psi = P_{d,1}\phi \in N$, $\psi \in X \cap N \subset U$, and

$$\begin{aligned} \phi'(0) &= (R_{d,1}\psi)'(0) = \psi'(0) = f(\psi) \quad (\text{since } \psi \in X) \\ &= f(P_{d,1}R_{d,1}\psi) \\ &\text{(with (lbd), } \psi \in N, P_{d,1}R_{d,1}\psi \in N, \text{ and } \psi(s) = P_{d,1}R_{d,1}\psi(s) \text{ on } [-d,0]) \\ &= f(P_{d,1}\phi) = f_d(\phi), \end{aligned}$$

which gives $\phi \in X_d$. □

Proposition 2.4. *For every* $\phi \in X \cap N \cap R^{-1}_{d,1}(U_d)$,

$$T_{R_{d,1}\phi}X_d = R_{d,1}T_\phi X.$$

Proof. Let $\phi \in X \cap N \cap R_{d,1}^{-1}(U_d)$ be given. Using Proposition 2.3 we infer $R_{d,1}\phi \in X_d$ and

$$R_{d,1}T_\phi X = DR_{d,1}(\phi)T_\phi X \subset T_{R_{d,1}\phi}X_d,$$

hence $\text{codim}\, R_{d,1}T_\phi X \geq \text{codim}\, T_{R_{d,1}\phi}X_d = n$. As $R_{d,1}$ is surjective and $\text{codim}\, T_\phi X = n$ we also get $n \geq \text{codim}\, R_{d,1}T_\phi X$. It follows that $R_{d,1}T_\phi X$ and $T_{R_{d,1}\phi}X_d$ have the same finite codimension n. Using the previous inclusion we obtain equality. □

Let $\Omega \subset X \times [0,\infty)$ and $\Omega_d \subset X_d \times [0,\infty)$ denote the domains of S and S_d, respectively. The unique maximal solutions to the initial value problems

$$x'(t) = f_d(x_t) \quad \text{for } t > 0, \quad x_0 = \chi \in X_d,$$

are denoted by x^χ (as in case of the initial value problem for (1.1) and data in X).

Proposition 2.5.
(i) For $(t,\phi) \in \Omega$ with $S([0,t] \times \{\phi\}) \subset N \cap R_{d,1}^{-1}(U_d)$,

$$(t, R_{d,1}\phi) \in \Omega_d \quad \text{and} \quad S_d(t, R_{d,1}\phi) = R_{d,1}S(t,\phi).$$

(ii) If $(t,\chi) \in \Omega_d$ and if $x : (-\infty, t] \to \mathbb{R}^n$ given by $x(s) = x^\chi(s)$ on $[-d,t]$ and by $x(s) = (P_{d,1}\chi)(s)$ for $s < -d$ satisfies $\{x_s : 0 \leq s \leq t\} \subset N$ then

$$(t, P_{d,1}\chi) \in \Omega \quad \text{and} \quad R_{d,1}S(t, P_{d,1}\chi) = S_d(t,\chi).$$

Proof. On (i): Let $x = x^\phi$ and set $y = x|_{[-d,t]}$. Each segment $y_s \in C_d^1$, $0 \leq s \leq t$, equals $R_{d,1}x_s \in R_{d,1}(X \cap N \cap R_{d,1}^{-1}(U_d)) = X_d$. In particular, $y_0 = R_{d,1}x_0 = R_{d,1}\phi \in U_d$, and for $0 \leq s \leq t$,

$$\begin{aligned} y'(s) &= x'(s) = f(x_s) \\ &= f(P_{d,1}R_{d,1}x_s) \\ &\quad \text{(by (lbd), using } x_s \in N, R_{d,1}x_s \in U_d \subset P_{d,1}^{-1}(N), P_{d,1}R_{d,1}x_s \in N, x_s(v) = P_{d,1}R_{d,1}x_s(v) \\ &\quad \text{on } [-d,0]) \\ &= f_d(R_{d,1}x_s) = f(y_s), \end{aligned}$$

which implies that the restriction $y = x|_{[-d,t]}$ satisfies (1.3) on $[0,t]$. Now the assertion becomes obvious.

On (ii): consider $(t,\chi) \in \Omega_d$ and the maximal solution x^χ of (1.3) and $x : (-\infty,t] \to \mathbb{R}^n$ as defined in assertion (ii) and assume the segments $x_s \in C^1$, $0 \leq s \leq t$, belong to N. For such s we have

$$\begin{aligned} x'(s) &= (x^\chi)'(s) = f_d(x_s^\chi) = f(P_{d,1}x_s^\chi) \\ &= f(x_s) \\ &\quad \text{(by (lbd), use } P_{d,1}x_s^\chi \in N, x_s \in N, (P_{d,1}x_s^\chi)(v) = x_s^\chi(v) = x^\chi(s+v) = x(s+v) = x_s(v) \\ &\quad \text{for } -d \leq v \leq 0), \end{aligned}$$

and $x_0 = P_{d,1}\chi \in N$. It follows that $(t, P_{d,1}\chi) \in \Omega$ and $x_s = S(s, P_{d,1}\chi)$ for all $s \in [0,t]$. Finally, observe $R_{d,1}x_s = x_s^\chi = S_d(s,\chi)$ for $0 \leq s \leq t$. □

Proposition 2.5 (i) shows that $\bar{\phi}_d$ is a stationary point of the semiflow S_d.

For $t \geq 0$ consider the operators $T_t = D_2S(t,\bar{\phi})$ on $T_{\bar{\phi}}X$ and $T_{d,t} = D_2S_d(t,\bar{\phi}_d)$ on $T_{\bar{\phi}_d}X_d$.

Corollary 2.6.
(i) For $(t,\phi)\in\Omega$ *as in Proposition 2.5 (i) and for all* $\chi\in T_\phi X$,

$$R_{d,1}\chi\in T_{R_{d,1}\phi}X_d \quad \text{and} \quad R_{d,1}D_2S(t,\phi)\chi = D_2S_d(t,R_{d,1}\phi)R_{d,1}\chi.$$

(ii) For all $\chi\in T_{\bar\phi}X$ *and for all* $t\geq 0$,

$$R_{d,1}\chi\in T_{\bar\phi_d}X_d \quad \text{and} \quad R_{d,1}T_t\chi = T_{d,t}R_{d,1}\chi.$$

Proof. On (i): for $\phi\in X$ with $S([0,t]\times\{\phi\})\subset N\cap R_{d,1}^{-1}(U_d)$ we have $S_d(t,R_{d,1}\phi)=R_{d,1}S(t,\phi)$, by Proposition 2.5. Let $\chi\in T_\phi X$ be given. By Proposition 2.4, $R_{d,1}\chi\in T_{R_{d,1}\phi}X_d$. By the chain rule, $D_2S_d(t,R_{d,1}\phi)R_{d,1}\chi = R_{d,1}D_2S(t,\phi)\chi$, which yields the assertion.

On (ii): we have $[0,\infty)\times\{\bar\phi\}\subset\Omega$, and for all $t\geq 0$, $S(t,\bar\phi)=\bar\phi\in N\cap R_{d,1}^{-1}(U_d)$, because of $\bar\phi\in N$ and $R_{d,1}\bar\phi=\bar\phi_d\in U_d$. Using part (i) we conclude that for all $t\geq 0$ and $\chi\in T_{\bar\phi}X$, $R_{d,1}\chi\in T_{\bar\phi_d}X_d$ and

$$T_{d,t}R_{d,1}\chi = D_2S_d(t,R_{d,1}\bar\phi)R_{d,1}\chi = R_{d,1}D_2S(t,\bar\phi)\chi = R_{d,1}T_t\chi. \qquad \square$$

3 Decompositions of tangent spaces

This section contains the decomposition of the Fréchet space $Y=T_{\bar\phi}X\subset C^1$ into the stable, center, and unstable spaces which in the subsequent sections will become the tangent spaces of the desired local invariant manifolds at $\bar\phi\in X$. The construction does not make use of spectral properties of the operators $T_t=D_2S(t,\bar\phi)$, $t\geq 0$, on Y, or of the generator of this semigroup, but exploits well-known properties of the strongly continuous semigroup on the Banach space C^0_d which arises from linearizing the semiflow S_d at $\bar\phi_d\in X_d$ as follows: in [6] it is shown that the derivatives $T_{d,t}=D_2S_d(t,\bar\phi_d)$, $t\geq 0$, form a strongly continuous semigroup on the Banach space

$$Y_d = T_{\bar\phi_d}X_d = \{\chi\in C^1_d : \chi'(0)=Df_d(\bar\phi_d)\chi\},$$

and they are given by the equations

$$T_{d,t}\chi = T_{d,e,t}\chi \quad \text{for } t\geq 0, \quad \chi\in Y$$

where $T_{d,e,t}\eta = v_t$ with the continuous solution $v:[-d,\infty)\to\mathbb{R}^n$ of the initial value problem

$$v'(t) = D_ef_d(\bar\phi_d)v_t \quad \text{for } t>0, \quad v_0=\eta\in C^0_d.$$

Here the term *continuous solution* means that v is continuous, differentiable for $t>0$, and satisfies the delay differential equation for $t>0$, as in [2,4]. – In the present section a symbol like v_t above always denotes a segment which is defined on $[-d,0]$.

The operators $T_{d,e,t}:C^0_d\to C^0_d$, $t\geq 0$, form a strongly continuous semigroup whose generator has a discrete spectrum $\sigma_{d,e}$ which consists of eigenvalues of finite algebraic multiplicity, with only a finite number of them in each halfplane $\{z\in\mathbb{C}: \operatorname{Re} z>u\}$, $u\in\mathbb{R}$. Then the stable, center, and unstable spaces of the semigroup are defined as the realified generalized eigenspaces $C^0_{d,s}, C^0_{d,c}, C^0_{d,u}$ which are given by the eigenvalues satisfying

$$\operatorname{Re} z<0, \quad \operatorname{Re} z=0, \quad \operatorname{Re} z>0,$$

respectively. The operators $T_{d,e,t}$, $t\geq 0$, map $C^0_{d,s}$ into itself and act on $C^0_{d,c}$ and on $C^0_{d,u}$ as isomorphisms. The center and unstable spaces are finite-dimensional. Initial data χ in $C^0_{d,c}$

and in $C^0_{d,u}$ uniquely define analytic solutions $v = v^{(\chi)}$ on $\mathbb{R}$ of the equation $v'(t) = D_e f_d(\bar{\phi}_d) v_t$ with $v_0 = \chi$ and with all segments $v_t : [-d, 0] \ni s \mapsto v(t+s) \in \mathbb{R}^n$, $t \in \mathbb{R}$, in $C^0_{d,c}$ and in $C^0_{d,u}$, respectively. From $\chi \in C^1_d$ and $\chi'(0) = D_e f_d(\bar{\phi}_d)\chi = Df_d(\bar{\phi}_d)\chi$ we have $\chi \in Y_d$. This yields $C^0_{d,c} \subset Y_d$, $C^0_{d,u} \subset Y_d$. For every $t \geq 0$ the operator $T_{d,t}$ given by $T_{d,e,t}$ acts as an ismorphism on $Y_{d,c} = C^0_{d,c}$ and on $Y_{d,u} = C^0_{d,u}$. With the closed space $Y_{d,s} = Y_d \cap C^0_{d,s}$,

$$Y_d = Y_{d,s} \oplus Y_{d,c} \oplus Y_{d,u} \quad \text{and} \quad T_{d,t} Y_{d,s} \subset Y_{d,s} \quad \text{for all } t \geq 0,$$

see [6]. The injective linear maps

$$I_c : C^0_{d,c} \ni \chi \mapsto v^{(\chi)}|_{(-\infty,0]} \in C^1 \quad \text{and} \quad I_u : C^0_{d,u} \ni \chi \mapsto v^{(\chi)}|_{(-\infty,0]} \in C^1$$

with finite-dimensional domains are continuous. Define

$$Y_c = I_c C^0_{d,c} = I_c Y_{d,c} \quad \text{and} \quad Y_u = I_u C^0_{d,u} = I_u Y_{d,u}.$$

