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A priori estimates of global solutions of superlinear parabolic systems

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Abstract. We consider the parabolic system $u_t - \Delta u = u^r v^p$, $v_t - \Delta v = u^q v^s$ in $\Omega \times (0, \infty)$, complemented by the homogeneous Dirichlet boundary conditions and the initial conditions $(u, v)(\cdot, 0) = (u_0, v_0)$ in Ω, where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $u_0, v_0 \in L^\infty(\Omega)$ are nonnegative functions. We find conditions on p, q, r, s guaranteeing a priori estimates of nonnegative classical global solutions. More precisely every such solution is bounded by a constant depending on suitable norm of the initial data. Our proofs are based on bootstrap in weighted Lebesgue spaces, universal estimates of auxiliary functions and estimates of the Dirichlet heat kernel.

Keywords: parabolic system, a priori estimates, bootstrap.

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1 Introduction

Superlinear parabolic problems represent important mathematical models for various phenomena occurring in physics, chemistry or biology. Therefore such problems have been intensively studied by many authors. Beside solving the question of existence, uniqueness, regularity etc. significant effort has been made to obtain a priori estimates of solutions. A priori estimates are important in the study of global solutions (i.e. solutions which exist for all positive times) or blow-up solutions (i.e. solutions whose L^∞-norm becomes unbounded in finite time); superlinear parabolic problems may possess both of these types of solutions. Uniform a priori estimates also play a crucial role in the study of so-called threshold solutions, i.e. solutions lying on the borderline between global existence and blow-up.

Stationary solutions of parabolic problems are particular global solutions and their a priori estimates are of independent interest since they can be used to prove the existence and/or multiplicity of steady states, for example. The proofs of such estimates are usually much easier than the proofs of estimates of time-dependent solutions. On the other hand, the methods of the proofs of a priori estimates of stationary solutions can often be modified to yield a priori estimates of global time-dependent solutions.

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In this paper we study global classical positive solutions of the model problem

$$\left.\begin{aligned} u_t - \Delta u &= u^r v^p, && (x,t) \in \Omega \times (0,\infty), \\ v_t - \Delta v &= u^q v^s, && (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = v(x,t) &= 0, && (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) &= u_0(x), && x \in \Omega, \\ v(x,0) &= v_0(x), && x \in \Omega, \end{aligned}\right\} \tag{1.1}$$

where $p,q,r,s \geq 0$ and

$$\Omega \subset \mathbb{R}^N \text{ is smooth and bounded, } u_0, v_0 \in L^\infty(\Omega) \text{ are nonnegative.} \tag{1.2}$$

In this case, sufficient conditions on the exponents p,q,r,s guaranteeing a priori estimates and existence of positive stationary solutions have been obtained in [3, 13, 16–19]. In particular, the conditions in [10] are valid for a large class of so-called very weak solutions, and they are optimal in this class. We find sufficient conditions on the exponents guaranteeing uniform a priori estimates of global classical solutions. Our method is in some sense similar to that used in [10] (both methods are based on bootstrap in weighted Lebesgue spaces and estimates of auxiliary functions of the form $u^a v^{1-a}$; the idea of using such auxiliary functions for elliptic systems seems to go back to a paper [12]) but our proofs are much more involved. In particular, we have to use precise estimates of the Dirichlet heat semigroup and several additional ad-hoc arguments. These difficulties cause that our sufficient conditions are quite technical and probably not optimal. On the other hand, our results are new and our approach is also new in the parabolic setting: Although the bootstrap in weighted Lebesgue spaces has been used many times in the case of superlinear elliptic problems (see the references in [10], for example), it has not yet been used to prove a priori estimates of global solutions of superlinear parabolic problems. In fact, the known methods for obtaining such estimates always require some special structure of the problem and cannot be used for system (1.1) in general. In addition, our method is quite robust: It can also be used if the problem is perturbed or if we replace the Dirichlet boundary conditions by the Neumann ones, for example.

Next we present our main results concerning problem (1.1). Beside (1.2), we will further assume that

$$p,q,r,s \geq 0; \qquad \text{if } q = 0 \text{ then either } r > 1 \text{ or } s \leq 1, \tag{1.3}$$

and we denote by $\|\cdot\|_{1,\delta}$ the norm in the weighted Lebesgue space $L^1(\Omega; \mathrm{dist}(x,\partial\Omega)\,\mathrm{d}x)$.

Theorem 1.1. *Assume* (1.2), (1.3) *and* $pq > (r-1)(s-1)$. *Assume also that either*

$$r > 1, \quad p > 0, \quad p + r < \frac{N+3}{N+1}, \quad s + \frac{2}{N+1}\frac{r-1}{p+r-1} < \frac{N+3}{N+1}$$

or

$$r \leq 1, \quad 0 < p < \frac{2}{N+1}, \quad s < \frac{N+3}{N+1}.$$

Let (u,v) *be a global solution of problem* (1.1). *Then there exists* $C = C(p,q,r,s,\Omega,\|u_0\|_\infty,\|v_0\|_\infty)$ *such that*

$$\|u(t)\|_\infty + \|v(t)\|_\infty \leq C, \quad t \geq 0.$$

Theorem 1.2. *Assume (1.2), (1.3) and either* $\max\{r,s\} > 1$ *or* $pq > (r-1)(s-1)$. *Assume also*

$$p \geq 1, \quad p+r < \frac{N+3}{N+1}, \quad s \leq 1, \quad (p+r)\left(p - \frac{2}{N+1}\right) + r < 1$$

and

$$0 < q < \frac{1-r}{p - \frac{2}{N+1}}\left(1 - \frac{N-1}{N+1}s\right).$$

Let (u,v) *be a global solution of problem (1.1). Then, given* $\tau > 0$, *there exists* $C = C(p,q,r,s,\Omega,\tau,$ $\|u(\tau)\|_{1,\delta}, \|v(\tau)\|_{1,\delta})$ *such that*

$$\|u(t)\|_\infty + \|v(t)\|_\infty \leq C, \quad t \geq \tau. \tag{1.4}$$

The constant C *may explode if* $\tau \to 0^+$, *and is bounded for* $\|u(\tau)\|_{1,\delta}$, $\|v(\tau)\|_{1,\delta}$ *bounded.*

One of the main applications of uniform a priori estimates of global positive solutions of (1.1) is the proof of global existence and boundedness of threshold solutions lying on the borderline between global existence and blow-up. Let us mention that our conditions on p,q,r,s from Theorems 1.1 and 1.2 guarantee that both global and blow-up solutions (hence also threshold solutions) of (1.1) exist; see [1,14]. See also [2,15,20] for other results on blow-up of positive solutions of (1.1).

As already mentioned, our approach is quite robust. It can also be used, for example, for the following problem with Neumann boundary conditions

$$\left.\begin{aligned} u_t - \Delta u &= u^r v^p - \lambda u, && (x,t) \in \Omega \times (0,\infty), \\ v_t - \Delta v &= u^q v^s - \lambda v, && (x,t) \in \Omega \times (0,\infty), \\ u_\nu(x,t) = v_\nu(x,t) &= 0, && (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) &= u_0(x), && x \in \Omega, \\ v(x,0) &= v_0(x), && x \in \Omega, \end{aligned}\right\} \tag{1.5}$$

where Ω, p,q,r,s and u_0, v_0 are as above, $\lambda > 0$ and ν is the outer unit normal on the boundary $\partial\Omega$. The terms $-\lambda u, -\lambda v$ with $\lambda > 0$ are needed in (1.5), since otherwise (1.5) cannot admit both global and blow-up positive solutions. Let us also note that in this case one has to work in standard (and not weighted) Lebesgue spaces and that the restrictions on the exponents p,q,r,s are less severe than in the case of Dirichlet boundary conditions: roughly speaking, one can replace N with $N-1$ in those restrictions (in particular, the condition $p+r < \frac{N+3}{N+1}$ becomes $p+r < \frac{N+2}{N}$ in this case). As other particular application of our method, we present the following theorem.

Theorem 1.3. *Consider problem*

$$\left.\begin{aligned} u_t - \Delta u &= uv - b_1 u, && (x,t) \in \Omega \times (0,\infty), \\ v_t - \Delta v &= b_2 u, && (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = v(x,t) &= 0, && (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) &= u_0(x), && x \in \Omega, \\ v(x,0) &= v_0(x), && x \in \Omega, \end{aligned}\right\} \tag{1.6}$$

where Ω *is a bounded domain with smooth boundary,* $N \leq 2$, $b_1 \geq 0$, $b_2 > 0$ *and* $u_0, v_0 \in L^\infty(\Omega)$. *Then there exists* $C = C(\Omega, b_1, b_2)$ *such that*

$$\limsup_{t\to\infty} \left(\|u(t)\|_\infty + \|v(t)\|_\infty\right) \leq C$$

for every global nonnegative solution (u,v) *of problem (1.6).*

More detailed proofs of Theorems 1.1–1.3 can be found in [9].

