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A LIAPUNOV FUNCTIONAL FOR A LINEAR INTEGRAL EQUATION

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Abstract. In this note we consider a scalar integral equation $x(t) = a(t) - \int_0^t C(t,s)x(s)ds$, together with its resolvent equation, $R(t,s) = C(t,s) - \int_s^t C(t,u)R(u,s)du$, where C is convex. Using a Liapunov functional we show that for fixed s then $|R(t,s) - C(t,s)| \to 0$ as $t \to \infty$ and $\int_s^\infty (R(t,s) - C(t,s))^2 dt < \infty$. We then show that the variation of parameters formula $x(t) = a(t) - \int_0^t R(t,s)a(s)ds$ can be replaced by $X(t) = a(t) - \int_0^t C(t,s)a(s)ds$ when $a \in L^1[0,\infty)$ and that $|X(t) - x(t)| \to 0$ as $t \to \infty$ and $\int_0^\infty (x(t) - X(t))^2 dt < \infty$. A mild nonlinear extension is given.

1. Introduction

The purpose of this note is to show that some of the most intricate properties of solutions of integral equations can be obtained from a Liapunov functional and that the resolvent can often be replaced by the kernel with impunity. We consider a scalar integral equation

$$x(t) = a(t) - \int_0^t C(t,s)x(s)ds \tag{1}$$

where C is convex:

$$C(t,s) \geq 0, C_s(t,s) \geq 0, C_{st}(t,s) \leq 0, C_t(t,s) \leq 0. \tag{2}$$

The subscripts denote the usual partial derivatives and a is continuous. Convex kernels for both integral and integrodifferential equations are commonly used in the study of many real-world problems. Discussions may be found in Volterra [13], Grippenberg-Londen-Staffans [7; see index], Londen [10], and throughout Burton [4], for example.

In 1963 Levin [8] followed a suggestion of Volterra and constructed a Liapunov functional for

$$x' = -\int_0^t C(t,s)g(x(s))ds \tag{3}$$

with C convex and $xg(x) > 0$ if $x \neq 0$. There are numerous papers along the same line by Levin and Levin and Nohel. In 1992 we [1] constructed a Liapunov functional for a nonlinear form of (1). That Liapunov functional has seen considerable use in the literature and

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much of the work is cited in [4]. Here, we will show how it can be used to yield very intricate properties of the solution, $R(t,s)$, of the resolvent equation as well as the solution of (1) in case $a \in L^2[0,\infty)$. This problem has also been studied in Burton-Dwiggins [5] and that paper, together with some significant extensions, was discussed at Equadiff 12 in Brno, Czech Republic in June 2009 and at a conference in Ekaterinburg, Russia in September 2009.

The resolvent equation for (1) is

$$\begin{aligned} R(t,s) &= C(t,s) - \int_s^t C(t,u)R(u,s)du \\ &= C(t,s) - \int_s^t R(t,u)C(u,s)du \end{aligned} \tag{4}$$

and the variation-of-parameters formula is

$$x(t) = a(t) - \int_0^t R(t,s)a(s)ds. \tag{5}$$

It follows from work of Ritt [12] and Kaplansky [11] that the solution of (1) can generally not be expressed in terms of intricate combinations of elementary functions, and the same is true for R. Yet, it will turn out that $R(t,s)$ converges to $C(t,s)$ so strongly that C can replace R in (5), yielding a function which converges to x both pointwise and in $L^2[0,\infty)$.

2. The main results

A Liapunov functional for (1) was constructed in [1] and several applications and extensions are discussed in the monograph [4]. An important extension to nonlinear systems is found in Zhang [14]. Very recently [4] it was extended to (4) and has the form

(6)

$$V(t) = \int_s^t C_v(t,v)\Big(\int_v^t R(u,s)du\Big)^2 dv + C(t,s)\Big(\int_s^t R(u,s)du\Big)^2.$$

The derivative of V along the unique solution of (4) satisfies

$$V'(t) \le 2R(t,s)[C(t,s) - R(t,s)]. \tag{7}$$

Moreover, if there is a positive constant B with

$$C(t,t) \le B \tag{8}$$

then it is true that

$$(C(t,s) - R(t,s))^2 \le 2BV(t). \tag{9}$$

It would be a distraction to derive all of those relations at this point, but we include the details in the appendix.