Notice that

$$\phi = I_c R_{d,1} \phi \quad \text{on } Y_c \quad \text{and} \quad \phi = I_u R_{d,1} \phi \quad \text{on } Y_u.$$

The finite-dimensional spaces Y_c and Y_u are both contained in Y, because of

$$\begin{aligned} (v^{(\chi)}|_{(-\infty,0]})'(0) &= \chi'(0) = Df_d(\bar{\phi}_d)\chi \\ &= Df(P_{d,1}\bar{\phi}_d) P_{d,1}\chi \\ &= Df(\bar{\phi})(v^{(\chi)}|_{(-\infty,0]}) \\ &\text{(by Proposition 2.1).} \end{aligned}$$

The spaces Y_c and Y_u serve as center and unstable spaces in Y.

Proposition 3.1 (Conjugacy, invariance)**.** *For every $t \geq 0$,*

$$\begin{aligned} T_t I_c \chi &= I_c T_{d,t} \chi \quad \textit{for all } \chi \in Y_{d,c} = C^0_{d,c} \quad \textit{and} \\ T_t I_u \chi &= I_u T_{d,t} \chi \quad \textit{for all } \chi \in Y_{d,u} = C^0_{d,u}, \\ &\textit{and} \quad T_t Y_c = Y_c \quad \textit{and} \quad T_t Y_u = Y_u. \end{aligned}$$

Proof. Let $\chi \in C^0_{d,c}$, $v = v^{(\chi)}$, $t \geq 0$. Then $v_t = T_{d,t}\chi \in C^0_{d,c}$. The translate $w = v(t + \cdot)$ of $v : \mathbb{R} \to \mathbb{R}^n$ also is an analytic solution of the linear equation given by $D_e f_d(\bar{\phi}_d) : C^0_d \to \mathbb{R}^n$, with initial value $w_0 = v_t \in C^0_{d,c}$. Hence $w|_{(-\infty,0]} = I_c v_t$. Next, $I_c\chi = v|_{(-\infty,0]}$, and for all $s > 0$,

$$\begin{aligned} v'(s) &= D_e f_d(\bar{\phi}_d) v_s = Df_d(\bar{\phi}_d) v_s = Df(P_{d,1}\bar{\phi}_d) P_{d,1} v_s \\ &= Df(\bar{\phi}) P_{d,1} v_s \\ &= Df(\bar{\phi})(v(s+\cdot)|_{(-\infty,0]}) \\ &\text{(by Proposition 2.1, with } (P_{d,1}v_s)(r) = v_s(r) = v(s+r) \text{ for } -d \leq r \leq 0\text{),} \end{aligned}$$

which gives $T_t(v|_{(-\infty,0]}) = v(t+\cdot)|_{(-\infty,0]} = w|_{(-\infty,0]}$. Altogether,

$$T_t I_c \chi = T_t(v|_{(-\infty,0]}) = w|_{(-\infty,0]} = I_c v_t = I_c T_{d,t}\chi.$$

The proof for $\chi \in C^0_{d,u}$ is analogous. The last assertions follow from the first and second assertion, respectively. □

Define the stable space in Y as the closed space

$$Y_s = Y \cap R_{d,1}^{-1} Y_{d,s}.$$

Proposition 3.2. *$Y = Y_s \oplus Y_c \oplus Y_u$ and $T_t Y_s \subset Y_s$ for all $t \geq 0$.*

Proof. 1. Proof of $Y \subset Y_s \oplus Y_c \oplus Y_u$: for $\phi \in Y$, $R_{d,1}\phi \in Y_d$, see Proposition 2.4. There exist $\chi_s \in Y_{d,s}, \chi_c \in Y_{d,c} = C^0_{d,c}, \chi_u \in Y_{d,u} = C^0_{d,u}$ so that $R_{d,1}\phi = \chi_s + \chi_c + \chi_u$. Hence

$$R_{d,1}(\phi - I_c\chi_c - I_u\chi_u) = \chi_s + \chi_c + \chi_u - R_{d,1}I_c\chi_c - R_{d,1}I_u\chi_u = \chi_s \in Y_{d,s},$$

which in combination with $\phi - I_c\chi - I_u\chi_u \in Y$ yields $\phi - I_c\chi_c - I_u\chi_u \in Y_s$.

2.1 Proof of $Y_s \cap Y_c \subset \{0\}$: for $\phi \in Y_s \cap Y_c = (Y \cap R_{d,1}^{-1}Y_{d,s}) \cap I_c Y_{d.c}$ we have $R_{d,1}\phi \in Y_{d,s} \cap Y_{d,c} = \{0\}$. Consequently, $R_{d,1}\phi = 0$, and thereby $\phi = I_c R_{d,1}\phi = 0$.

2.2 The proof of $Y_s \cap Y_u \subset \{0\}$ is analogous.

2.3 Proof of $Y_c \cap Y_u \subset \{0\}$: for $\phi \in Y_c \cap Y_u = I_c Y_{d.c} \cap I_u Y_{d,u}$, hence $R_{d,1}\phi \in Y_{d,c} \cap Y_{d,u} = \{0\}$. Consequently, $R_{d,1}\phi = 0$, and thereby $\phi = I_c R_{d,1}\phi = 0$.

3. Let $t \geq 0$, $\phi \in Y_s$. Then $R_{d,1}\phi \in Y_{d,s}$. Using this and Corollary 2.6 one finds

$$R_{d,1}T_t\phi = T_{d,t}R_{d,1}\phi \in Y_{d,s},$$

which gives $T_t\phi \in Y_s$. □

What will be used from this section in the sequel are only the definitions of the spaces Y_s, Y_c, Y_u and the inclusion

$$I_u C^0_{d,u} = Y_u \subset B^1_a$$

which follows from $v^{(\chi)}(t) \to 0$ and $(v^{(\chi)})'(t) \to 0$ as $t \to -\infty$ for all $\chi \in C^0_{d,u}$.

4 The local stable manifold

We begin with the local stable manifold $W^s_d \subset X_d$ of the semiflow S_d at the stationary point $\bar{\phi}_d \in X_d \subset C^1_d$ as it was obtained in [6]. It is easy to see that W^s_d is a continuously differentiable submanifold of the Banach space C^1_d which is locally positively invariant under S_d, with tangent space

$$T_{\bar{\phi}_d} W^s_d = Y_{d,s}$$

at $\bar{\phi}_d$, and that it has the following properties (I) and (II), for some $\beta > 0$ chosen with

$$-\beta > \operatorname{Re} z \quad \text{for all } z \in \sigma_{d,e} \text{ with } \operatorname{Re} z < 0$$

and for some $\gamma > \beta$.

(I) There are an open neighbourhood $\tilde{W}^s_d$ of $\bar{\phi}_d$ in W^s_d such that $[0,\infty) \times \tilde{W}^s_d \subset \Omega_d$ and $S_d([0,\infty) \times \tilde{W}^s_d) \subset W^s_d$, and a constant $\tilde{c} > 0$ such that for all $\psi \in \tilde{W}^s_d$ and all $t \geq 0$,

$$|S_d(t,\psi) - \bar{\phi}_d|_{d,1} \leq \tilde{c}\, e^{-\gamma t} |\psi - \bar{\phi}_d|_{d,1}.$$

(II) There exists a constant $\bar{c} > 0$ such that each $\psi \in X_d$ with $[0,\infty) \times \{\psi\} \subset \Omega_d$ and

$$e^{\beta t}|S_d(t,\psi) - \bar{\phi}_d|_{d,1} < \bar{c} \quad \text{for all } t \geq 0$$

belongs to W^s_d.

The codimension of W^s_d in C^1_d is equal to

$$n + \dim Y_{d,c} + \dim Y_{d,u} = n + \dim C^0_{d,c} + \dim C^0_{d,u}.$$

As the continuous linear map $R_{d,1} : C^1 \to C^1_d$ is surjective we can apply Proposition 7.3 from the Appendix and obtain an open neighbourhood V of $\bar{\phi}$ in $N \subset U \subset C^1$ so that

$$W^s = W^s(\bar{\phi}) = V \cap R^{-1}_{d,1}(W^s_d)$$

is a continuously differentiable submanifold of C^1 with codimension $n + \dim C^0_{d,c} + \dim C^0_{d,u}$ and tangent space

$$T_{\bar{\phi}} W^s = R^{-1}_{d,1}(T_{\bar{\phi}_d} W^s_d) = R^{-1}_{d,1}(Y_{d,s}).$$

The next propositions show that W^s is the desired local stable manifold of S at $\bar{\phi}$.

Proposition 4.1. *$W^s \subset X$ and $T_{\bar{\phi}} W^s = Y_s$, and W^s is locally positively invariant.*

Proof. 1. Let $\phi \in W^s$. Then $\phi \in V \subset N$ and $R_{d,1}\phi \in W^s_d \subset X_d \subset U_d \subset P^{-1}_{d,1}(N)$ and $\phi(t) = P_{d,1}R_{d,1}\phi(t)$ on $[-d, 0]$. Using $R_{d,1}\phi \in X_d$, the definition of f_d, and property (lbd) we infer

$$\phi'(0) = (R_{d,1}\phi)'(0) = f_d(R_{d,1}\phi) = f(P_{d,1}R_{d,1}\phi) = f(\phi)$$

which means $\phi \in X$.