If $r = s = 0$ and $p, q > 1$, then a very easy argument in [6] yields a universal estimate of $\|u(\tau)\|_{1,\delta}, \|v(\tau)\|_{1,\delta}$ for all $\tau \geq 0$, hence Theorem 1.2 guarantees estimate (1.4) with $C = C(p, q, \Omega, \tau)$. The same estimate was obtained in [6] under the assumption $p, q \in \left(1, \frac{N+3}{N+1}\right)$ which is different from that in Theorem 1.2 (we do not require $q < \frac{N+3}{N+1}$, for example). Of course, if $r = s = 0$, then one could also use different methods for obtaining a priori estimates, e.g. the parabolic Liouville theorems in [5] together with scaling and doubling arguments to prove qualitative universal estimates. The main advantage of our results and proofs is the fact that we do not need the assumption $r = s = 0$.

2 Preliminaries

We introduce some notation we will use frequently. Denote $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ for $x \in \Omega$, and for $1 \leq p \leq \infty$ define the weighted Lebesgue spaces $L^p_\delta = L^p_\delta(\Omega) := L^p(\Omega; \delta(x)\,\mathrm{d}x)$. If $1 \leq p < \infty$, then the norm in L^p_δ is defined by $\|u\|_{p,\delta} = \left(\int_\Omega |u(x)|^p \delta(x)\,\mathrm{d}x\right)^{1/p}$. Recall that $L^\infty_\delta = L^\infty(\Omega; \delta(x)\,\mathrm{d}x)$ with $\|u\|_{\infty,\delta} = \|u\|_\infty$. We will use the notation $\|\cdot\|_p$ for the norm in $L^p(\Omega)$ for $p \in [1, \infty)$, as well.

Let λ_1 be the first eigenvalue of the problem

$$\left.\begin{aligned} -\Delta\phi &= \lambda\phi, \qquad x \in \Omega, \\ \phi &= 0, \qquad x \in \partial\Omega, \end{aligned}\right\}$$

and φ_1 to be the corresponding positive eigenfunction satisfying $\|\varphi_1\|_2 = 1$. There holds

$$C(\Omega)\delta(x) \leq \varphi_1(x) \leq C'(\Omega)\delta(x) \quad \text{for all } x \in \Omega. \tag{2.1}$$

Therefore the norm $\|u\|_{p,\varphi_1} = \left(\int_\Omega |u(x)|^p \varphi_1(x)\,\mathrm{d}x\right)^{1/p}$ is equivalent to the norm $\|u\|_{p,\delta}$ in $L^p_\delta(\Omega)$ for $1 \leq p < \infty$.

Let (u, v) be a solution of system (1.1). Then (u, v) solves the system of integral equations

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s')\Delta} u^r v^p(s')\,\mathrm{d}s', \quad v(t) = e^{t\Delta}v_0 + \int_0^t e^{(t-s')\Delta} u^q v^s(s')\,\mathrm{d}s' \tag{2.2}$$

where $t \geq 0$ and $\left(e^{t\Delta}\right)_{t\geq 0}$ is the Dirichlet heat semigroup in Ω. In the following lemma we recall some basic properties of the semigroup $\left(e^{t\Delta}\right)_{t\geq 0}$, which we will use often. The corresponding proofs can be found e.g. in [6].

Lemma 2.1. *Let Ω be arbitrary bounded domain.*

(i) If $\phi \in L^1_\delta(\Omega)$, $\phi \geq 0$, then $e^{t\Delta}\phi \geq 0$.

(ii) $\|e^{t\Delta}\phi\|_{1,\varphi_1} = e^{-\lambda_1 t}\|\phi\|_{1,\varphi_1}$ for $t \geq 0$, $\phi \in L^1_\delta(\Omega)$.

(iii) If $p \in (1, \infty]$, then $\|e^{t\Delta}\phi\|_{p,\delta} \leq C(\Omega)e^{-\lambda_1 t}\|\phi\|_{p,\delta}$ for $t \geq 0$, $\phi \in L^p_\delta(\Omega)$.

(iv) Let Ω be of the class C^2. For $1 \leq p < q \leq \infty$, there exists constant $C = C(\Omega)$ such that, for all $\phi \in L^p_\delta(\Omega)$, it holds

$$\|e^{t\Delta}\phi\|_{q,\delta} \leq C(\Omega)t^{-\frac{N+1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\phi\|_{p,\delta}, \quad t > 0.$$

Assertions (iii) and (iv) from Lemma 2.1 for $1 \le p < q \le \infty$, $t > 0$ and $\varepsilon \in (0,1)$ imply

$$\|e^{t\Delta}\phi\|_{q,\delta} \le C(\Omega)e^{-\lambda_1 \varepsilon t}((1-\varepsilon)t)^{-\frac{N+1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\phi\|_{p,\delta}, \quad \phi \in L^p_\delta(\Omega).$$

If we multiply the equations in (2.2) by φ_1 and integrate on Ω, then assertions (i) and (ii) from Lemma 2.1 imply

$$\|u(t)\|_{1,\varphi_1} \ge e^{-\lambda_1 t}\|u_0\|_{1,\varphi_1}, \quad \|v(t)\|_{1,\varphi_1} \ge e^{-\lambda_1 t}\|v_0\|_{1,\varphi_1}. \tag{2.3}$$

We will also use the following estimate of the semigroup $\left(e^{t\Delta}\right)_{t\ge 0}$; see e.g. [11].

Lemma 2.2. *Let Ω be smooth bounded domain. For every $f \in L^1_\delta(\Omega)$, $f \ge 0$, there holds*

$$(e^{t\Delta}f)(x) \ge C(t)\delta(x)\|f\|_{1,\delta}, \quad x \in \Omega,$$

where the constant C may be arbitrarily small if $t \to 0^+$, and is positive for t bounded.

Let (u,v) be a solution of system (1.5). Then (u,v) solves the system of integral equations similar to (2.2) with e^{tL} instead of $e^{t\Delta}$, where $e^{tL} := e^{-\lambda t}e^{t\Delta_N}$, $t \ge 0$ is the semigroup corresponding to operator $L := \Delta - \lambda$ with homogeneous Neumann boundary condition and $\left(e^{t\Delta_N}\right)_{t\ge 0}$ is the Neumann heat semigroup in Ω. For the Neumann semigroup, estimates similar to those from Lemma 2.1 are true; see [4, 8]. One can also obtain inequalities similar to (2.3) with φ_1 replaced by 1 and λ_1 replaced by λ.

In the following we will use the notation from [10]. We set

$$A := \begin{cases} [a_r, a_s] \cap (0,1) & \text{if } pq \ge (r-1)(s-1) \text{ or } \min\{r,s\} \le 1, \\ [a_s, a_r] \cap (0,1) & \text{if } pq < (r-1)(s-1) \text{ and } r,s > 1, \end{cases}$$

where

$$a_r := \begin{cases} \frac{r-1}{p+r-1} & \text{if } r > 1, \\ 0 & \text{if } r \le 1, \end{cases} \qquad a_s := \begin{cases} \frac{q}{q+s-1} & \text{if } s > 1, \\ 1 & \text{if } s \le 1. \end{cases}$$

Note that the set A is nonempty provided there holds

$$\left.\begin{array}{l} \text{if } p = 0, \text{ then either } s > 1 \text{ or } r \le 1, \\ \text{if } q = 0, \text{ then either } r > 1 \text{ or } s \le 1. \end{array}\right\} \tag{2.4}$$

The following lemma is an adaptation of [10, Lemma 7] to systems (1.1) and (1.5):

Lemma 2.3. *Assume $p,q,r,s \ge 0$, $pq \ne (1-r)(1-s)$ and (2.4). For given $a \in A$, there exists $\kappa \ge 0$ and $C = C(p,q,r,s,a)$ such that any global nonnegative solution of (1.1) satisfies*

$$(u^a v^{1-a})_t - \Delta(u^a v^{1-a}) \ge F_a(u,v) \ge C(u^a v^{1-a})^\kappa, \quad t \in (0,\infty), \tag{2.5}$$

where

$$\begin{aligned} F_a(u,v) &:= a u^{a-1} v^{1-a}(u_t - \Delta u) + (1-a)u^a v^{-a}(v_t - \Delta v) \\ &= a u^{r+a-1} v^{p+1-a} + (1-a)u^{q+a} v^{s-a}, \quad t \in (0,\infty). \end{aligned}$$

Similarly, for any global nonnegative solution of (1.5), there holds

$$(u^a v^{1-a})_t - \Delta(u^a v^{1-a}) + \lambda(u^a v^{1-a}) \ge C(u^a v^{1-a})^\kappa, \quad t \in (0,\infty). \tag{2.6}$$

If

$$\max\{r,s\} > 1 \quad \text{or} \quad pq > (r-1)(s-1), \tag{2.7}$$

then $\kappa > 1$.