The natural consequence of (7) is

$$V'(t) \le -R^2(t,s) + C^2(t,s), \tag{10}$$

yielding

$$V(t) \le V(s) - \int_s^t R^2(u,s)du + \int_s^t C^2(u,s)du \tag{11}$$

so that in view of (9) we have the relation

$$\frac{1}{2B}(C(t,s) - R(t,s))^2 + \int_s^t R^2(u,s)du \le \int_s^t C^2(u,s)du. \tag{12}$$

This is parallel to what is done in [1–6, 14]. Our new work begins here. It is such a small step to refrain from passing from (7) to the inequality (10) and, instead, write the equality

$$\begin{aligned} V'(t) &\le 2R(t,s)[C(t,s) - R(t,s)] \\ &= -(C(t,s) - R(t,s))^2 - R^2(t,s) + C^2(t,s). \end{aligned} \tag{13}$$

Integration of that last line will lead to very exact properties of both $R(t,s)$ and $x(t)$. Concerning terminology, when we say that f converges to g in $L^2[s,\infty)$ we mean that $\int_s^\infty (f(t) - g(t))^2 dt < \infty$.

Theorem 2.1. *Let (2) and (8) hold so that (13) also holds. Then for* $0 \le s \le t < \infty$,

$$\begin{aligned} \frac{1}{2B}(C(t,s) - R(t,s))^2 + \int_s^t (C(u,s) - R(u,s))^2 du + \int_s^t R^2(u,s)du \\ \le \int_s^t C^2(u,s)du. \end{aligned} \tag{14}$$

If, in addition,

$$\sup_{0 \le s \le t < \infty} \int_s^t C^2(u,s)du < \infty, \tag{15}$$

then for fixed s*,*

$$\int_s^\infty (R(u,s) - C(u,s))^2 du < \infty. \tag{16}$$

If, in addition,

$$\sup_{0 \le s \le t < \infty} \int_s^t [C^2(t,u) + C_t^2(t,u)]du < \infty \tag{17}$$

then for fixed s

$$|R(t,s) - C(t,s| \to 0 \text{ as } t \to \infty. \tag{18}$$

Proof. Relation (14) is obtained by integrating (13) from s to t, observing that $V(s) = 0$, and then using (9). Next, if (15) holds then (16) is immediate from (14); moreover,

$$(C(t,s) - R(t,s))^2 \text{ is bounded for } 0 \le s \le t < \infty. \tag{19}$$

If we can show that this quantity has a bounded derivative with respect to t, then the fact that $\int_s^t (C(u,s) - R(u,s))^2 du$ converges for fixed s will imply that the integrand tends to zero as $u \to \infty$ for fixed s. To that end, we note that from (4) we have

$$R_t(t,s) - C_t(t,s) = -C(t,t)R(t,s) - \int_s^t C_t(t,u)R(u,s)du \tag{20}$$

and

$$|R(t,s)| \le |C(t,s)| + \sqrt{\int_s^t C^2(t,u)du \int_s^t R^2(u,s)du}. \tag{21}$$

Since $C(t,s) \ge 0$ and $C_t(t,s) \le 0$, for fixed s it follows that $C(t,s)$ is bounded. From (17) and (12) we now have that $|R(t,s)|$ is bounded. It follows from (20) that

$$|R_t(t,s) - C_t(t,s)| \le |C(t,t)R(t,s)| + \sqrt{\int_s^t C_t^2(t,u)du \int_s^t R^2(u,s)du}$$

which is bounded. This, together with (19) yields a bounded t-derivative for $(C(t,s) - R(t,s))^2$ and that will show that $C(t,s) - R(t,s) \to 0$ as $t \to \infty$ for fixed s. □

Parts of this result had been obtained in [4] and [5] using (10) instead of (13) which required several additional conditions.

As R converges to C we now show that in (5) we can replace the totally unknown function R by the clearly given function C and have an excellent approximation to x.

Theorem 2.2. *Suppose that all conditions of the previous theorem hold and let $a \in L^1[0,\infty)$. If*

$$X(t) = a(t) - \int_0^t C(t,s)a(s)ds \tag{22}$$

and if x solves (1), then

$$|X(t) - x(t)| \to 0 \text{ as } t \to \infty \text{ and } \int_0^\infty (X(t) - x(t))^2 dt < \infty. \tag{23}$$

Proof. Set $\int_0^\infty |a(s)|ds = A$ and let $\int_s^t (C(u,s) - R(u,s))^2 du \le L$ for positive constants A and L. By the Schwarz inequality we have

$$(x(t) - X(t))^2 \le A \int_0^t [C(t,s) - R(t,s)]^2 |a(s)| ds.$$

Integration and interchange of the order of integration yields

$$\begin{aligned}\int_0^t (x(u) - X(u))^2 du &\le A \int_0^t \int_0^u [C(u,s) - R(u,s)]^2 |a(s)| ds du \\ &= A \int_0^t \int_s^t [C(u,s) - R(u,s)]^2 du |a(s)| ds \\ &\le AL \int_0^t |a(s)| ds \\ &\le A^2 L.\end{aligned}$$

We also have that $C_t(t,s) - R_t(t,s)$ is bounded so if we notice that $C(t,t) = R(t,t)$ then

$$(x(t) - X(t))' = [C(t,t) - R(t,t)]a(t) + \int_0^t [C_t(t,s) - R_t(t,s)]a(s) ds$$

is bounded. Since $\int_0^t (x(u) - X(u))^2 du$ converges, the integrand tends to zero. □

The properties of R are important in other contexts such as nonlinear perturbation problems of the type considered in [4; Section 2.8].