2. The first assertion yields $T_{\bar{\phi}} W^s \subset T_{\bar{\phi}} X = Y \subset C^1$. Hence

$$T_{\bar{\phi}} W^s = Y \cap R^{-1}_{d,1}(Y_{d,s}) = Y_s.$$

3. (On local positive invariance) Choose an open neighbourhood V_d of $\bar{\phi}_d$ according to local positive invariance of W^s_d. Then choose an open neighbourhood $\hat{V} \subset V$ of $\bar{\phi}$ with $R_{d,1}\hat{V} \subset V_d$. Consider $t \geq 0$ and $\phi \in W^s \cap \hat{V}$ with $S([0,t] \times \{\phi\}) \subset \hat{V}$. Then $R_{d,1}S([0,t] \times \{\phi\}) \subset V_d$ and $R_{d,1}\phi \in W^s_d \cap V_d$. For $0 \leq s \leq t$ the solution $x : (-\infty, t] \to \mathbb{R}^n$ of the initial value problem (1.2) satisfies

$$\begin{aligned} x'(s) = f(x_s) = f(P_{d,1}R_{d,1}x_s) \quad & \text{(with (lbd); we have} \\ & R_{d,1}x_s \in U_d, P_{d,1}R_{d,1}x_s \in N, x_s \in \hat{V} \subset N, P_{d,1}R_{d,1}x_s = x_s \text{ on } [-d,0]) \\ & = f_d(R_{d,1}x_s), \end{aligned}$$

which shows that $y = x|_{[-d,t]}$ is a solution of (1.3) on $[0, t]$, with initial value $y_0 = R_{d,1}\phi \in W^s_d \cap V_d$ and with the segments $y_s = R_{d,1}x_s$, $0 \leq s \leq t$, in $R_{d,1}\hat{V} \subset V_d$. By local positive invariance of W^s_d, $y_s = R_{d,1}x_s \in W^s_d$ for $0 \leq s \leq t$. It follows that for such s, $x_s \in \hat{V} \cap R^{-1}_{d,1}(W^s_d) \subset V \cap R^{-1}_{d,1}(W^s_d) = W^s$. □

Proposition 4.2.
(i) There are an open neighbourhood $\tilde{V}$ of $\bar{\phi}$ in V with $[0,\infty) \times (\tilde{V} \cap W^s) \subset \Omega$ and a constant $\tilde{c} > 0$ such that for all $\phi \in \tilde{V} \cap W^s$ the solution $x : \mathbb{R} \to \mathbb{R}^n$ of the initial value problem (1.2) satisfies

$$|x(t) - \bar{\phi}(0)| + |x'(t)| \leq \tilde{c} e^{-\gamma t} |R_{d,1}\phi - \bar{\phi}_d|_{d,1} \quad \textit{for all } t \geq 0.$$

(ii) There are an open neighbourhood $\hat{V}$ of $\bar{\phi}$ in V and a constant $\hat{c} > 0$ such that for every solution $x : \mathbb{R} \to \mathbb{R}^n$ of the initial value problem (1.2) with $\phi \in \hat{V} \cap X$ and

$$|x(t) - \bar{\phi}(0)| + |x'(t)| \leq \hat{c}\, e^{-\beta t} \quad \textit{for all } t \geq 0$$

we have $\phi \in W^s$.

Proof. 1. Consider $\gamma > \beta > 0$ and $\tilde{W}^s_d, \tilde{c}, \bar{c}$ from statements (I) and (II) above. There is an open neighbourhood $\tilde{V}_d \subset U_d$ of $\bar{\phi}_d$ with $W^s_d \cap \tilde{V}_d = \tilde{W}^s_d$.

2. On (i). Choose an integer $j \geq d$ so that for all $\chi \in C^1_d$ with $|\chi - \bar{\phi}_d|_{d,1} < \frac{1}{j}$ we have $\chi \in U_d$, and for all $\psi \in C^1$ with $|\psi - \bar{\phi}|_{1,j} < \frac{1}{j}$ we have $\psi \in N$. Choose an open neighbourhood $\tilde{V} \subset V$ of $\bar{\phi}$ so that for all $\phi \in \tilde{V}$ we have

$$|\phi - \bar{\phi}|_{1,j} < \frac{1}{2j(\tilde{c}+1)} \quad \text{and} \quad R_{d,1}\tilde{V} \subset \tilde{V}_d.$$

For $\phi \in W^s \cap \tilde{V}$ we obtain $R_{d,1}\phi \in W^s_d \cap \tilde{V}_d = \tilde{W}^s_d$. By statement (I), $[0,\infty) \times \{R_{d,1}\phi\} \subset \Omega_d$ and for all $t \geq 0$,

$$\begin{aligned}|S_d(t, R_{d,1}\phi) - \bar{\phi}_d|_{d,1} &\leq \tilde{c}\, e^{-\gamma t}|R_{d,1}\phi - \bar{\phi}_d|_{d,1}\\ &\leq \tilde{c}\, e^{-\gamma t}|\phi - \bar{\phi}|_{1,j} < \frac{1}{2j}.\end{aligned}$$

Then the solution $y : [-d,\infty) \to \mathbb{R}^n$ on $[0,\infty)$ of (1.3) with initial value $y_0 = R_{d,1}\phi \in \tilde{W}^s_d \subset W^s_d$ satisfies

$$|y(s) - \bar{\phi}(0)| + |y'(s)| < \frac{1}{2j} \quad \text{for all } s \geq -d.$$

The map $x : \mathbb{R} \to \mathbb{R}^n$ given by $x(t) = y(t)$ for $t \geq -d$ and $x(t) = \phi(t)$ for $t < -d$ is continuously differentiable. Using $x(s) = \phi(s)$ for $s \leq 0$, $\phi \in \tilde{V}$, and the previous estimate we infer

$$|x(s) - \bar{\phi}(0)| + |x'(s)| < \frac{1}{2j} \quad \text{for all } s \geq -j,$$

which yields $|x_t - \bar{\phi}|_{1,j} < \frac{1}{j}$ for all segments $x_t : (-\infty,0] \ni u \mapsto x(t+u) \in \mathbb{R}^n$, $t \geq 0$. Consequently, $x_t \in N$ for all $t \geq 0$. Using

$$|R_{d,1}x_t - R_{d,1}\bar{\phi}|_{d,1} \leq |x_t - \bar{\phi}|_{1,j} < \frac{1}{j} \quad \text{for all } t \geq 0$$

we obtain for all $t \geq 0$ that $R_{d,1}x_t \in U_d$, hence $P_{d,1}R_{d,1}x_t \in N$, and

$$x'(t) = y'(t) = f_d(y_t) = f_d(R_{d,1}x_t) = f(P_{d,1}R_{d,1}x_t) = f(x_t) \text{ (with (lbd)).}$$

It follows that x is the solution of the initial value problem (1.2), and for every $t \geq 0$,

$$\begin{aligned}|x(t) - \bar{\phi}(0)| + |x'(t)| &= |y(t) - \bar{\phi}(0)| + |y'(t)|\\ &\leq |S_d(t, R_{d,1}\phi) - \bar{\phi}_d|_{d,1} \leq \tilde{c}\, e^{-\gamma t}|R_{d,1}\phi - \bar{\phi}_d|_{d,1}.\end{aligned}$$

3. On (ii). Choose an integer $j \geq d$ with

$$\frac{1}{j} < \frac{\bar{c}}{2}e^{-\beta d}$$

so that

$$\left\{\phi \in C^1 : |\phi - \bar{\phi}|_{1,j} < \frac{2}{j}\right\} \subset V \quad \text{and} \quad \left\{\chi \in C^1_d : |\chi - \bar{\phi}_d|_{d,1} < \frac{2}{j}\right\} \subset U_d.$$

Set

$$\hat{V} = \left\{\phi \in C^1 : |\phi - \bar{\phi}|_{1,j} < \frac{1}{j}\right\} \quad \text{and choose } \hat{c} > 0 \text{ with } \hat{c}\, e^{\beta d} < \frac{1}{j}.$$

Let $\phi \in X \cap \hat{V}$ be given with $[0,\infty) \times \{\phi\} \subset \Omega$ and assume that the solution $x : \mathbb{R} \to \mathbb{R}^n$ of the initial value problem (1.2) satisfies

$$|x(t) - \bar{\phi}(0)| + |x'(t)| \le \hat{c}\, e^{-\beta t} \quad \text{for all } t \ge 0.$$

Notice that all segments $x_t : (-\infty, 0] \ni u \mapsto x(t+u) \in \mathbb{R}^n$, $t \ge 0$, belong to V. Next, consider the restriction $y = x|_{[-d,\infty)}$. For $t \ge 0$ and $-d \le s \le 0$ we have in case $t + s \le 0$ that

$$|y(t+s) - \bar{\phi}(0)| + |y'(t+s)| \le |\phi - \bar{\phi}|_{1,j} < \frac{1}{j} \le \frac{\bar{c}}{2} e^{-\beta d} \le \frac{\bar{c}}{2} e^{-\beta t}$$

while for $0 < t + s$,

$$|y(t+s) - \bar{\phi}(0)| + |y'(t+s)| = |x(t+s) - \bar{\phi}(0)| + |x'(t+s)| \le \hat{c}\, e^{-\beta(t+s)}$$
$$\le \hat{c}\, e^{\beta d} e^{-\beta t} < \frac{\bar{c}}{2} e^{-\beta t}.$$

It follows that for all segments $y_t : [-d, 0] \ni u \mapsto y(t+u) \in \mathbb{R}^n$, $t \ge 0$, we have $e^{\beta t}|y_t - \bar{\phi}_d|_{d,1} \le \bar{c}$. Notice that we also obtained

$$|y_t - \bar{\phi}_d|_{d,1} < \frac{2}{j} \quad \text{for all } t \ge 0,$$

which yields $y_t \in U_d$ for all $t \ge 0$, hence $P_{d,1} R_{d,1} x_t = P_{d,1} y_t \in N$ for all $t \ge 0$. As $x_t \in V \subset N$ we can apply (lbd) and find

$$y'(t) = x'(t) = f(x_t) = f(P_{d,1} R_{d,1} x_t) = f_d(R_{d,1} x_t) = f_d(y_t) \quad \text{for all } t \ge 0.$$

In particular, $y_0 \in X_d$. It follows that $y_t = S_d(t, y_0)$ for all $t \ge 0$. Now statement (II) gives $R_{d,1}\phi = y_0 \in W_d^s$. Consequently, $\phi \in V \cap R_{d,1}^{-1}(W_d^s) = W^s$. □

5 The local unstable manifold

In this section segments x_t are always defined on $(-\infty, 0]$. Fix some $a > 0$ and consider the Banach space $B_a^1 \subset C^1$ introduced in Section 1, and let $B_a^0 \subset C^0$ denote the Banach space of continuous maps $\phi : (-\infty, 0] \to \mathbb{R}^n$ satisfying

$$\lim_{s \to -\infty} \phi(s) e^{as} = 0,$$

with the norm given by $|\phi|_{a,0} = \sup_{s \le 0} |\phi(s)| e^{as}$. It is easy to see that the linear inclusion maps

$$j_0 : B_a^0 \to C^0 \quad \text{and} \quad j_1 : B_a^1 \to C^1$$

are continuous, as well as the restriction and prolongation maps

$$R_{a,d,1} : B_a^1 \ni \phi \mapsto R_{d,1}\phi \in C_d^1 \quad \text{and} \quad P_{a,d,1} : C_d^1 \ni \chi \mapsto P_{d,1}\chi \in B_a^1.$$

The set $U_a = j_1^{-1}(N) \cap R_{a,d,1}^{-1}(U_d) \subset B_a^1$ is open and contains $\bar{\phi}$, and the map

$$f_a : U_a \to \mathbb{R}^n, \quad f_a(\phi) = f(j_1 \phi),$$

satisfies $f_a(\bar{\phi}) = 0$. Notice that every solution of the equation

$$x'(t) = f_a(x_t) \tag{5.1}$$

on some interval also is a solution of (1.1) on this interval.