Let (u, v) be a global nonnegative solution of system (1.1). Denote $w = w(t) := \int_\Omega u^a v^{1-a}(t)\varphi_1 \,\mathrm{d}x$. The following estimates are based on ideas from [7]. Let $a \in A$ and condition (2.7) be true (then $\kappa > 1$). Then due to Lemma 2.3 and due to Jensen's inequality, it holds

$$w_t + \lambda_1 w \geq C \int_\Omega u^{a\kappa} v^{(1-a)\kappa}(t)\varphi_1 \,\mathrm{d}x \geq C w^\kappa, \quad t \in (0, \infty), \tag{2.8}$$

where $C = C(\Omega, p, q, r, s, a)$ is independent of w. Since w is global and satisfies the inequality (2.8) for all $t > 0$, it holds

$$w(t) = \int_\Omega u^a v^{1-a}(t)\varphi_1 \,\mathrm{d}x \leq \left(\frac{\lambda_1}{C}\right)^{\frac{1}{\kappa-1}} \quad \text{for all } t \geq 0 \text{ and } a \in A. \tag{2.9}$$

Lemma 2.3 also implies

$$w_t(s') + \lambda_1 w(s') \geq C \int_\Omega u^{r+a-1} v^{p+1-a}(s')\varphi_1 \,\mathrm{d}x, \quad s' \in (0, \infty). \tag{2.10}$$

Multiplying inequality (2.10) by $e^{\lambda_1 s'}$, integrating on interval $[0, t]$ with respect to s' and using $0 \leq w \leq C$, we deduce that

$$\int_0^t e^{-\lambda_1(t-s')} \int_\Omega u^{r+a-1} v^{p+1-a}(s')\varphi_1 \,\mathrm{d}x \,\mathrm{d}s' \leq C. \tag{2.11}$$

Since there holds $e^{-\lambda_1(t-s')} \geq e^{-\lambda_1 t}$ for $s' \in [0, t]$, the ineqality (2.11) implies

$$\int_0^t \int_\Omega u^{r+a-1} v^{p+1-a}(s')\varphi_1 \,\mathrm{d}x \,\mathrm{d}s' \leq C e^{\lambda_1 t} \leq C', \tag{2.12}$$

where $C' = C'(\Omega, p, q, r, s, a, t)$.

Let (u, v) be a global nonnegative solution of system (1.5). Since (u, v) satisfies homogeneous Neumann boundary conditions, so does $u^a v^{1-a}$ and hence Green's formula implies $\int_\Omega \Delta(u^a v^{1-a}(t)) \,\mathrm{d}x = 0$ for $t \geq 0$ and $a \in A$. We obtain estimates similar to (2.9), (2.11), (2.12) with φ_1, λ_1 replaced by $1, \lambda$, respectively, in (2.9), (2.11), (2.12) if (2.7) is true.

3 Proofs of Theorems 1.1–1.3

In the following proofs, every constant may depend on Ω, p, q, r, s, however we do not denote this dependence. The constants may vary from step to step.

For $0 < p < \frac{2}{N+1}$, $r \leq 1$ denote

$$\begin{aligned}
&\widehat{K} : \left[1, \frac{p+1}{p}\right) \longrightarrow \mathbb{R} \cup \{\infty\}, \\
&\widehat{K}(M) = \begin{cases} \frac{M(p+1)(N+1)}{(p+1)(N+1)-2M}, & M \in \left[1, \frac{(p+1)(N+1)}{2}\right), \\ \infty, & M \in \left[\frac{(p+1)(N+1)}{2}, \frac{p+1}{p}\right), \end{cases} \\
&\widehat{k} : \left[1, \frac{p+1}{p}\right) \longrightarrow \mathbb{R}, \\
&\widehat{k}(M) = \frac{M(p+r)}{M - (M-1)(p+1)}.
\end{aligned} \tag{3.1}$$

For $r>1,\ p+r<\frac{N+3}{N+1}$ denote

$$\begin{aligned}
K' &: \left[1, \frac{p+r}{p+r-1}\right) \longrightarrow \mathbb{R}\cup\{\infty\},\\
K'(M) &= \begin{cases} \frac{M(p+r)(N+1)}{(p+r)(N+1)-2M}, & M\in\left[1,\frac{(p+r)(N+1)}{2}\right),\\ \infty, & M\in\left[\frac{(p+r)(N+1)}{2},\frac{p+r}{p+r-1}\right),\end{cases}\\
k' &: \left[1, \frac{p+r}{p+r-1}\right) \longrightarrow \mathbb{R},\\
k'(M) &= \frac{M(p+r)}{M-(M-1)(p+r)}.
\end{aligned} \tag{3.2}$$

Observe that

$$\widehat{K}(M) > \max\{M,\widehat{k}(M)\} \quad \text{for all} \quad M\in\left[1,\frac{p+1}{p}\right), \tag{3.3}$$

since $p<\frac{2}{N+1}$ and

$$K'(M)>k'(M)>M \quad \text{for all} \quad M\in\left[1,\frac{p+r}{p+r-1}\right), \tag{3.4}$$

since $p+r<\frac{N+3}{N+1}$.

Lemma 3.1. *Let $p+r<\frac{N+3}{N+1}$, $p>0$ and conditions (2.4), (2.7) be true. Let (u,v) be a global nonnegative solution of problem (1.1).*

(i) Assume $r>1$. Then for $\gamma\in[p+r,\infty]$ and $T\geq 0$, there exists $C=C(p,q,r,s,\Omega,T)$ such that

$$\sup_{s'\in[0,T]}\|u(s')\|_{\gamma,\delta}\leq C\|u_0\|_{\gamma,\delta}.$$

(ii) Assume $r>1,\ pq>(r-1)(s-1)$ or $r\leq 1,\ p<\frac{2}{N+1}$. Then for $\gamma\in\left[\max\{1,p+r\},\frac{N+3}{N+1}\right)$, there exists $C=C(p,q,r,s,\Omega)$ such that

$$\sup_{s'\in[0,T]}\|u(s')\|_{\gamma,\delta}\leq C(1+\|u_0\|_{\gamma,\delta}),\quad T\geq 0.$$

(iii) Assume $r\leq 1,\ p<\frac{2}{N+1}$. Then for $\gamma\in[\max\{1,p+r\},\infty]$ and $T\geq 0$, there exists $C=C(p,q,r,s,\Omega,T)$ such that

$$\sup_{s'\in[0,T]}\|u(s')\|_{\gamma,\delta}\leq C(1+\|u_0\|_{\gamma,\delta}).$$

Remark 3.2. In the assertion (i) of Lemma 3.1, the constant C is bounded for T bounded.

Proof. Let $\gamma\in\left[\max\{1,p+r\},\frac{N+3}{N+1}\right)$, $a\in A$ and $\varepsilon\in\left(0,1-\frac{p}{p+1-a}\right)$. Denote $\kappa:=\frac{(p+r)(1-a)}{p+1-a}$. For $T\geq 0, t\in[0,T]$ we estimate

$$\begin{aligned}
\|u(t)\|_{\gamma,\delta} &\leq C\left[\|u_0\|_{\gamma,\delta}+\int_0^t e^{-\lambda_1\left(\frac{p}{p+1-a}+\varepsilon\right)(t-s')}(t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)}\|u^r v^p(s')\|_{1,\delta}\,\mathrm{d}s'\right]\\
&\leq C\left[\|u_0\|_{\gamma,\delta}+\int_0^t\int_\Omega\left[e^{-\lambda_1\left(\frac{p}{p+1-a}\right)(t-s')}u^{r-\kappa}v^p(s')\right]\left[f u^\kappa(s')\varphi_1\,\mathrm{d}x\,\mathrm{d}s'\right]\right]
\end{aligned}$$

where $f = f(s') := e^{-\lambda_1 \varepsilon (t-s')}(t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)}$. Now, using Hölder's inequality we obtain

$$\|u(t)\|_{\gamma,\delta} \le C\left[\|u_0\|_{\gamma,\delta} + \left(\int_0^t g\,\mathrm{d}s'\right)^{\frac{p}{p+1-a}}\left(\int_0^t f^{\frac{p+1-a}{1-a}}\|u^{p+r}(s')\|_{1,\delta}\,\mathrm{d}s'\right)^{\frac{1-a}{p+1-a}}\right],$$

where $g = g(s') := e^{-\lambda_1(t-s')}\|u^{r+a-1}v^{p+1-a}(s')\|_{1,\delta}$. We use (2.11) to estimate

$$\|u(t)\|_{\gamma,\delta} \le C\left[\|u_0\|_{\gamma,\delta} + I^{\frac{1-a}{p+1-a}}\left(\sup_{s'\in[0,T]}\|u(s')\|_{\gamma,\delta}\right)^{\kappa}\right], \tag{3.5}$$

where $I = I(t) := \int_0^t e^{-\lambda_1 \varepsilon \frac{p+1-a}{1-a}(t-s')}(t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)\frac{p+1-a}{1-a}}\,\mathrm{d}s'$.