3. A direct result

The work here has been rather indirect for we wanted to derive important properties of the resolvent. However, we can start with (1), (2), and (8), define a Liapunov functional

$$W(t) = \int_0^t C_s(t,s) \left(\int_s^t x(u) du \right)^2 ds + C(t,0) \left(\int_0^t x(u) du \right)^2 \tag{24}$$

and find that the derivative of W along the solution of (1) satisfies

$$W'(t) \le 2x(t)[a(t) - x(t)] = -(a(t) - x(t))^2 - x^2(t) + a^2(t) \tag{25}$$

and that

$$(x(t) - a(t))^2 \le 2BW(t). \tag{26}$$

The details are easily obtained from the presentation in the appendix and are explicitly given in [4; pp. 64-66]. We gather the results as follows.

Theorem 3.1. *If (2) and (8) hold, then*

$$\frac{1}{2B}(a(t)-x(t))^2+\int_0^t (a(s)-x(s))^2ds+\int_0^t x^2(s)ds \le \int_0^t a^2(u)du. \tag{27}$$

If, in addition, $a \in L^2[0,\infty)$ then $\int_0^\infty (x(t)-a(t))^2 dt < \infty$, while $a(t)$ bounded implies $x(t)$ bounded. If, in addition, $\int_0^t C_t^2(t,s)ds$ is bounded, then $|x(t)-a(t)| \to 0$ as $t \to \infty$.

The conditions in the last sentence will show that $(a(t)-x(t))'$ is bounded so convergence of $\int_0^t (a(s)-x(s))^2 ds$ will show that the integrand tends to zero as $s \to \infty$.

These simple Liapunov functionals show us quite precisely both the solution of (1) for large t and the properties of the resolvent.

4. A nonlinear extension

We now consider a nonlinear problem

$$x(t) = a(t) - \int_0^t C(t,s)g(s,x(s))ds \tag{28}$$

in which $g : [0,\infty) \times \Re \to \Re$ is continuous, (2) is satisfied, and

$$xg(t,x) > 0 \text{ if } x \neq 0. \tag{29}$$

Define a Liapunov functional

$$\begin{aligned} Z(t) = &\int_0^t C_s(t,s)\left(\int_s^t g(u,x(u))du\right)^2 ds \\ &+ C(t,0)\left(\int_0^t g(u,x(u))du\right)^2, \end{aligned} \tag{30}$$

follow the differentiation shown in the appendix (or see [4; p. 191]), and conclude that the derivative of Z along any solution of (28) satisfies

$$Z'(t) \le 2g(t,x(t))[a(t)-x(t)] \tag{31}$$

and that

$$(x(t)-a(t))^2 \le 2C(t,t)Z(t). \tag{32}$$

There are several ways to obtain a counterpart of (13), but in this linear context we ask that

$$|g(t,x)| \le |x|. \tag{33}$$

This permits us to say that $Z'(t) \le 2g(t,x)[a(t)-g(t,x)]$ from which we obtain

$$Z'(t) \le -(a(t)-g(t,x(t))^2 - g^2(t,x(t)) + a^2(t). \tag{34}$$

Theorem 4.1. *Let (2), (29), and (33) hold. If $a \in L^2[0,\infty)$, so are $(a(t) - g(t,x(t)))$ and $g(t,x(t))$, while $a(t)$ and $C(t,t)$ bounded yield $x(t)$ bounded.*

Proof. An integration of (34) from 0 to t yields the integrability properties so that $Z(t)$ is bounded. That and $C(t,t)$ bounded in (32) yield $(x(t)-a(t))^2$ bounded, yielding $x(t)$ bounded when $a(t)$ is bounded. $\square$

While it was simple to show $(x(t)-a(t))'$ bounded, it is not so easy to show $(g(t,x(t))-a(t))'$ bounded. There is another way to reach the conclusion that $|x(t)-a(t)| \to 0$ as $t \to \infty$.