Proposition 5.1. *For all ϕ and ψ in U_a with $\phi(s)=\psi(s)$ on $[-d,0]$ we have $f_a(\phi)=f_a(\psi)$. The map f_a is continuously differentiable (F), each derivative $Df_a(\phi):B_a^1\to\mathbb{R}^n$, $\phi\in U_a$, has a linear extension $D_ef_a(\phi):B_a^0\to\mathbb{R}^n$, and the map*

$$U_a\times B_a^0\ni(\phi,\chi)\mapsto D_ef_a(\phi)\chi\in\mathbb{R}^n$$

is continuous.

Proof. 1. For ϕ,ψ in U_a with $\phi(s)=\psi(s)$ on $[-d,0]$ we have $j_1\phi\in N$, $j_1\psi\in N$, and $R_{a,d,1}\phi=R_{a,d,1}\psi\in U_d\subset P_{d,1}^{-1}(N)$. Using (lbd) we infer

$$f_a(\phi)=f(j_1\phi)=f(P_{d,1}R_{a,d,1}\phi)=f(P_{d,1}R_{a,d,1}\psi)=f(j_1\psi)=f_a(\psi).$$

2. Using $f_a(\phi)=f(P_{d,1}R_{a,d,1}\phi)=f_d(R_{a,d,1}\phi)$ we see that f_a is continuously differentiable (F).

3. For $\phi\in U_a$ and $\chi\in B_a^0$ define $D_ef_a(\phi)\chi=D_ef(j_1\phi)j_0\chi$. □

Now results from [17] show that $X_a=\{\phi\in U_a:\phi'(0)=f_a(\phi)\}$ is a continuously differentiable submanifold of $U_a\subset B_a^1$, that the solutions of (5.1) define a continuous semiflow $S_a:[0,\infty)\times X_a\supset\Omega_a\to X_a$, and that there is a local unstable manifold $W_a^u\subset X_a$ at the stationary point $\bar\phi\in X_a$, which has the following properties: W_a^u is a continuously differentiable submanifold of B_a^1, $\bar\phi\in W_a^u$, each $\phi\in W_a^u$ is a solution on $(-\infty,0]$ of (5.1) with $\phi_s\to\bar\phi$ as $s\to-\infty$, and

$$T_{\bar\phi}W_a^u=Y_u.$$

(In order to verify the last equation observe that in [17] the tangent space of W_a^u at $\bar\phi$ is obtained as the vector space of all maps $\hat\chi:(-\infty,0]\to\mathbb{R}^n$ with $\hat\chi_0=\chi\in C_{d,u}^0$ which for some $t>0$ and for all integers $j<0$ satisfy

$$\hat\chi_{jt}=\Lambda^j\chi$$

where $\Lambda:C_{d,u}^0\to C_{d,u}^0$ is the isomorphism whose inverse is given by $T_{d,e,-t}$. The maps in the vector space $I_uC_{d,u}^0=Y_u$ share the said property. The dimension of both vector spaces equals $\dim C_{d,u}^0$.)

Moreover, there exist $\overline{\beta}>\overline{\gamma}>0$ and $c_u>0$ so that
(I) $|\phi_s-\bar\phi|_{a,1}\le c_ue^{\bar\beta s}|\phi-\bar\phi|_{a,1}$ for all $\phi\in W_a^u$ and $s\le0$,
and
(II) for every solution $\psi\in B_a^1$ of (5.1) on $(-\infty,0]$ with

$$\sup_{s\le0}|\psi_s-\bar\phi|_{a,1}e^{-\bar\gamma s}<\infty$$

there exists $s_\psi\le0$ with $\psi_s\in W_a^u$ for all $s\le s_\psi$.

From a manifold chart at $\bar\phi$ we obtain $\epsilon>0$ and a continuously differentiable (F) map

$$w_a^u:Y_u(\epsilon)\to B_a^1,\quad Y_u(\epsilon)=\{\phi\in Y_u:|\phi|_{a,1}<\epsilon\},$$

with $w_a^u(0)=\bar\phi$, $w_a^u(Y_u(\epsilon))$ an open subset of W_a^u, and $Dw_a^u(0)\eta=\eta$ for all $\eta\in Y_u$. Proposition 7.4 applies to the continuously differentiable (MB) map $j_1\circ w_a^u$. So we may assume that

$$W^u=W^u(\bar\phi)=j_1w_a^u(Y_u(\epsilon))$$

is a continuously differentiable submanifold of C^1 with

$$T_{\bar\phi}W^u=j_1Dw_a^u(0)Y_u=Y_u.$$

Proposition 5.2.
(i) Every $\phi \in W^u$ *is a solution of* (1.1) *on* $(-\infty, 0]$*, with* $\phi_s \to \bar{\phi}$ *as* $s \to -\infty$*, and for all* $s \leq 0$*,*

$$|\phi(s) - \bar{\phi}(0)| \leq c_u e^{\bar{\beta}s} |\phi - \bar{\phi}|_{a,1} \quad \textit{and} \quad |\phi'(s)| \leq c_u e^{\bar{\beta}s} |\phi - \bar{\phi}|_{a,1}.$$

(ii) For every $\psi \in X$ *which is a solution of* (1.1) *on* $(-\infty, 0]$ *with*

$$\sup_{s \leq 0} e^{-\bar{\gamma}s} |\psi(s) - \bar{\phi}(0)| < \infty \quad \textit{and} \quad \sup_{s \leq 0} e^{-\bar{\gamma}s} |\psi'(s)| < \infty$$

there exists $s(\psi) \leq 0$ *with* $\psi_s \in W^u$ *for all* $s \leq s(\psi)$*.*

Proof. 1. On (i). Let $\phi \in W^u \subset j_1 W_a^u = W_a^u \subset B_a^1$ be given. From the properties of W_a^u combined with the remark preceding Proposition 5.1 we infer that ϕ is a solution of (1.1) on $(-\infty, 0]$. Using $\phi_s \to \bar{\phi}$ for $s \to -\infty$ in B_a^1 and continuity we get $\phi_s \to \bar{\phi}$ for $s \to -\infty$ also in C^1. The exponential estimate in assertion (i) is obvious from the exponential estimate of $|\phi_s - \bar{\phi}|_{a,1}$ for $s \leq 0$ in statement (I).

2. On (ii). Consider $\psi \in C^1$ which is a solution of (1.1) on $(-\infty, 0]$ and assume there is some $c \geq 0$ with

$$e^{-\bar{\gamma}s} |\psi(s) - \bar{\phi}(0)| \leq c \quad \text{and} \quad e^{-\bar{\gamma}s} |\psi'(s)| \leq c \quad \text{for all } s \leq 0.$$

Then ψ and ψ' are bounded, hence $\psi_s \in B_a^1$ for all $s \leq 0$. For each $s \leq 0$ we have

$$\begin{aligned} |\psi_s - \bar{\phi}|_{a,1} &= \sup_{v \leq 0} |\psi(s+v) - \bar{\phi}(0)| e^{av} + \sup_{v \leq 0} |\psi'(s+v)| e^{av} \\ &\leq \sup_{v \leq 0} |\psi(s+v) - \bar{\phi}(0)| + \sup_{v \leq 0} |\psi'(s+v)| \\ &\leq 2c \sup_{v \leq 0} e^{\bar{\gamma}(s+v)} \leq 2c\, e^{\bar{\gamma}s}, \end{aligned}$$

hence $\psi_s \to \bar{\phi}$ in B_a^1 as $s \to -\infty$. Choose $s_1 \leq 0$ with $\psi_s \in U_a$ for all $s \leq s_1$. For $s \leq s_1$ we also have

$$\begin{aligned} \psi'(s) &= (\psi_s)'(0) = f(\psi_s) \quad (\text{since } \psi \text{ is a solution of } (1.1)) \\ &= f_a(\psi_s) \quad (\text{since } \psi_s \in U_a), \end{aligned}$$

hence ψ also is a solution of (5.1) on $(-\infty, s_1]$. It follows that $\bar{\psi} = \psi_{s_1} \in B_a^1$ is a solution of (5.1) on $(-\infty, 0]$. For $s \leq 0$ we get

$$\begin{aligned} e^{-\bar{\gamma}s} |\bar{\psi}_s - \bar{\phi}|_{a,1} &= e^{-\bar{\gamma}s} |\psi_{s_1+s} - \bar{\phi}|_{a,1} \\ &\leq 2c\, e^{\bar{\gamma}s_1} < \infty. \end{aligned}$$

Property (II) shows that there exists $s_2 \leq 0$ with $\bar{\psi}_s \in W_a^u$ for all $s \leq s_2$. Using $\psi_s \to \bar{\phi}$ in B_a^1 for $s \to -\infty$ once again we find $s_3 \leq s_2$ with $\bar{\psi}_s \in w_a^u(Y_u(\epsilon))$ for all $s \leq s_3$. For such s, $\psi_{s_1+s} = j_1 \psi_{s_1+s} = j_1 \bar{\psi}_s \in j_1 w_a^u(Y_u(\epsilon)) = W^u$. □

6 Local center manifolds

In this section we assume

$$\{0\} \neq C_{d,c}^0 = Y_{d,c}.$$

First we perform the translation $\bar{\phi}_d \to 0$, in order to use constructions from Section 4.2 in [6] and from [8]. Consider the continuously differentiable (F) map

$$g_d : C^1_d \supset V_d \to \mathbb{R}^n, \quad V_d = U_d - \bar{\phi}_d, \quad g_d(\phi) = f_d(\phi + \bar{\phi}_d),$$

which satisfies $g_d(0) = 0$ and $Dg_d(\phi) = Df_d(\phi + \bar{\phi}_d)$ for all $\phi \in V_d$. In particular, $Dg_d(0) = Df_d(\bar{\phi}_d)$. Setting $D_e g_d(\phi) = D_e f_d(\phi + \bar{\phi}_d)$ for $\phi \in V_d$ we observe that the derivatives of g_d have an extension property as in Proposition 2.2. The set

$$X_{g_d} = \{\phi \in V_d : \phi'(0) = g_d(\phi)\} = X_d - \bar{\phi}_d$$

is a continuously differentiable submanifold of X_d, with codimension n, and a map $x : [-d,0] + I \to \mathbb{R}^n$, $I \subset \mathbb{R}$ an interval, is a solution of

$$x'(t) = g_d(x_t) \tag{6.1}$$

on I if and only if the map $[-d,0] + I \ni t \mapsto x(t) + \bar{\phi}(0) \in \mathbb{R}^n$ is a solution of (1.3) on I. It follows that the relations

$$\Omega_{g_d} = \{(t,\phi) \in [0,\infty) \times X_{g_d} : (t, \phi + \bar{\phi}_d) \in \Omega_d\}, \quad S_{g_d}(t,\phi) = S_d(t, \phi + \bar{\phi}_d)$$

define a continuous semiflow on X_{g_d}, with all solution operators $S_{g_d}(t,\cdot)$ continuously differentiable (F). Now $0 \in X_{g_d}$ is a stationary point of S_{g_d}, the tangent space of X_{g_d} at 0 is

$$Y_{g_d} = T_0 X_{g_d} = \{\chi \in C^1_d : \chi'(0) = Dg_d(0)\chi\} = \{\chi \in C^1_d : \chi'(0) = Df_d(\bar{\phi}_d)\chi\} = Y_d,$$

and the derivatives $D_2 S_{g_d}(t,0) : Y_{g_d} \to Y_{g_d}$, $t \geq 0$, are given by

$$T_{g_d,t} = D_2 S_{g_d}(t,0) = D_2 S(t,\bar{\phi}_d) = T_{d,t}.$$

For every $\chi \in Y_{g_d}$ and for all $t \geq 0$, $T_{g_d,t}\chi = v^\chi_t$ with the continuously differentiable solution $v^\chi : \mathbb{R} \to \mathbb{R}^n$ of the initial value problem

$$v'(t) = Dg_d(0)v_t = Df_d(\bar{\phi}_d)v_t \quad \text{for } t > 0,\ v_0 = \chi \in Y_{g_d} = Y_d. \tag{6.2}$$