We prove that the function I is finite in $[0,\infty)$, i.e. due to our assumptions on p,q,r,s, there holds

$$\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)\frac{p+1-a}{1-a} < 1 \tag{3.6}$$

for some $a \in A$.

In fact, in the following proof we will choose

$$a = \frac{r-1}{p+r-1} \qquad \text{in case (i),} \tag{3.7}$$

$$a > \frac{r-1}{p+r-1} \quad \text{sufficiently close to} \quad \frac{r-1}{p+r-1} \qquad \text{in case (ii) for } r>1, \tag{3.8}$$

$$a > 0 \quad \text{sufficiently small} \qquad \text{in case (iii) or (ii) for } r \le 1. \tag{3.9}$$

The choice (3.8) is possible, since due to the assumptions $pq > (r-1)(s-1)$ and $p > 0$, we have $a \in A$. If a is defined by (3.7) or (3.8) then $\frac{p+1-a}{1-a}$ is close to $p+r$ and condition $p+r < \frac{N+3}{N+1}$ implies the inequality (3.6). If a is defined by (3.9), then $\frac{p+1-a}{1-a}$ is close to $p+1$ and condition $p < \frac{2}{N+1}$ implies the inequality (3.6). Note that the function I is bounded by a constant independent of T.

First we prove (ii). In the estimate (3.5) we choose a defined by (3.8), if $r > 1$, or by (3.9), if $r \le 1$. In both cases we have $\kappa < 1$, hence the assertion (ii) follows from Young's inequality.

Assertion (iii) for $\gamma \in \left[\max\{1, p+r\}, \frac{N+3}{N+1}\right)$ follows from assertion (ii).

To prove (i) for $\gamma \in \left[p+r, \frac{N+3}{N+1}\right)$ we choose a defined by (3.7) in estimate (3.5). Then $\kappa = 1$ and the assertion (i) for $\gamma \in \left[p+r, \frac{N+3}{N+1}\right)$ and T small enough follows from the estimate (3.5). The assertion (i) for $\gamma \in \left[p+r, \frac{N+3}{N+1}\right)$ actually holds for every $T \ge 0$.

Now we prove the assertion (i) for $\gamma \in \left[\frac{N+3}{N+1}, \infty\right]$. Fix $K \in \left[\frac{N+3}{N+1}, \infty\right)$. Then there exists $M \in \left[1, \frac{(p+r)(N+1)}{2}\right]$ such that $K'(M) > K > k = k'(M)$ (where functions K', k' are defined by (3.2)). For $t \in [0,T]$ and a defined by (3.7) we estimate

$$\|u(t)\|_{K,\delta} \le C\left[\|u_0\|_{K,\delta} + \int_0^t (t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)}\|u^r v^p(s')\|_{M,\delta}\,\mathrm{d}s'\right]. \tag{3.10}$$

Observe that $M < \frac{p+1-a}{p}$, since $M \le \frac{(p+r)(N+1)}{2} < \frac{p+r}{p+r-1}$ (the last inequality is true due to the assumption $p+r < \frac{N+3}{N+1}$). Hence Hölder's inequality yields

$$\begin{aligned}\|u^r v^p(s')\|_{M,\delta} &= \left(\int_\Omega \left[u^{pM\frac{r+a-1}{p+1-a}}v^{pM}(s')\right]\left[u^{M\kappa}(s')\right]\varphi_1\,\mathrm{d}x\right)^{\frac{1}{M}} \\ &\le \left(\int_\Omega u^{r+a-1}v^{p+1-a}(s')\varphi_1\,\mathrm{d}x\right)^{\frac{p}{p+1-a}}\left(\int_\Omega u^k(s')\varphi_1\,\mathrm{d}x\right)^{\frac{p+1-a-pM}{M(p+1-a)}},\end{aligned} \tag{3.11}$$

since $k = M\frac{(p+r)(1-a)}{p+1-a-pM}$ due to our choice of a. We use (3.11) and Hölder's inequality to obtain

$$\|u(t)\|_{K,\delta} \leq C\left[\|u_0\|_{K,\delta} + \left(\sup_{s'\in[0,T]} \|u(s')\|_{k,\delta}\right)^{\kappa} \left(\int_0^t \int_\Omega u^{r+a-1}v^{p+1-a}(s')\varphi_1 \,\mathrm{d}x \,\mathrm{d}s'\right)^{\frac{p}{p+1-a}} J\right],$$

where

$$J = J(t) := \left(\int_0^t (t-s')^{-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)\frac{p+1-a}{1-a}} \,\mathrm{d}s'\right)^{\frac{1-a}{p+1-a}}.$$

Notice that κ is equal to 1. Observe that I_0 is finite on $[0,\infty)$, since $\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)\frac{p+1-a}{1-a} < 1$. This follows from the definition (3.2) of function K' and our choice of K. Since $k < K$, we can use (2.12) to obtain

$$\|u(t)\|_{K,\delta} \leq C\left[\|u_0\|_{K,\delta} + C(T)\sup_{s'\in[0,T]} \|u(s')\|_{K,\delta}\right].$$

This estimate implies the assertion (i) with $\gamma \in \left[\frac{N+3}{N+1},\infty\right)$ for T small, hence this assertion is actually true for every $T \geq 0$.

If $M \in \left(\frac{(p+r)(N+1)}{2}, \frac{p+r}{p+r-1}\right)$, then we can choose $K = \infty$ and $k \in (k'(M),\infty)$.

The proof of the assertion (iii) for $\gamma \in \left[\frac{N+3}{N+1},\infty\right]$ is similar to the proof of the assertion (i) for $\gamma \in \left[\frac{N+3}{N+1},\infty\right]$. One would use (3.1), (3.3) instead of (3.2), (3.4). □

Lemma 3.3. *Let $p+r < \frac{N+3}{N+1}$, $p > 0$ and conditions (2.4), (2.7) be true. Let (u,v) be a global nonnegative solution of problem (1.1).*

(i) Assume $r > 1$. Then for $\gamma \in \left(1, \frac{1}{2-(p+r)}\right]$ and $T \geq 0$, there exists $C = C(p,q,r,s,\Omega,T)$ such that

$$\int_0^T \|u(s')\|_{\gamma,\delta} \,\mathrm{d}s' \leq C\|u(T)\|_{1,\delta}.$$

(ii) Assume $r \leq 1$, $p+r > 1$. Then for $\gamma \in \left(1, \frac{1}{2-(p+r)}\right]$ and $T \geq 0$, there exists $C = C(p,q,r,s,\Omega,T)$ such that

$$\int_0^T \|u(s')\|_{\gamma,\delta} \,\mathrm{d}s' \leq C(1+\|u(T)\|_{1,\delta}).$$

If $p+r \leq 1$, then this estimate is true for $\gamma \in \left[1, \frac{N+1}{N-1}\right)$.

Proof. We define exponent $\gamma = \frac{1}{2-(p+r)}$. The conditions $1 < p+r < \frac{N+3}{N+1}$ imply $p+r < \gamma < \frac{N+1}{N-1}$. For $T \geq 0$ and $t \in (0,T]$ we estimate

$$\|u(t)\|_{\gamma,\delta} \leq C\left[t^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)}\|u_0\|_{1,\delta} + \int_0^t (t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)}\|u^r v^p(s')\|_{1,\delta} \,\mathrm{d}s'\right].$$

Integrating this estimate on interval $[0,T]$ with respect to t and using Fubini's theorem we obtain

$$\int_0^T \|u(t)\|_{\gamma,\delta} \,\mathrm{d}t \leq CT^{1-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)}\left[\|u_0\|_{1,\delta} + \int_0^T \|u^r v^p(s')\|_{1,\delta} \,\mathrm{d}s'\right]. \tag{3.12}$$

Note that $\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right) < 1$, since $\gamma < \frac{N+1}{N-1}$.