Theorem 4.2. *Let (2), (29), and (33) hold with $a \in L^2[0,\infty)$ so that $\int_0^\infty g^2(s,x(s)ds =: L < \infty$. Let*

$$\sup_{0\le t<\infty} \int_0^\infty C^2(t,s)ds =: M < \infty$$

and suppose that for each $T > 0$ then $\lim_{t\to\infty} \int_0^T C^2(t,s)ds = 0$. Then any solution $x(t)$ of (28) satisfies $|x(t)-a(t)| \to 0$ as $t \to \infty$.

Proof. For any $T > 0$ we have from (28) that

$$\begin{aligned}
|x(t)-a(t)| &\le \int_0^t |C(t,s)||g(s,x(s))|ds \\
&= \int_0^T |C(t,s)||g(s,x(s))|ds \\
&+ \int_T^t |C(t,s)||g(s,x(s))|ds \\
&\le \sqrt{\int_0^T C^2(t,s)ds \int_0^T g^2(s,x(s))ds} \\
&+ \sqrt{\int_T^t C^2(t,s)ds \int_T^t g^2(s,x(s))ds} \\
&\le \sqrt{L\int_0^T C^2(t,s)ds} + \sqrt{M\int_T^t g^2(s,x(s))ds}.
\end{aligned}$$

Now, for a given $\epsilon > 0$ choose T so large that

$$M\int_T^\infty g^2(s,x(s))ds < \epsilon^2/4.$$

Having chosen T, take t so large that

$$L\int_0^T C^2(t,s)ds < \epsilon^2/4.$$

This completes the proof. $\square$

5. APPENDIX

We now want to show how V is constructed and find the derivative of V along the solution of (4).

Theorem 5.1. *Let (2) hold and define*

$$V(t) = \int_s^t C_v(t,v)\Big(\int_v^t R(u,s)du\Big)^2 dv + C(t,s)\Big(\int_s^t R(u,s)du\Big)^2.$$

Then along the solution of (4) we have

$$V'(t) \le 2R(t,s)[C(t,s) - R(t,s)]$$

If there is a $B > 0$ with $C(t,t) \le B$ then

$$\frac{1}{2B}(R(t,s) - C(t,s))^2 \le V(t).$$

Proof. We first prove that for this V we will have the required derivative along the unique solution of (4). Begin by differentiating V using Leibnitz's rule. Thus,

$$\begin{aligned} V'(t) = &\int_s^t C_{vt}(t,v)\Big(\int_v^t R(u,s)du\Big)^2 dv \\ &+ C_t(t,s)\Big(\int_s^t R(u,s)du\Big)^2 \\ &+ 2R(t,s)\int_s^t C_v(t,v)\int_v^t R(u,s)dudv \\ &+ 2R(t,s)C(t,s)\int_s^t R(u,s)du. \end{aligned}$$

If we integrate the next to last term by parts we have

$$\begin{aligned} &2R(t,s)\Big[C(t,v)\int_v^t R(u,s)du\Big|_s^t + \int_s^t C(t,v)R(v,s)dv\Big] \\ &= 2R(t,s)\Big[-C(t,s)\int_s^t R(u,s)du + \int_s^t C(t,v)R(v,s)dv\Big]. \end{aligned}$$

Hence, taking into account sign conditions we have

$$\begin{aligned} V'(t) &\le 2R(t,s)\int_s^t C(t,v)R(v,s)dv \\ &= 2R(t,s)[C(t,s) - R(t,s)] \quad \text{from (4).} \end{aligned}$$

Notice that

$$C(t,s) + \int_s^t C_u(t,u)du = C(t,t) \le B$$

and we want to show that

$$\frac{1}{2B}(R(t,s) - C(t,s))^2 \le V(t).$$

At the same time we will see how the Liapunov functional is constructed. Squaring (4) yields

$$\begin{aligned}
&\Big(R(t,s) - C(t,s)\Big)^2 = \Big(-\int_s^t C(t,u)R(u,s)du\Big)^2 \\
&= \Big(C(t,u)\int_u^t R(v,s)dv\Big|_s^t - \int_s^t C_u(t,u)\int_u^t R(v,s)dvdu\Big)^2 \\
&= \Big(-C(t,s)\int_s^t R(v,s)dv - \int_s^t C_u(t,u)\int_u^t R(v,s)dvdu\Big)^2 \\
&\le 2\Big[C^2(t,s)\Big(\int_s^t R(v,s)dv\Big)^2 \\
&+ \int_s^t C_u(t,u)du \int_s^t C_u(t,u)\Big(\int_u^t R(v,s)dv\Big)^2 du\Big] \\
&\le 2\Big[C(t,s) + \int_s^t C_u(t,u)du\Big]V(t) \\
&\le 2BV(t).
\end{aligned}$$

$\square$

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