In particular,

$$T_{g_d,t}\chi = T_{g_d,e,t}\chi$$

with the operator $T_{g_d,e,t} = T_{d,e,t}$ from the strongly continuous semigroup on the space C^0_d which is given by the continuous solutions of the initial value problem

$$v'(t) = D_e g_d(0)v_t = D_e f_d(\bar{\phi}_d)v_t \quad \text{for } t > 0,\ v_0 = \eta \in C^0_d. \tag{6.3}$$

The operators $T_{g_d,e,t} = T_{d,e,t}$, $t \geq 0$, leave the stable space $C^0_{d,s}$ invariant and define isomorphisms of the center and unstable spaces $C^0_{d,c}, C^0_{d,u}$. In the sequel we need certain constants related to the behaviour of the semigroup on these invariant spaces: there are $K \geq 1, a < 0, b > 0$ and $\epsilon \in (0, \min\{-a, b\})$ such that

$$|T_{d,e,t}\phi|_{d,0} \leq K e^{at} |\phi|_{d,0} \qquad \text{for all } \phi \in C^0_{d,s},\ t \geq 0,$$
$$|T_{d,e,t}\phi|_{d,0} \leq K e^{\epsilon|t|} |\phi|_{d,0} \qquad \text{for all } \phi \in C^0_{d,c},\ t \in \mathbb{R},$$
$$|T_{d,e,t}\phi|_{d,0} \leq K e^{bt} |\phi|_{d,0} \qquad \text{for all } \phi \in C^0_{d,u},\ t \leq 0.$$

Next we recall the first part of the construction of a continuously differentiable local center manifold of S_{g_d} at $0 \in X_{g_d}$ from [8]. (Be aware of different notation: the space C in [8] corresponds to the space C^0_d in the present paper, etc.)

Setting $C^1_{d,s} = C^1_d \cap C^0_{d,s}$ we obtain a decomposition

$$C^1_d = C^1_{d,s} \oplus C^0_{d,c} \oplus C^0_{d,u} \tag{6.4}$$

of the space C^1_d into closed subspaces. The associated projections $C^1_d \to C^1_d$ onto $C^1_{d,s}, C^0_{d,c}, C^0_{d,u}$ are denoted by $P^1_{d,s}, P^1_{d,c}, P^1_{d,u}$, respectively.

Following [8] we choose a norm $|\cdot|_{d,c}$ on the finite-dimensional space $C^0_{d,c}$ whose restriction to $C^0_{d,c} \setminus \{0\}$ is C^∞-smooth. Then

$$||\phi||_{d,1} = \max\{|P^1_{d,c}\phi|_{d,c}, |(P^1_{d,s} + P^1_{d,u})\phi|_{d,1}\}$$

defines a norm on C^1_d which is equivalent to $|\cdot|_{d,1}$. The continuously differentiable (F) remainder

$$r_d : V_d \ni \phi \mapsto g_d(\phi) - Dg_d(0)\phi \in \mathbb{R}^n$$

satisfies $r_d(0) = 0$ and $Dr_d(0) = 0$. Using analogues of the formulae for r_δ in [8] we introduce a family of maps

$$r_{d,\delta} : C^1_d \to \mathbb{R}^n, \quad 0 < \delta \le \delta_1,$$

which are defined on the whole space C^1_d and have the property that for all $\delta \in (0, \delta_1]$ and for all $\phi \in C^1_d$ with $\|\phi\|_{d,1} < \delta$,

$$\phi \in V_d \quad \text{and} \quad r_{d,\delta}(\phi) = r_d(\phi).$$

(The preceding property is obvious from the definition of $r_{d,\delta}$ but was not stated for r_δ in [8].) In particular, $r_{d,\delta}(0) = r_d(0) = 0$. There is a continuous non-decreasing function $\mu : [0, \delta_1] \to [0, 1]$ with $\mu(0) = 0$ such that for all $\delta \in (0, \delta_1]$ and all ϕ, ψ in C^1_d we have

$$|r_{d,\delta}(\phi)| \le \delta\, \mu(\delta)$$
and
$$|r_{d,\delta}(\phi) - r_{d,\delta}(\psi)| \le \mu(\delta)\|\phi - \psi\|_{d,1}.$$

(For the construction of μ compare, e. g., the proof of [9, Proposition II.2].)

For a given Banach space E and $\eta > 0$ let E_η denote the Banach space of all continuous maps $u : \mathbb{R} \to E$ satisfying $\sup_{t\in\mathbb{R}} e^{-\eta|t|}|u(t)| < \infty$, with the norm given by

$$|u|_{E,\eta} = \sup_{t\in\mathbb{R}} e^{-\eta|t|}|u(t)|.$$

For $E = C^0_d$ and $E = C^1_d$ abbreviate $C^0_{d,\eta} = (C^0_d)_\eta$ and $C^1_{d,\eta} = (C^1_d)_\eta$, $|\cdot|_{d,0,\eta} = |\cdot|_{C^0_d,\eta}$ and $|\cdot|_{d,1,\eta} = |\cdot|_{C^1_d,\eta}$, respectively. In case $\eta > \epsilon$ the map

$$S_\eta : C^0_{d,c} \to C^1_{d,\eta}, \quad (S_\eta\phi)(t) = T_{d,e,t}\phi,$$

is injective, linear, and continuous. This follows easily from the facts that the operators $T_{d,e,t}, t \ge 0$, form a strongly continuous semigroup and define isomorphisms of the space $C^0_{d,c}$, in combination with the growth estimate on $C^0_{d,c}$ and with the equivalence of all norms on the finite-dimensional space $C^0_{d,c} \subset C^1_d$.

In the proof of [8, Theorem 2.1] it is shown that there exist $\Delta \in (0,\delta_1]$ and $\eta_1 > \eta_0$ in $(\epsilon, \min\{-a,b\})$ with the following properties: for every $\phi \in C^0_{d,c}$ there is a uniquely determined curve $u = u(\phi) \in C^1_{d,\eta_1}$ which satisfies the integral equation

$$u(t) = T_{d,e,t-s}u(s) + \int_s^t T^{\odot\star}_{d,e,t-\tau} l(r_{d,\Delta}(u(\tau)))d\tau, \quad -\infty < s \le t < \infty, \tag{6.5}$$

and the condition $P^1_{d,c}u(\phi)(0) = \phi$. (For the correct interpretation of (6.5), for the integral in it, and for the maps $T^{\odot\star}_{d,e,t-\tau}$ and l see [8].) We have $u(0) = 0$, and the map $u : C^0_{d,c} \ni \phi \mapsto u(\phi) \in C^1_{d,\eta_1}$ is continuously differentiable (F) with $Du(0) = S_{\eta_1}$.

Because of $P^1_{d,c}u(\phi)(0) = \phi$ the map $u : C^0_{d,c} \ni \phi \mapsto u(\phi) \in C^1_{d,\eta_1}$ is injective.

It is important to notice that in the preceding statement Δ can be chosen so small that, taking into account the Lipschitz constant $\mu(\Delta)$ of r_Δ and the equivalence of the norms $\|\cdot\|_{d,1}$ and $|\cdot|_{d,1}$, we also get

$$|r_{d,\Delta}(\phi) - r_{d,\Delta}(\psi)| \le \lambda|\phi - \psi|_{d,1} \quad \text{for all } \phi \in C^1_d, \psi \in C^1_d$$

with a constant $\lambda = \lambda(\Delta) \ge 0$ strictly less than 1. (In [8] only the related estimate $\mu(\delta) \le 1$ occurs, which is not enough for the present purpose. We need $\lambda < 1$ for the application of Proposition 7.1 in Part 1 of the proof of Proposition 6.4 below.)

(6.5) is equivalent to the differential equation

$$x'(t) = Dg_d(0)x_t + r_{d,\Delta}(x_t) \tag{6.6}$$

in a certain sense, see [6, Section 4.2]. We only need that given $\phi \in C^0_{d,c}$ there is a continuously differentiable function $x^{[\phi]} : \mathbb{R} \to \mathbb{R}^n$ which satisfies (6.6) for all $t \in \mathbb{R}$ and

$$x^{[\phi]}_t = u(\phi)(t) \quad \text{for all } t \in \mathbb{R},$$

and conversely, that for every continuously differentiable function $x : \mathbb{R} \to \mathbb{R}^n$ which satisfies (6.6) for all $t \in \mathbb{R}$ the continuous curve $\mathbb{R} \ni t \mapsto x_t \in C^1_d$ satisfies (6.5).

Define open balls

$$\begin{aligned} C^0_{d,c,\Delta} &= \{\phi \in C^0_{d,c} : \|\phi\|_{d,1} < \Delta\}, \\ C^1_{d,su,\Delta} &= \{\phi \in C^1_{d,s} \oplus C^0_{d,u} : \|\phi\|_{d,1} < \Delta\}, \\ N_\Delta &= C^0_{d,c,\Delta} + C^1_{d,su,\Delta} \quad (= \{\phi \in C^1_d : \|\phi\|_{d,1} < \Delta\}). \end{aligned}$$

From here on, we deviate from the proof of [8, Theorem 2.1]. The next aim is to show that the map $C^0_{d,c} \ni \phi \mapsto x^{[\phi]}|_{(-\infty,0]} \in C^1$ is continuously differentiable (MB) with the derivative at $\phi = 0$ given by the map

$$I_c : C^0_{d,c} \to C^1$$

from Section 3, with image $I_c C^0_{d,c} = Y_c \subset Y = T_{\bar\phi}X$. This requires some preparation.

Notice that the differentiation and evaluation maps $\partial_d : C^1_d \to C^0_d$, $\partial_d\phi = \phi'$, and $ev_s : C^0_d \ni \phi \mapsto \phi(s) \in \mathbb{R}^n$, $-d \le s \le 0$, are linear and continuous.

Proposition 6.1.
(i) The set

$$Z = \{z \in C^1_{d,\eta_1} : \text{for all } t \in \mathbb{R} \text{ and } s \in [-d,0],\ z(t)(s) = z(t+s)(0)\}$$

is a closed linear subspace of C^1_{d,η_1}.

(ii) For each $z \in C^1_{d,\eta_1}$, $\partial_d \circ z \in C^0_{d,\eta_1}$, *and the linear map*

$$C^1_{d,\eta_1} \ni z \mapsto \partial_d \circ z \in C^0_{d,\eta_1}$$

is continuous.