As in the proof of Lemma 3.1 we use (3.11) with $M = 1, k = p + r$ to obtain

$$\int_0^T \|u(t)\|_{\gamma,\delta}\,\mathrm{d}t \le C\left[\|u_0\|_{1,\delta} + C(T)\left(\int_0^T \|u(s')\|_{p+r,\delta}^{p+r}\,\mathrm{d}s'\right)^{\frac{1-a}{p+1-a}}\right]. \tag{3.13}$$

Notice that $\frac{\gamma(p+r-1)}{\gamma-1} = 1$. We use the interpolation inequality

$$\|u(s')\|_{p+r,\delta}^{p+r} \le \|u(s')\|_{1,\delta}^{\frac{\gamma-(p+r)}{\gamma-1}}\,\|u(s')\|_{\gamma,\delta}^{\frac{\gamma(p+r-1)}{\gamma-1}}, \quad s' \in [0,T]$$

and Young's inequality to deduce

$$\int_0^T \|u(t)\|_{\gamma,\delta}\,\mathrm{d}t \le C(T)\left[\|u_0\|_{1,\delta} + \left(\sup_{s'\in[0,T]} \|u(s')\|_{1,\delta}\right)^{\beta}\right],$$

where $\beta = \frac{\gamma-(p+r)}{\gamma-1}\frac{1-a}{p}$. Using this estimate we are ready to prove the assertions of the Lemma. First we prove the assertion (i). If $r > 1$, then we choose $a = \frac{r-1}{p+r-1}$ in the definition of β, hence $\beta = 1$. Finally, we use (2.3) to obtain the assertion (i).

To prove the assertion (ii) for $p + r > 1$, we choose arbitrary $a \in A$ in the definition of β, hence $\beta < 1$. One can use Young's inequality to obtain the assertion.

If $p + r \le 1$, then for $\gamma \in \left[1, \frac{N+1}{N-1}\right)$ we obtain estimate similar to (3.13) a then we use Young's inequality. The proof of Lemma 3.3 is complete. □

In Lemma 3.4 we will use the following notation. For $r > 1$ denote

$$\begin{gathered} K_0' : \left[1, \frac{p+r}{p+r-1}\right) \longrightarrow \mathbb{R} \cup \{\infty\}, \\ K_0'(M) = \begin{cases} \frac{M(N+1)}{(N+1)-2M}, & M \in \left[1, \frac{N+1}{2}\right), \\ \infty, & M \in \left[\frac{N+1}{2}, \frac{p+r}{p+r-1}\right). \end{cases} \end{gathered} \tag{3.14}$$

Lemma 3.4. *Let $p + r < \frac{N+3}{N+1}$, $p > 0$ and conditions (2.4), (2.7) be true. Let (u, v) be a global nonnegative solution of problem (1.1).*

(i) Assume $r > 1$. Then for $T \ge 0$, there exists $C = C(p, q, r, s, \Omega, T)$ such that

$$\int_0^T \|u(s')\|_{K,\delta}\,\mathrm{d}s' \le C\|u_0\|_{k,\delta}$$

for $K_0'(M) > K > k = k'(M)$, $M \in \left[1, \frac{N+1}{2}\right]$. If $M \in \left(\frac{N+1}{2}, \frac{p+r}{p+r-1}\right)$, then we can take $K = \infty$.

(ii) Assume $r \le 1$, $\frac{2}{N+1} > p$. Then for $T \ge 0$, there exists $C = C(p, q, r, s, \Omega, T)$ such that

$$\int_0^T \|u(s')\|_{K,\delta}\,\mathrm{d}s' \le C(1 + \|u_0\|_{\max\{M,k\},\delta})$$

for $K_0(M) > K > k > \widehat{k}(M)$, $k \ge 1$, $M \in \left[1, \frac{N+1}{2}\right]$. If $M \in \left(\frac{N+1}{2}, \frac{p+1}{p}\right)$, then we can take $K = \infty$.

Proof. We choose a as follows

$$a = \frac{r-1}{p+r-1} \qquad \text{for part (i),} \tag{3.15}$$

$$a > 0 \text{ sufficiently close to } 0 \qquad \text{for part (ii).} \tag{3.16}$$

We only prove (i), since the proof of (ii) is similar. Observe that $\frac{N+1}{2} < \frac{p+r}{p+r-1}$ and $K_0'(M) > K'(M)$ for every $M \in \left[1, \frac{p+r}{p+r-1}\right)$ due to conditions $1 < p+r < \frac{N+3}{N+1}$ (see the definition (3.2) of functions K', k' and the definition (3.14) of K_0'). Hence (3.4) implies

$$K_0'(M) > k'(M) > M \quad \text{for all} \quad M \in \left[1, \frac{p+r}{p+r-1}\right). \tag{3.17}$$

Let $K_0'(M) > K > k = k'(M)$, $M \in \left[1, \frac{N+1}{2}\right]$, $T \geq 0$ and $t \in (0, T]$. Then, there holds $\frac{N+1}{2}\left(\frac{1}{M} - \frac{1}{K}\right) < 1$. As in the proof of Lemma 3.3 we obtain

$$\int_0^T \|u(t)\|_{K,\delta}\,\mathrm{d}t \leq C T^{1-\frac{N+1}{2}\left(\frac{1}{M}-\frac{1}{K}\right)} \left[\|u_0\|_{M,\delta} + \int_0^T \|u^r v^p(s')\|_{M,\delta}\,\mathrm{d}s'\right].$$

Using Lemma 3.1 (i), (3.11) and similar arguments as in the proof of Lemma 3.1 (with a is defined by (3.15)) we have

$$\int_0^T \|u(t)\|_{K,\delta}\,\mathrm{d}t \leq C(T)\left(\|u_0\|_{M,\delta} + \|u_0\|_{k,\delta}\right) \leq C(T)\|u_0\|_{k,\delta}, \tag{3.18}$$

since $k > M$.

If $M \in \left(\frac{N+1}{2}, \frac{p+r}{p+r-1}\right)$, then in previous estimates, we can choose $K = \infty$ and $k'(M) < k < \infty$. Hence we proved (i). $\square$

Lemma 3.5. *Let $p + r < \frac{N+3}{N+1}$, $p > 0$ and conditions (2.4), (2.7) be true. Let (u, v) be a global nonnegative solution of problem (1.1).*

(i) Assume $r > 1$. Then for every $\tau > 0$, there exists $C = C(p, q, r, s, \Omega, \tau)$ such that

$$\|u(t)\|_\infty \leq C\|u(t)\|_{1,\delta}$$

for every $t \geq \tau$.

(ii) Assume $r \leq 1, p < \frac{2}{N+1}$. Then for every $\tau > 0$, there exists $C = C(p, q, r, s, \Omega, \tau)$ such that

$$\|u(t)\|_\infty \leq C(1 + \|u(t)\|_{1,\delta})$$

for every $t \geq \tau$.

Remark 3.6. The constant C from both assertions of Lemma 3.5 may explode if $\tau \to 0^+$.

Proof. We prove only (i). Let $\gamma = \frac{1}{2-(p+r)}$. Conditions $1 < p+r < \frac{N+3}{N+1}$ imply $p+r < \gamma < \frac{N+1}{N-1}$. Fix $1 > \tau_0 > 0$ and let $t > 0$ be arbitrary. Note that there exists $\tau' \in [\tau_0 + t, 2\tau_0 + t]$ such that $\|u(\tau')\|_{\gamma,\delta} = \tau_0^{-1}\int_{\tau_0+t}^{2\tau_0+t} \|u(s')\|_{\gamma,\delta}\,\mathrm{d}s'$. Obviously, this τ' may depend on t and u. Note that $2\tau_0 + t \in [\tau', \tau' + \tau_0]$. We use Lemma 3.3 (i) and Lemma 3.1 (i) to obtain

$$\sup_{s' \in [\tau', \tau'+\tau_0]} \|u(s')\|_{\gamma,\delta} \leq C\|u(\tau')\|_{\gamma,\delta} \leq C\int_{\tau_0+t}^{2\tau_0+t} \|u(s')\|_{\gamma,\delta}\,\mathrm{d}s' \leq C\|u(2\tau_0 + t)\|_{1,\delta}$$

where $C = C(\tau_0)$ is independent of τ'. Finally, above estimates imply

$$\|u(2\tau_0 + t)\|_{\gamma,\delta} \le C(\tau_0)\|u(2\tau_0 + t)\|_{1,\delta}.$$

Now fix $l \in \mathbb{N}, l > 1$ and K, k such that $K_0'(M) > K > k = k'(M)$, $M \in \left(1, \frac{N+1}{2}\right]$ (see the definition (3.2), (3.14) of function k', K_0', respectively). This choice is possible due to inequality (3.17). As in the previous part of the proof we obtain

$$\|u((l+1)\tau_0 + t)\|_{K,\delta} \le C(\tau_0)\|u(l\tau_0 + t)\|_{k,\delta}. \tag{3.19}$$