(iii) The linear maps

$$em_{0,s} : C^0_{d,\eta_1} \ni z \mapsto (ev_s \circ z)|_{(-\infty,0]} \in C^0, \quad -d \le s \le 0,$$

are continuous.

(iv) For every $z \in Z$ *the map* $ev_0 \circ z$ *is continuously differentiable with*

$$(ev_0 \circ z)'(t) = (ev_0 \circ \partial_d \circ z)(t)$$

for all $t \in \mathbb{R}$.

(v) The linear map

$$em_Z : C^1_{d,\eta_1} \supset Z \ni z \mapsto (ev_0 \circ z)|_{(-\infty,0]} \in C^1$$

is continuous.

Proof. 1. On assertion (i). For every $t \in \mathbb{R}$ and $s \in [-d,0]$ the maps $C^1_{d,\eta_1} \ni z \mapsto z(t)(s) \in \mathbb{R}^n$ and $C^1_{d,\eta_1} \ni z \mapsto z(t+s)(0) \in \mathbb{R}^n$ are linear and continuous. The set Z is the intersection of the kernels of their differences.

2. In order to prove assertion (ii) recall the norm on C^0_{d,η_1} and use that given $z \in C^1_{d,\eta_1}$ and $t \in \mathbb{R}$,

$$\begin{aligned}
|(\partial_d \circ z)(t)|_{d,0} e^{-\eta_1|t|} &= |(z(t))'|_{d,0} e^{-\eta_1|t|} \\
&\le (|z(t)|_{d,0} + |(z(t))'|_{d,0}) e^{-\eta_1|t|} = |z(t)|_{d,1} e^{-\eta_1|t|} \\
&\le |z|_{d,1,\eta_1}
\end{aligned}$$

3. Proof of assertion (iii). Let $-d \le s \le 0$ and $j \in \mathbb{N}$. For every $z \in C^0_{d,\eta_1}$ and for all $t \in [-j,0]$,

$$\begin{aligned}
|(ev_s \circ z)(t)| &= |z(t)(s)| e^{-\eta_1|t|} e^{\eta_1|t|} \\
&\le |z(t)|_{d,0} e^{-\eta_1|t|} e^{\eta_1 j} \\
&\le |z|_{d,0,\eta_1} e^{\eta_1 j},
\end{aligned}$$

which shows

$$|em_{0,s}(z)|_{0,j} = |(ev_s \circ z)|_{(-\infty,0]}|_{0,j} \le e^{\eta_1 j} |z|_{d,0,\eta_1} \quad \text{for all } z \in C^0_{d,\eta_1}.$$

Now continuity follows easily.

4. Proof of assertion (iv). Let $z \in Z$ and $t \in \mathbb{R}$ be given. For $h \in \mathbb{R}$ with $0 < |h| < \frac{d}{2}$,

$$\begin{aligned}
& z\left(t+\frac{d}{2}\right)\left(-\frac{d}{2}+h\right) - z\left(t+\frac{d}{2}\right)\left(-\frac{d}{2}\right) \\
&= z\left(t+\frac{d}{2}-\frac{d}{2}+h\right)(0) - z\left(t+\frac{d}{2}-\frac{d}{2}\right)(0) \\
&\left(\text{since } z \in Z \text{ and } -\frac{d}{2}+h \in [-d,0] \ni -\frac{d}{2}\right) \\
&= (ev_0 \circ z)(t+h) - (ev_0 \circ z)(t)
\end{aligned}$$

which shows that

$$\frac{1}{h}((ev_0\circ z)(t+h)-(ev_0\circ z)(t)) \quad \to \quad z\left(t+\frac{d}{2}\right)'\left(-\frac{d}{2}\right)$$

as $h\to 0$, and $ev_0\circ z$ is differentiable. For $-d\le h<0$ we have

$$\begin{aligned}(ev_0\circ z)(t+h)-(ev_0\circ z)(t) &= z(t+h)(0)-z(t)(0)\\ &= z(t)(h)-z(t)(0)\\ &\text{(as } z\in Z \text{ and } -d\le h\le 0).\end{aligned}$$

This yields $(ev_0\circ z)'(t)=(z(t))'(0)=\partial_d(z(t))(0)=(ev_0\circ\partial_d\circ z)(t)$. The formula shows that $(ev_0\circ z)'$ is continuous.

5. Proof of assertion (v). From assertion (iii) in combination with the continuity of the inclusion map

$$C^1_{d,\eta_1}\to C^0_{d,\eta_1}$$

we infer that the map

$$C^1_{d,\eta_1}\supset Z\ni z\mapsto em_{0,0}(z)\in C^0$$

is continuous, and for every $z\in Z$, $em_Z(z)=(ev_0\circ z)|_{(-\infty,0]}=em_{0,0}(z)$. According to assertion (iv), each map $em_Z(z)$, $z\in Z$, is continuously differentiable with

$$(em_Z(z))'(t)=(ev_0\circ z)'(t)=(ev_0\circ(\partial_d\circ z))(t)=(em_{0,0}(\partial_d\circ z))(t)$$

for all $t\le 0$, or $(em_Z(z))'=em_{0,0}(\partial_d\circ z)$. Using assertions (ii) and (iii) we conclude that the map

$$C^1_{d,\eta_1}\supset Z\ni z\mapsto (em_Z(z))'\in C^0$$

is continuous. Recall $|\phi|_{1,j}=|\phi|_{0,j}+|\phi'|_{0,j}$ for all $j\in\mathbb{N}$ and all $\phi\in C^1$. Now it follows easily that the map

$$em_Z: C^1_{d,\eta_1}\supset Z\to C^1$$

is continuous. □

Observe that we have $S_{\eta_1}C^0_{d,c}\subset Z$ as for all $\chi\in C^0_{d,c}$, $t\in\mathbb{R}$ and $s\in[-d,0]$,

$$((S_{\eta_1}\chi)(t))(s)=(T_{d,e,t}\chi)(s)=v_t^{(\chi)}(s)=v^{(\chi)}(t+s)=v_{t+s}^{(\chi)}(0)=((S_{\eta_1}\chi)(t+s))(0).$$

Also, for every $\chi\in C^0_{d,c}$ and for all $t\le 0$,

$$ev_0(S_{\eta_1}(\chi)(t))=ev_0(T_{d,e,t}\chi)=ev_0(v_t^{(\chi)})=v^{(\chi)}(t)=(I_c\chi)(t).$$

Corollary 6.2. *The map* $J: C^0_{d,c}\ni\phi\mapsto\bar{\phi}+x^{[\phi]}|_{(-\infty,0]}\in C^1$ *is continuously differentiable (MB) with* $DJ(0)=I_c$.

Proof. For every $\phi\in C^0_{d,c}$ we have $u(\phi)\in Z$, because of

$$(u(\phi)(t))(s)=x_t^{[\phi]}(s)=x^{[\phi]}(t+s)=x_{t+s}^{[\phi]}(0)=(u(\phi)(t+s))(0)$$

for all $t\in\mathbb{R}$ and $s\in[-d,0]$. For each $t\le 0$,

$$x^{[\phi]}(t)=(u(\phi)(t))(0)=ev_0(u(\phi)(t))=(ev_0\circ u(\phi))(t)=(em_Z(u(\phi)))(t).$$

Hence $J(\phi) = \bar{\phi} + em_Z(u(\phi))$ for all $\phi \in C^0_{d,c}$. An application of the chain rule to the linear continuous map em_Z from Proposition 6.1 (vi) and to the continuously differentiable (F) map

$$C^0_{d,c} \ni \phi \mapsto u(\phi) \in Z \subset C^1_{d,\eta_1}$$

yields that the map J is continuously differentiable (MB) with $DJ(0)\chi = em_Z(S_{\eta_1}\chi)$ for all $\chi \in C^0_{d,c}$. For such χ and for all $t \le 0$,

$$(em_Z(S_{\eta_1}\chi))(t) = ev_0((S_{\eta_1}\chi)(t)) = v_t^{(\chi)}(0) = v^{(\chi)}(t) = I_c\chi(t),$$

see the statement preceding the corollary. □

As $J(0) = \bar{\phi} \in N \subset U$ and $DJ(0)$ is injective Proposition 7.4 applies. It follows that there is an open neighbourhood $N^0_{d,c}$ of 0 in $C^0_{d,c}$ so that the set

$$W^c = J(N^0_{d,c}) \subset N \subset U$$

is a continuously differentiable submanifold of C^1, with

$$T_{\bar{\phi}}W^c = I_c C^0_{d,c} = Y_c.$$

As $u(0) = 0$ and as the map $C^0_{d,c} \ni \phi \mapsto u(\phi)(0) \in C^1_d$ is continuous we may assume that for every $\phi \in N^0_{d,c}$ we have

$$\|u(\phi)(0)\|_{d,1} < \Delta, \quad \text{or equivalently,} \quad u(\phi)(0) \in N_\Delta,$$

which implies $u(\phi)(0) \in V_d$.

Proposition 6.3.

$$W^c \subset X$$

Proof. For $\phi \in N^0_{d,c}$ and $x = J(\phi) = \bar{\phi} + x^{[\phi]}|_{(-\infty,0]} \in W^c \subset N \subset U$,

$$\begin{aligned}
x'(0) = (x^{[\phi]})'(0) &= Dg_d(0)x_0^{[\phi]} + r_{d,\Delta}(x_0^{[\phi]})\\
&\text{(by (6.6), with the segment } x_0^{[\phi]} \text{ defined on } [-d,0])\\
&= Dg_d(0)x_0^{[\phi]} + r_d(x_0^{[\phi]})\\
&\text{(as } \|x_0^{[\phi]}\|_{d,1} = \|u(\phi)(0)\|_{d,1} < \Delta)\\
&= g_d(x_0^{[\phi]}) = f_d(\bar{\phi} + x_0^{[\phi]}) \quad (\text{as } x_0^{[\phi]} \in V_d = U_d - \bar{\phi})\\
&= f(P_{d,1}(\bar{\phi} + x_0^{[\phi]}))\\
&= f(x) \quad \text{(by (lbd), with arguments in } N \subset C^1)\\
&= f(x_0) \quad \text{(with the segment defined on } (-\infty,0]).
\end{aligned}$$

□

Choose an open neighbourhood U_* of $\bar{\phi}$ in $N \subset U$ so small that

$$R_{d,1}U_* \subset U_d \cap (N_\Delta + \bar{\phi}_d)$$

and for all $\psi \in U_*$,

$$P^1_{d,c}R_{d,1}(\psi - \bar{\phi}) \in N^0_{d,c}.$$

Proposition 6.4 (Local positive invariance). *For every* $(t,\psi)\in\Omega$ *with* $\psi\in W^c\subset X$ *and* $S([0,t]\times\{\psi\})\subset U_*$ *we have* $S([0,t]\times\{\psi\})\subset W^c$.

Proof. 1. Let $(t,\psi)\in\Omega$ with $\psi\in W^c\subset X$ and $S([0,t]\times\{\psi\})\subset U_*$ be given. Let $s\in[0,t]$. We have to show $S(s,\psi)\in W^c$. There exists $\chi\in N^0_{d,c}$ with $\psi=J(\chi)=\bar\phi+x^{[\chi]}|_{(-\infty,0]}$. Consider the maximal continuously differentiable solution $y:(-\infty,t_y)\to\mathbb{R}^n$ of (1.1) on $(0,t_y)$, $0<t_y\le\infty$, with $y_0=\psi\in X$. Then $t<t_y$ and $y_v=S(v,\psi)$ for $0\le v<t_y$. Obviously,

$$y(v)-\bar\phi(0)=\psi(v)-\bar\phi(0)=x^{[\chi]}(v)\quad\text{for all } v\le 0.$$

Proof of $y(v)-\bar\phi(0)=x^{[\chi]}(v)$ for $0<v\le t$: consider the map

$$z:[-d,t_y)\ni v\mapsto y(v)-\bar\phi(0)\in\mathbb{R}^n.$$

For $0<v\le t$,

$z'(v)=y'(v)=f(y_v)$

(with the segment y_v defined on $(-\infty,0]$)

$=f(P_{d,1}R_{d,1}y_v)$

((lbd) applies since $y_v=S(v,\psi)\in U_*\subset N\cap R^{-1}_{d,1}(U_d)$, $R_{d,1}y_v\in U_d$, $P_{d,1}R_{d,1}y_v\in N$)

$=f_d(R_{d,1}y_v)$ (since $R_{d,1}y_v\in U_d$)

$=g_d(R_{d,1}y_v-\bar\phi_d)$

(with $V_d=U_d-\bar\phi_d$ and the definition of g_d)

$=Dg_d(0)(R_{d,1}y_v-\bar\phi_d)+r_d(R_{d,1}y_v-\bar\phi_d)$

$=Dg_d(0)(R_{d,1}y_v-\bar\phi_d)+r_{d,\Delta}(R_{d,1}y_v-\bar\phi_d)$

(using $R_{d,1}y_v\in N_\Delta+\bar\phi_d$, $\|R_{d,1}y_v-\bar\phi_d\|_{1,d}<\Delta$),

and $R_{d,1}y_v-\bar\phi_d$ is the segment $z_v:[-d,0]\ni s\mapsto y(v+s)-\bar\phi(0)\in\mathbb{R}^n$. Proposition 7.1 now yields $y(v)-\bar\phi(0)=z(v)=x^{[\chi]}(v)$ for $0<v\le t$.

2. Due to autonomy the shifted copy

$$\xi:\mathbb{R}\ni v\mapsto x^{[\chi]}(v+s)\in\mathbb{R}^n$$

of $x^{[\chi]}$ satisfies (6.6) for all $t\in\mathbb{R}$. The continuous curve

$$\mathbb{R}\ni v\mapsto \xi_v\in C^1_d$$

is a solution of (6.5) and belongs to the space C^1_{d,η_1} as we have the estimate

$$\begin{aligned}|\xi_v|_{d,1}&=|x^{[\chi]}_{v+s}|_{d,1}=|u(\chi)(v+s)|_{d,1}e^{-\eta_1|v+s|}e^{\eta_1|v+s|}\\&\le e^{\eta_1|v|}e^{\eta_1 s}\sup_{w\in\mathbb{R}}|u(\chi)(w)|_{d,1}e^{-\eta_1|w|}\quad\text{for all } v\in\mathbb{R}.\end{aligned}$$

It follows that for all $v\in\mathbb{R}$,

$$\xi_v=u(\phi)(v)=x^{[\phi]}_v$$

with

$$\phi=P^1_{d,c}\xi_0,$$

and we observe that $\xi = x^{[\phi]}$. - In order to show that ϕ belongs to the domain $N^0_{d,c}$ of J, notice that Part 1 yields

$$\xi(v) = x^{[\chi]}(v+s) = y(v+s) - \bar{\phi}(0) \quad \text{for all } v \le 0.$$

Using this in combination with the fact that the segment $y_s : (-\infty, 0] \ni v \mapsto y(s+v) \in \mathbb{R}^n$ belongs to U_*, and the choice of U_* we infer

$$\phi = P^1_{d,c}\xi_0 = P^1_{d,c}R_{d,1}(\xi|_{(-\infty,0]}) = P^1_{d,c}R_{d,1}(y_s - \bar{\phi}) \in N^0_{d,c}.$$

Finally,

$$y_s = \bar{\phi} + \xi|_{(-\infty,0]} = \bar{\phi} + x^{[\phi]}|_{(-\infty,0]} = J(\phi) \in W^c.$$

□

Proposition 6.5. *For every solution $y : \mathbb{R} \to \mathbb{R}^n$ of (1.1) on $\mathbb{R}$ with $y_t \in U_*$ for all $t \in \mathbb{R}$ we have $y_t \in W^c$ for all $t \in \mathbb{R}$.*

Proof. Let a solution $y : \mathbb{R} \to \mathbb{R}^n$ of (1.1) on $\mathbb{R}$ with $y_t \in U_*$ for all $t \in \mathbb{R}$ be given. Because of the autonomy of (1.1) it suffices to show $y_0 \in W^c$. Define $x : \mathbb{R} \to \mathbb{R}^n$ by $x(t) = y(t) - \bar{\phi}(0)$. Then $R_{d,1}x_t = R_{d,1}(y_t - \bar{\phi}) = R_{d,1}y_t - R_{d,1}\bar{\phi} = R_{d,1}y_t - \bar{\phi}_d \in R_{d,1}U_* - \bar{\phi}_d \subset N_\Delta$ for all $t \in \mathbb{R}$. The continuously differentiable map x is a solution of (6.6) on $\mathbb{R}$ because similar as in Part 1 of the proof of Proposition 6.4 we have

$$\begin{aligned}
x'(t) &= y'(t) = f(y_t) = f(P_{d,1}R_{d,1}y_t) \\
&\quad \text{(with } y_t \in U_* \subset N, R_{d,1}y_t \in U_d, P_{d,1}R_{d,1}y_t \in N \text{ and (lbd))} \\
&= f_d(R_{d,1}y_t) \text{ (since } R_{d,1}y_t \in U_d) \\
&= g_d(R_{d,1}y_t - \bar{\phi}_d) \\
&\quad \text{(with } R_{d,1}y_t \in U_d = V_d + \bar{\phi}_d \text{ and the definition of } g_d) \\
&= Dg_d(0)(R_{d,1}y_t - \bar{\phi}_d) + r_d(R_{d,1}y_t - \bar{\phi}_d) \\
&= Dg_d(0)(R_{d,1}y_t - \bar{\phi}_d) + r_{d,\Delta}(R_{d,1}y_t - \bar{\phi}_d) \\
&\quad \text{(with } R_{d,1}y_t - \bar{\phi}_d \in N_\Delta) \\
&= Dg_d(0)(R_{d,1}x_t) + r_{d,\Delta}(R_{d,1}x_t)
\end{aligned}$$

for every $t \in \mathbb{R}$, and $R_{d,1}x_t$ is the segment $[-d, 0] \ni s \mapsto x(t+s) \in \mathbb{R}^n$ for each $t \in \mathbb{R}$. Then the curve $\mathbb{R} \ni t \mapsto R_{d,1}x_t \in C^1_d$ is continuous and solves (6.5). As all $R_{d,1}x_t = R_{d,1}y_t - \bar{\phi}_d \in N_\Delta$ are uniformly bounded the curve belongs to the space $C^1_{d,\eta_1} = C_{\eta_1}(\mathbb{R}, C^1_d)$. It follows that

$$R_{d,1}x_t = u(\chi)(t) = x^{[\chi]}_t \quad \text{for all } t \in \mathbb{R}$$

with

$$\chi = P^1_{d,c}R_{d,1}x_0 = P^1_{d,c}R_{d,1}(y_0 - \bar{\phi}) \in N^0_{d,c}.$$

Notice that $x(t) = (R_{d,1}x_t)(0) = x^{[\chi]}_t(0) = x^{[\chi]}(t)$ for all $t \in \mathbb{R}$. Finally,

$$y_0 = \bar{\phi} + x|_{(-\infty,0]} = \bar{\phi} + x^{[\chi]}|_{(-\infty,0]} = J(\chi) \in W^c.$$

□

Some comments on the choice of the different methods used in Sections 5 and 6 seem in order. The proof in Section 5 embeds a local unstable manifold $W^u_a \subset X_a \subset B^1_a$ into the space C^1. The basic unstable manifold W^u_a is taken from [17]. Center manifolds in X_a are not addressed in [17], and seem unavailable anywhere else. This precludes the application of the embedding technique from Section 5 in the present Section 6. Instead the construction in Section 6 borrows from the work in [6,8] on center manifolds in $X_d \subset C^1_d$.

Let us mention that the present construction of a center manifold in $X \subset C^1$ can be modified in order to establish first the local center manifolds in $X_a \subset B^1_a$ which are missing in [17]. Upon that, one could use the embedding technique from Section 5 and proceed to local center manifolds in $X \subset C^1$.

It also is possible to establish unstable manifolds in $X \subset C^1$ in the same way as here in Section 6, without recourse to results from [17] and starting from the construction of unstable manifolds in $X_d \subset C^1_d$ in [7]. The different route chosen in Section 5 is much shorter. Moreover it saves us from a discussion how to change technical details in the proof in [7], in order to get rid of unnecessary hypotheses which are hyperbolicity and the assumption that the functional in the delay differential equation considered is the restriction of a map on a subset of C^0_d.

7 Appendix on uniqueness, preimages and embeddings

Proposition 7.1. *Suppose $L : C^1_d \to \mathbb{R}^n$ is linear and continuous with a linear continuous extension $L_e : C^0_d \to \mathbb{R}^n$, and $r : C^1_d \to \mathbb{R}^n$ satisfies $|r(\phi) - r(\psi)| \le \lambda|\phi - \psi|_{d,1}$ for all ϕ, ψ in C^1_d, with $0 \le \lambda < 1$. Then any two continuously differentiable maps $x : [-d, t_e) \to \mathbb{R}^n$ and $y : [-d, t_e) \to \mathbb{R}^n$, $0 < t_e \le \infty$, satisfying $x(t) = y(t)$ on $[-d, 0]$ and*

$$z'(t) = Lz_t + r(z_t) \quad \text{for } 0 < t < t_e$$

coincide.

Proof. 1. Assume $x(t') \ne y(t')$ for some $t' \in (0, t_e)$. Let $t_0 = \inf\{t \in [0, t_e) : x(t) \ne y(t)\}$. Then $0 \le t_0 < t_e$, $x(t) = y(t)$ on $[-d, t_0]$ and for every $\epsilon > 0$ there exists $t' \in (t_0, t_0 + \epsilon)$ with $t' < t_e$ and $x(t') \ne y(t')$. The curves $[0, t_e) \ni s \mapsto x_s \in C^1_d$ and $[0, t_e) \ni s \mapsto y_s \in C^1_d$ are continuous. Let $c = |L_e|_{L_c(C^0_d,\mathbb{R}^n)} = \sup_{|\chi|_{d,0}\le 1} |L_e\chi|$.