If we choose $\frac{p+r}{p+r-1} > M_0 > \frac{N+1}{2}$, then in (3.19), we can take $K = \infty$ and some $\infty > k > k'(M_0)$. Now we apply bootstrap argument on (3.19) to finish the proof. □

Corollary 3.7. *Assume $p + r < \frac{N+3}{N+1}, \frac{2}{N+1} > p > 0$ and let conditions (2.4), (2.7) be true. If $r > 1$, then assume also $pq > (1-r)(1-s)$. Let (u, v) be a global nonnegative solution of problem (1.1). Then there exists $C = C(p, q, r, s, \Omega)$ such that*

$$\sup_{s' \in [0,\infty)} \|u(s')\|_\infty \le C(1 + \|u_0\|_\infty). \tag{3.20}$$

Proof. This follows from Lemma 3.5 (i), Lemma 3.1 (i) (if $r > 1$) or Lemma 3.5 (ii), Lemma 3.1 (iii) (if $r \le 1$) and Lemma 3.1 (ii). □

Lemma 3.8. *Assume $p + r < \frac{N+3}{N+1}, \frac{2}{N+1} > p > 0$, $s < \frac{N+3}{N+1}$, (2.4) and (2.7). If $r > 1$, then assume also $pq > (1-r)(1-s)$. Let (u, v) be a global nonnegative solution of problem (1.1). Then for $T \ge 0$, there exists $C = C(p, q, r, s, \Omega, \|u_0\|_\infty, \|v_0\|_\infty, \sup_{s' \in [0,T]} \|v(s')\|_{1,\delta})$ such that*

$$\sup_{s' \in [0,T]} \|v(s')\|_\infty \le C, \quad T \ge 0.$$

Proof. Due to Corollary 3.7 we can write $u(x,t) \le C(\|u_0\|_\infty)$ for $(x,t) \in \Omega \times [0,\infty)$ (note that the constant C in (3.20) is independent of T). Then v satisfies

$$v_t - \Delta v \le C(\|u_0\|_\infty)^q v^s, \quad (x,t) \in \Omega \times [0,\infty),$$

where $s < \frac{N+3}{N+1}$.

Assume $s > 1$. We choose arbitrary γ such that $\frac{1}{2-s} < \gamma < \frac{N+1}{N-1}$. Note that $\frac{1}{2-s} < \frac{N+1}{N-1}$, since $s < \frac{N+3}{N+1}$. For fixed $T > 0$ and $t \in [0,T]$ we estimate

$$\begin{aligned} \|v(t)\|_{\gamma,\delta} &\le C(\|u_0\|_\infty)\left[\|v_0\|_{\gamma,\delta} + \int_0^t e^{-\frac{\lambda_1}{2}(t-s')}(t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)} \int_\Omega v^s(s')\varphi_1 \, dx \, ds'\right] \\ &\le C(\|u_0\|_\infty, \|v_0\|_\infty)\left[1 + \sup_{s' \in [0,T]} \|v(s')\|_{s,\delta}^s\right]. \end{aligned} \tag{3.21}$$

As in the proof of Lemma 3.3 we use the interpolation inequality and Young's inequality to obtain

$$\|v(s')\|_{s,\delta}^s \le \|v(s')\|_{1,\delta}^{\frac{\gamma-s}{\gamma-1}} \|v(s')\|_{\gamma,\delta}^{\frac{\gamma(s-1)}{\gamma-1}} \le C_\varepsilon \|v(s')\|_{1,\delta}^{\frac{\gamma-s}{(\gamma-1)(1-\theta)}} + \varepsilon\|v(s')\|_{\gamma,\delta}^{\frac{\gamma(s-1)}{(\gamma-1)\theta}},$$

where $\theta \in (0,1)$. Due to our choice of γ there holds $\frac{\gamma(s-1)}{\gamma-1} < 1$, hence there exists $\theta \in (0,1)$ such that $\frac{\gamma(s-1)}{(\gamma-1)\theta} = 1$. Choosing ε sufficiently small yields

$$\sup_{s' \in [0,T]} \|v(s')\|_{\gamma,\delta} \le C(\|u_0\|_\infty, \|v_0\|_\infty, \sup_{s' \in [0,T]} \|v(s')\|_{1,\delta}), \quad \gamma \in \left(1, \frac{N+1}{N-1}\right). \tag{3.22}$$

For $0 \le s \le 1$ the assertion (3.22) follows from estimates analogous to (3.21). To finish the proof (i.e. to prove (3.22) for $\gamma = \infty$) we now use obvious bootstrap argument. $\square$

Proof of Theorem 1.1. We only deal with case $s \ge 1$, since case $s < 1$ can be done easily by using Corollary 3.7 and Young's inequality. For $\eta \in [0, 1-a)$ denote $\gamma(\eta) := \frac{s-a}{1-\eta-a}$, $\varepsilon(\eta) := \frac{(q+a)(s-1+\eta)}{s-a}$. The assumption $s \ge 1$ guarantees that $\varepsilon(\eta) > 0$ for all $\eta \in (0, 1-a)$.

In the following proof we will choose

$$a = \frac{r-1}{p+r-1} \qquad \text{in case } r > 1, \tag{3.23}$$

$$a > 0 \quad \text{sufficiently small} \qquad \text{in case } r \le 1. \tag{3.24}$$

If a is defined by (3.23),, then the condition $pq > (r-1)(s-1)$ implies $\varepsilon(0) < q$ and the condition $s + \frac{2}{N+1}\frac{r-1}{p+r-1} < \frac{N+3}{N+1}$ implies $\gamma(0) < \frac{N+3}{N+1}$. Hence

$$1 < \gamma(\eta) < \frac{N+3}{N+1}, \quad \varepsilon(\eta) < q \tag{3.25}$$

for $\eta > 0$ sufficiently small. If a is chosen by (3.24), then there holds (3.25) for small $\eta > 0$. The choice (3.24) of a may vary from step to step.

Now we choose η such that in the both cases (3.23) and (3.24) there holds (3.25) and for the rest of the proof denote $\gamma' := \gamma(\eta)$, $\varepsilon' := \varepsilon(\eta)$. For $t \in [0, T]$ and $\varepsilon \in \left(0, \frac{1}{\gamma'}\right)$ we estimate

$$\|v(t)\|_{\gamma',\delta} \le C\left[\|v_0\|_{\gamma',\delta} + \int_0^t f \int_\Omega u^{\varepsilon'} v^s u^{q-\varepsilon'}(s')\varphi_1 \,\mathrm{d}x\,\mathrm{d}s'\right],$$

where $f = f(s') := e^{-\lambda_1\left(1-\frac{1}{\gamma'}+\varepsilon\right)(t-s')}(t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma'}\right)}$. The term $u^{q-\varepsilon'}$ can be estimated by a constant depending on $\|u_0\|_\infty$ due to Corollary 3.7. Since $s \ge 1$ and $\eta \in (0, 1-a)$, we have $0 < \frac{s-1+\eta}{s-a} < 1$. Thus the following Hölder's inequality is true

$$\int_\Omega [u^{\varepsilon'} v^{s-1+\eta}(s')][v^{1-\eta}(s')]\varphi_1 \,\mathrm{d}x \le \left(\int_\Omega u^{q+a} v^{s-a}(s')\varphi_1 \,\mathrm{d}x\right)^{1-\frac{1}{\gamma'}} \left(\int_\Omega v^{(1-\eta)\gamma'}(s')\varphi_1 \,\mathrm{d}x\right)^{\frac{1}{\gamma'}}.$$

This estimate and Hölder's inequality then imply the boundedness of $\sup_{s' \in [0,T]} \|v(s')\|_{\gamma',\delta}$ by constant depending on $\|u_0\|_\infty$, $\|v_0\|_\infty$. Hence the assertion follows with help of Corollary 3.7 and Lemma 3.8. $\square$

Lemma 3.9. *Let $p \ge 1$, $p + r < \frac{N+3}{N+1}$ and conditions (2.4), (2.7) be true. Let (u, v) be a global nonnegative solution of problem (1.1). Moreover assume*

$$(p+r)\left(p - \frac{2}{N+1}\right) + r < 1. \tag{3.26}$$

Then for $\gamma \in \left(p + r, \frac{1-r}{p-\frac{2}{N+1}}\right)$, there exists $C = C(p, q, r, s, \Omega)$ such that

$$\|u(t)\|_{\gamma,\delta} \le C\left(1 + \|u_0\|_{\gamma,\delta}\right) \quad \textit{for } t \ge 0.$$

Proof. We choose

$$\gamma \in \left(p + r, \frac{1-r}{p - \frac{2}{N+1}} \right). \tag{3.27}$$

This choice is possible due to the assumption (3.26). Let $a \in A$. We introduce the following exponents $\alpha_1, \alpha_2, \alpha_3$ satisfying conditions

$$\alpha_1, \alpha_2, \alpha_3 > 0, \quad \alpha_1 + \alpha_2 + \alpha_3 = 1, \tag{3.28}$$
$$(1-a)\alpha_1 + (p+1-a)\alpha_2 = p, \tag{3.29}$$
$$\frac{r+\gamma(p-1)}{\gamma p - (1-r)} < \alpha_2 < \frac{p}{p+1}, \quad \alpha_2 \text{ sufficiently close to } \frac{r+\gamma(p-1)}{\gamma p - (1-r)}. \tag{3.30}$$

The condition $\gamma > p + r$ implies $\frac{r+\gamma(p-1)}{\gamma p-(1-r)} < \frac{p}{p+1}$. It is easy to check that there exist $\alpha_1, \alpha_2, \alpha_3$ such that the conditions (3.28)–(3.30) are true, if a is sufficiently small.