2. A preliminary estimate. For $t_0 \le v \le t < t_e$ with $t \le t_0 + d$, we have

$$\begin{aligned}
|x(v) - y(v)| &= \left|\int_{t_0}^{v} (x'(s) - y'(s))ds\right| = \left|\int_{t_0}^{v} (L(x_s - y_s) + r(x_s) - r(y_s))ds\right| \\
&= \left|\int_{t_0}^{v} (L_e(x_s - y_s) + r(x_s) - r(y_s))ds\right| \\
&\le (v - t_0)(c \max_{t_0\le s\le v} |x_s - y_s|_{d,0} + \lambda \max_{t_0\le s\le v} |x_s - y_s|_{d,1}) \\
&= (v - t_0)((c + \lambda) \max_{t_0\le s\le v} |x_s - y_s|_{d,0} + \lambda \max_{t_0\le s\le v} |(x')_s - (y')_s|_{d,0}) \\
&\le (t - t_0)((c + \lambda)|x_t - y_t|_{d,0} + \lambda|(x')_t - (y')_t|_{d,0})
\end{aligned}$$

where the last estimate follows from $t_0 \le v \le t \le t_0 + d$ and $x(s) = y(s)$ on $[-d, t_0]$. Using this once more we get

$$|x_t - y_t|_{d,0} \le (t - t_0)((c + \lambda)|x_t - y_t|_{d,0} + \lambda|(x')_t - (y')_t|_{d,0})$$

for $t_0 \le t < t_e$ with $t \le t_0 + d$.

3. Estimate of derivatives. For $t_0 \le v \le t < t_e$ with $t \le t_0 + d$, we have

$$\begin{aligned}
|x'(v) - y'(v)| &\le c|x_v - y_v|_{d,0} + \lambda(|x_v - y_v|_{d,0} + |(x')_v - (y')_v|_{d,0}) \\
&\le (c + \lambda)|x_t - y_t|_{d,0} + \lambda|(x')_t - (y')_t|_{d,0},
\end{aligned}$$

where the last estimate follows from $v \le t \le t_0 + d$ and $x(s) = y(s)$ on $[-d, t_0]$. Using this once more we see that in case $t_0 \le t < t_e$ and $t_0 \le t \le t_0 + d$ we have

$$|(x')_t - (y')_t|_{d,0} \le (c+\lambda)|x_t - y_t|_{d,0} + \lambda|(x')_t - (y')_t|_{d,0},$$

hence

$$|(x')_t - (y')_t|_{d,0} \le \frac{c+\lambda}{1-\lambda}|x_t - y_t|_{d,0}.$$

4. The result of part 3 inserted into the result of part 2 yields

$$|x_t - y_t|_{d,0} \le (t - t_0)\left((c+\lambda)|x_t - y_t|_{d,0} + \lambda\frac{c+\lambda}{1-\lambda}|x_t - y_t|_{d,0}\right)$$

for $t_0 \le t < t_e$ with $t \le t_0 + d$. It follows that $|x_t - y_t|_{d,0} = 0$ for $t > t_0$ sufficiently small, hence $x(u) = y(u)$ on $[-d, t]$ for some $t > t_0$ which is a contradiction to the properties of t_0. □

We turn to maps in Fréchet spaces which are continuously differentiable (MB). For the proof of a local transversality result which is familiar in case of continuously differentiable (F) maps in Banach spaces we need the following implicit function theorem.

Theorem 7.2. *Let a Fréchet space F and finite-dimensional normed spaces B and E and a continuously differentiable map (MB) $f : F \times B \supset U \to E$, U open, be given with $f(x, y) = 0$ and assume that $D_2 f(x, y) : B \to E$ is an isomorphism. Then there exist convex open neighbourhoods N_F of x in F and N_B of y in B and a continuously differentiable (MB) map $g : N_F \to N_B$ with $y = g(x)$ and*

$$(N_F \times N_B) \cap f^{-1}(0) = \{(z, b) \in N_F \times N_B : b = g(z)\}$$

For a proof see [3], or [18, Theorem 7.3] in combination with the remark preceding this theorem.

Proposition 7.3. *Let F, G be Fréchet spaces, $U \subset F$ open, $f : U \to G$ continuously differentiable (MB), and consider a continuously differentiable submanifold $M \subset G$ of finite codimension m. Assume that f and M are transversal at a point $x \in f^{-1}(M)$ in the sense that*

$$G = Df(x)F + T_{f(x)}M.$$

Then there is an open neighbourhood V of x in U so that $V \cap f^{-1}(M)$ is a continuously differentiable submanifold of codimension m in F, and $T_x(f^{-1}(M) \cap V) = Df(x)^{-1}T_xM$.

Proof. 1. There are an open neighbourhood N_g of $f(x)$ in G and a continuously differentiable (MB) diffeomorphism $g : N_g \to G$ onto an open set $U_g \subset G$ such that $g(f(x)) = 0$, $g(N_g \cap M) = U_g \cap T_{f(x)}M$, and $Dg(f(x)) = id$. (The last property can always be achieved by replacing g with $Dg(f(x))^{-1} \circ g$. Notice that $Dg(f(x))$ maps $T_{f(x)}M$ onto itself.)

2. By transversality and codim $M = m$ we find a subspace $Q \subset Df(x)F$ of dimension m which complements $T_{f(x)}M$ in G,

$$G = T_{f(x)}M \oplus Q.$$

The projection $P : G \to Q$ along $T_{f(x)}M$ onto Q is linear and continuous (see [13, Theorem 5.16]), and $PDg(f(x))Df(x) = PDf(x)$ is surjective. The preimage $U_f = f^{-1}(N_g)$ is open, with $x \in U_f \subset U$. For $z \in U_f$ we have

$$z \in f^{-1}(M) \cap U_f \Leftrightarrow f(z) \in M \cap N_g \Leftrightarrow Pg(f(z)) = 0.$$

For the continuously differentiable (MB) map $h = P \circ g \circ (f|_{U_f})$ we infer $f^{-1}(M) \cap U_f = h^{-1}(0)$. The derivative $Dh(x) : F \to Q$ is surjective. It follows that there is a subspace R of F with $\dim R = \dim Q = m$ and

$$F = Dh(x)^{-1}(0) \oplus R.$$

The restriction $Dh(x)|_R$ is an isomorphism.

3. The continuously differentiable (MB) map

$$H : \{(z,r) \in Dh(x)^{-1}(0) \times R : x + z + r \in U_f\} \ni (z,r) \mapsto h(x+z+r) \in Q$$

satisfies $H(0,0) = 0$. Because of $D_2H(0,0)\hat{r} = Dh(x)\hat{r}$ for all $\hat{r} \in R$ and $\dim R = \dim Q$ the map $D_2H(0,0)$ is an isomorphism. Theorem 7.2 yields convex open neighbourhoods V_H of 0 in $Dh(x)^{-1}(0)$ and V_R of 0 in R, with $x + V_H + V_R \subset U_f$, and a continuously differentiable (MB) map $w : V_H \to V_R$ with $w(0) = 0$ and

$$(V_H \times V_R) \cap H^{-1}(0) = \{(z,r) \in V_H \times V_R : r = w(z)\}.$$

For every $y \in x + V_H + V_R$, $y = x + z + r$ with $z \in V_H$ and $r \in V_R$, we have

$$y \in f^{-1}(M) \cap U_f \Leftrightarrow h(y) = 0 \Leftrightarrow h(x+z+r) = 0 \Leftrightarrow H(z,r) = 0 \Leftrightarrow r = w(z),$$

and $f^{-1}(M) \cap (x + V_H + V_R)$ is a shifted continuously differentiable (MB) graph, hence a continuously differentiable submanifold of F, with codimension equal to $\dim R = \dim Q = m$. Set $V = x + V_H + V_R$.

4. (On tangent spaces) From $f^{-1}(M) \cap U_f = h^{-1}(0)$ and $h(x) = 0$ we get $h(f^{-1}(M) \cap V) = \{0\}$, hence $Dh(x)T_x(f^{-1}(M) \cap V) = \{0\}$, or

$$T_x(f^{-1}(M) \cap V) \subset Dh(x)^{-1}(0).$$

As both spaces have the same codimension m they are equal. For every $v \in F$ we have

$$\begin{aligned} v \in Dh(x)^{-1}(0) &\Leftrightarrow Dh(x)v = 0 \Leftrightarrow PDf(x)v = 0 \\ &\Leftrightarrow Df(x)v \in P^{-1}(0) = T_{f(x)}M \Leftrightarrow v \in Df(x)^{-1}T_xM. \end{aligned}$$

Using this we obtain

$$T_x(f^{-1}(M) \cap V) = Dh(x)^{-1}(0) = Df(x)^{-1}T_xM. \qquad \square$$

Proposition 7.4. *Suppose W is an open subset of a finite-dimensional normed space V, and $j : W \to F$, F a Fréchet space, is continuously differentiable (MB), $b \in W$, and $Dj(b)$ is injective. Then there is an open neighbourhood N of $j(b)$ such that $N \cap j(W)$ is a continuously differentiable submanifold of F, with $T_{j(b)}(N \cap j(W)) = Dj(b)V$ (hence $\dim(N \cap j(W)) = \dim V$).*

Proof. 1. The finite-dimensional subspace $Y = Dj(b)V$ has a closed complementary space $Z \subset F$, see [13, Lemma 4.21], and the projection $P : F \to F$ along Z onto Y is continuous ([13, Theorem 5.16]). The map $P \circ j$ is continuously differentiable (MB) and defines a continuously differentiable (F) map $W \to Y$ since V and Y are finite-dimensional. Its derivative at b is an isomorphism $V \to Y$ (use $Py = y$ on Y and the injectivity of $Dj(b)$). The Inverse Mapping Theorem yields a continuously differentiable (F) map $g : Y \cap U \to V$, U open in F and $P(j(b)) \in Y \cap U$, such that $g(P(j(b))) = b$, and an open neighbourhood $W_1 \subset W$ of b in V such that $g(Y \cap U) = W_1$, $(P \circ j)(W_1) = Y \cap U$, $(g \circ (P \circ j))(v) = v$ on W_1, and $((P \circ j) \circ g)(y) = y$ on $Y \cap U$. It follows that the map $h : Y \cap U \to Z$ given by

$$h(y) = ((id_F - P) \circ j \circ g)(y)$$

is continuously differentiable (MB).

2. Proof of $j(W_1) = \{y + h(y) : y \in Y \cap U\}$: (a) For $y \in Y \cap U$,

$$\begin{aligned} y + h(y) &= y + ((id_F - P) \circ j \circ g)(y) \\ &= ((P \circ j) \circ g)(y) + (j \circ g)(y) - ((P \circ j) \circ g)(y) = j(g(y)) \in j(W_1). \end{aligned}$$

(b) For $x \in j(W_1)$ there exists $y \in Y \cap U$ with

$$\begin{aligned} x &= j(g(y)) = ((P \circ j) \circ g)(y) + j(g(y)) - (P \circ j)(g(y)) \\ &= y + ((id_F - P) \circ j \circ g)(y) = y + h(y). \end{aligned}$$

The graph representation of $j(W_1)$ now yields that it is a continuously differentiable submanifold of F. □

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