We define exponent

$$\kappa = r - a\alpha_1 - (r + a - 1)\alpha_2.$$

For a small, there holds $0 < \kappa < 1$. For $\varepsilon \in (0, 1-\alpha_2)$, $T \geq 0, t \in [0, T]$ we estimate

$$\|u(t)\|_{\gamma,\delta} \leq C \left[\|u_0\|_{\gamma,\delta} + \int_0^t e^{-\lambda_1(\alpha_2+\varepsilon)(t-s')} (t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)} \|u^r v^p(s')\|_{1,\delta} \, ds' \right].$$

The equality (3.29), Hölder's inequality and the estimate (2.9) imply

$$\begin{aligned} \|u^r v^p(s')\|_{1,\delta} &= \int_\Omega \left[u^a v^{1-a} \right]^{\alpha_1} \left[u^{r+a-1} v^{p+1-a}(s') \right]^{\alpha_2} \left[u^{\frac{\kappa}{\alpha_3}}(s') \right]^{\alpha_3} \varphi_1 \, dx \\ &\leq C \left(\int_\Omega u^{r+a-1} v^{p+1-a}(s') \varphi_1 \, dx \right)^{\alpha_2} \left(\int_\Omega u^{\frac{\kappa}{\alpha_3}}(s') \varphi_1 \, dx \right)^{\alpha_3}. \end{aligned}$$

Now we prove

$$\frac{\kappa}{\alpha_3} < \gamma \tag{3.31}$$

for a small. Due to the equality (3.29), $\alpha_1 + (p+1)\alpha_2$ is close to p and so $\gamma(1 + p\alpha_2 - p)$ is close to $\gamma(1 - \alpha_1 - \alpha_2) = \gamma\alpha_3$. The condition $\alpha_2 > \frac{r+\gamma(p-1)}{\gamma p-(1-r)}$ implies $r + \alpha_2(1-r) < \gamma(1 + p\alpha_2 - p)$, hence $\frac{r+\alpha_2(1-r)}{\alpha_3} < \gamma$. Thus we proved (3.31) for a small.

The inequality (3.31), Hölder's inequality and (2.11) then yield

$$\|u(t)\|_{\gamma,\delta} \leq C \left[\|u_0\|_{\gamma,\delta} + \sup_{s' \in [0,T]} \|u(s')\|^{\kappa}_{\gamma,\delta} \left(\int_0^t e^{-\frac{\lambda_1 \varepsilon}{1-\alpha_2}(t-s')} (t-s')^{-\frac{N+1}{2}\left(1-\frac{1}{\gamma}\right)\frac{1}{1-\alpha_2}} \, ds' \right)^{1-\alpha_2} \right].$$

We prove that for some α_2 close to $\frac{r+\gamma(p-1)}{\gamma p-(1-r)}$, there holds

$$\frac{N+1}{2} \left(1 - \frac{1}{\gamma} \right) \frac{1}{1-\alpha_2} < 1. \tag{3.32}$$

For α_2 sufficiently close to $\frac{r+\gamma(p-1)}{\gamma p-(1-r)}$, is $\left(1 - \frac{1}{\gamma}\right)\frac{1}{1-\alpha_2}$ close to $\frac{\gamma p-(1-r)}{\gamma}$. Our choice of γ implies $\frac{\gamma p-(1-r)}{\gamma} < \frac{2}{N+1}$, hence the inequality (3.32) is true. Then Young's inequality concludes the proof. □

Lemma 3.10. *Let $p \geq 1$, $p + r < \frac{N+3}{N+1}$ and conditions (2.4), (2.7) be true. Let (u, v) be a global nonnegative solution of problem (1.1). Then there holds*

$$\int_0^T \|u(t)\|_{\gamma',\delta} \,\mathrm{d}t \leq C\,(1 + \|u(T)\|_{1,\delta}) \quad \text{for } \gamma' \in \left[1, \frac{N+1}{N-1}\right),$$

where $C = C(p, q, r, s, \Omega, T)$.

Proof. We choose $\frac{1}{2-(p+r)} < \gamma' < \frac{N+1}{N-1}$. We introduce the following exponents $\alpha_1, \alpha_2, \alpha_3$ satisfying conditions (3.28), (3.29) with $a \in A$ and

$$\frac{r + \gamma'(p-1)}{\gamma' p - (1-r)} < \alpha_2 < \frac{1-r}{2-r} \leq \frac{p}{p+1}. \tag{3.33}$$

Note that the condition $\gamma' > \frac{1}{2-(p+r)}$ implies $\frac{r+\gamma'(p-1)}{\gamma' p-(1-r)} < \frac{1-r}{2-r}$ and $p + r \geq 1$ implies $\frac{1-r}{2-r} \leq \frac{p}{p+1}$. Observe that there exist $\alpha_1, \alpha_2, \alpha_3$ such that the conditions (3.28), (3.29) and (3.33) are true, if a is small.

Let $a \in A$ and $\kappa = r - a\alpha_1 - (r + a - 1)\alpha_2$. Due to the condition (3.28) and equality $u^r v^p(s') = \left[u^a v^{1-a}(s')\right]^{\alpha_1} \left[u^{r+a-1} v^{p+1-a}(s')\right]^{\alpha_2} \left[u^{\frac{\kappa}{\alpha_3}}(s')\right]^{\alpha_3}$, estimate analogous to (3.12) (with γ replaced by γ') Hölder's inequality, (2.9) and (2.12) yield

$$\begin{aligned}
\int_0^T \|u(t)\|_{\gamma',\delta} \,\mathrm{d}t &\leq C \left[\|u_0\|_{1,\delta} + \int_0^T \left[\int_\Omega f \varphi_1 \,\mathrm{d}x\right]^{\alpha_2} \left[\int_\Omega u^{\frac{\kappa}{\alpha_3}}(s')\varphi_1 \,\mathrm{d}x\right]^{\alpha_3} \mathrm{d}s'\right] \\
&\leq C \left[\|u_0\|_{1,\delta} + \left(\int_0^T \|u(s')\|^{\frac{\kappa}{1-\alpha_2}}_{\frac{\kappa}{\alpha_3},\delta} \,\mathrm{d}s'\right)^{1-\alpha_2}\right],
\end{aligned}$$

where $f = f(s') := u^{r+a-1} v^{p+1-a}(s')$. Observe that the inequality $\alpha_2 > \frac{r+\gamma'(p-1)}{\gamma' p-(1-r)}$ implies $\frac{\kappa}{\alpha_3} < \gamma'$ (cf. the proof of inequality (3.31)) and $\alpha_2 < \frac{1-r}{2-r}$ implies $\frac{\kappa}{1-\alpha_2} < 1$. Hence Jensen's and Young's inequalities imply the assertion of the Lemma. □

Lemma 3.11. *Let $p \geq 1$, $p + r < \frac{N+3}{N+1}$, $s \leq 1$ and condition (2.7) be true. Let (u, v) be a global nonnegative solution of problem (1.1). Moreover assume (3.26) and*

$$0 < q < \frac{1-r}{p - \frac{2}{N+1}} \left(1 - \frac{N-1}{N+1} s\right) \tag{3.34}$$

Then for $\tau > 0$, there exists $C = C(p, q, r, s, \Omega, \tau, \|u(\tau)\|_{1,\delta}, \|v(\tau)\|_{1,\delta})$ such that

$$\|u(t)\|_{k,\delta} + \|v(t)\|_{k,\delta} \leq C \quad \text{for } k \in \left[1, \frac{N+1}{N-1}\right),\ t \geq \tau.$$

Remark 3.12. The constant C from Lemma 3.11 may explode if $\tau \to 0^+$, and is bounded for $\|u(\tau)\|_{1,\delta}$, $\|v(\tau)\|_{1,\delta}$ bounded.

Proof. We use Lemmas 3.9 and 3.10 and arguments as in the proof of Lemma 3.5 to obtain

$$\sup_{s' \in [\tau, \tau+T]} \|u(s')\|_{\gamma,\delta} \leq C(\tau)\,(1 + \|u(\tau)\|_{1,\delta}) \leq C_0 \tag{3.35}$$

for $\tau \in (0, 1)$, $T \geq 0$, $1 \leq \gamma < \frac{1-r}{p-\frac{2}{N+1}}$ and $C_0 = C_0(\tau, \|u(\tau)\|_{1,\delta})$. C_0 may vary from step to step, but always depends on parameters in brackets. The constant $C(\tau)$ in (3.35) may explode if $\tau \to 0^+$. We prove the following assertion

$$\sup_{s' \in [\tau, \tau+T]} \|v(s')\|_{k,\delta} \leq C_0\,(1 + \|v(\tau)\|_{k,\delta}), \quad T \geq 0 \tag{3.36}$$

for $k < \frac{N+1}{N-1}$ close to $\frac{N+1}{N-1}$. For $\tau > 0, T \geq 0, t \in [0,T]$ we estimate

$$\|v(\tau+t)\|_{k,\delta} \leq C\left[\|v(\tau)\|_{k,\delta} + \int_{\tau}^{\tau+t} f \int_{\Omega} u^q v^s(s')\varphi_1 \,\mathrm{d}x\,\mathrm{d}s'\right], \tag{3.37}$$

where $f = f(s') := e^{-\frac{\lambda_1}{2}(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}(1-\frac{1}{k})}$. Assume $s \in (0,1)$ and let $s' \in [\tau, \tau+T]$. We use Hölder's inequality to obtain

$$\int_{\Omega} u^q v^s(s')\varphi_1 \,\mathrm{d}x \leq \left(\int_{\Omega} u^{\frac{q}{\theta}}(s')\varphi_1 \,\mathrm{d}x\right)^{\theta}\left(\int_{\Omega} v^{\frac{s}{1-\theta}}(s')\varphi_1 \,\mathrm{d}x\right)^{1-\theta}, \tag{3.38}$$

where $\theta = 1 - \frac{s}{k} \in \left(0, 1 - \frac{N-1}{N+1}s\right)$. Due to the assumption (3.34) for $k < \frac{N+1}{N-1}$ sufficiently close to $\frac{N+1}{N-1}$ there holds $q < \frac{(1-r)(1-\frac{s}{k})}{p-\frac{2}{N+1}}$, hence

$$\frac{q}{\theta} < \frac{1-r}{p-\frac{2}{N+1}}. \tag{3.39}$$

Thus there exists some $\gamma > \frac{q}{\theta}$ satisfying the condition (3.27) and we can use (3.35) and (3.38) to estimate

$$\|v(\tau+t)\|_{k,\delta} \leq C_0\left[\|v(\tau)\|_{k,\delta} + \sup_{s' \in [\tau,\tau+T]} \|u(s')\|^s_{k,\delta}\right].$$

Now we use Young's inequality and the assertion (3.36) follows.

If $s = 1$, then from (3.37) we deduce

$$\begin{aligned}\|v(\tau+t)\|_{k,\delta} \leq C\Bigg[\|v(\tau)\|_{k,\delta} &+ \int_{\tau}^{\tau+t} e^{-\frac{\lambda_1}{2}(\tau+t-s')}(\tau+t-s')^{-\frac{N+1}{2}(1-\frac{1}{k})} \times \\ &\times \int_{\Omega}\left[u^a v^{1-a}(s')\right]^{\varepsilon}\left[u^{\frac{q-a\varepsilon}{\theta}}(s')\right]^{\theta}\left[v^{\frac{1-(1-a)\varepsilon}{1-\varepsilon-\theta}}(s')\right]^{1-\varepsilon-\theta}\varphi_1 \,\mathrm{d}x\,\mathrm{d}s'\Bigg],\end{aligned}$$

where $a \in A$ and $0 < \varepsilon < \varepsilon' < 1$ for ε' such that $\theta := \theta(\varepsilon') = 1 - \varepsilon' - \frac{1-(1-a)\varepsilon'}{k} \in (0,1)$ (this is possible, if $k > 1$). Hence there holds $\frac{1-(1-a)\varepsilon}{1-\varepsilon-\theta} < k$. Note that $\frac{q-a\varepsilon}{\theta} > 0$ for $\varepsilon > 0$ small, since $q > 0$. Due to the assumption (3.34) for $k < \frac{N+1}{N-1}$ sufficiently close to $\frac{N+1}{N-1}$, there holds $q < \frac{(1-r)(1-\frac{1}{k})}{p-\frac{2}{N+1}}$, hence $\frac{q-a\varepsilon}{\theta(\varepsilon')} < \frac{1-r}{p-\frac{2}{N+1}}$ for ε' sufficiently small. Thus there exists some $\gamma > \frac{q-a\varepsilon}{\theta}$ satisfying the condition (3.27). We can use (2.9) and (3.35) and obvious arguments to finish the proof of (3.36) for $s = 1$.

The proof of the assertion (3.36) for $s = 0$ is obvious.

We prove

$$\int_{\tau}^{\tau+T} \|v(t)\|_{k,\delta}\,\mathrm{d}t \leq C(T, C_0)\left(1 + \|v(\tau+T)\|_{1,\delta}\right) \tag{3.40}$$

for $k \in \left[1, \frac{N+1}{N-1}\right)$. As in the proof of Lemma 3.3 we estimate

$$\int_{\tau}^{\tau+T} \|v(t)\|_{k,\delta}\,\mathrm{d}t \leq CT^{1-\frac{N+1}{2}(1-\frac{1}{k})}\left[\|v(\tau)\|_{1,\delta} + \int_{\tau}^{\tau+t}\int_{\Omega} u^q v^s(s')\varphi_1 \,\mathrm{d}x\,\mathrm{d}s'\right],$$

where $\tau > 0, T \geq 0, t \in (\tau, T+\tau]$. Let $s \in (0,1]$. We apply Hölder's inequality (3.38) with $\theta = 1 - \frac{s}{k}$. There holds (3.39) for $k < \frac{N+1}{N-1}$ close to $\frac{N+1}{N-1}$ and there exists $\gamma > \frac{q}{\theta}$ satisfying (3.27). Thus due to (3.35) and the definition of θ we have

$$\int_{\tau}^{\tau+T} \|v(t)\|_{k,\delta}\,\mathrm{d}t \leq C_0 T^{1-\frac{N+1}{2}(1-\frac{1}{k})}\left[\|v(\tau)\|_{1,\delta} + \int_{\tau}^{\tau+T} \|v(s')\|^s_{k,\delta}\,\mathrm{d}s'\right].$$

If $s \in (0,1)$, then Young's inequality and (2.3) yield the assertion (3.40). If $s = 1$, then (2.3) implies the assertion (3.40) for T sufficiently small. Assertion (3.40) is then true for every $T \geq 0$ fixed (with help of (2.3)).

If $s = 0$, then the proof of assertion (3.40) is obvious.

Combining (3.36) and (3.40) (using arguments as in the proof of Lemma 3.5) we obtain

$$\sup_{s' \in [\tau, \tau+T]} \|v(s')\|_{k,\delta} \leq C_0(1 + \|v(\tau)\|_{1,\delta}) \leq C_1, \tag{3.41}$$

where $k \in \left[1, \frac{N+1}{N-1}\right)$ and $C_1 = C_1(\tau, \|u(\tau)\|_{1,\delta}, \|v(\tau)\|_{1,\delta})$. Using arguments from previous part of the proof, one can prove estimates similar to (3.36), (3.40), (3.41) with v, C_0 replaced by u, C_1, respectively. Note that always $r < 1$ due to our assumptions on p, r. The constant C_1 then may vary in such estimates, but always depends on τ, $\|u(\tau)\|_{1,\delta}$ and $\|v(\tau)\|_{1,\delta}$. $\square$

Proof of Theorem 1.2. Denote $C_0 = C_0(\tau, \|u(\tau)\|_{1,\delta}, \|v(\tau)\|_{1,\delta})$. C_0 may vary from step to step, but always depends on parameters in brackets. Let $\tau > 0, T \geq 0$. Assume that there holds

$$\sup_{s' \in [\tau, \tau+T]} \|u(s')\|_{k,\delta} + \sup_{s' \in [\tau, \tau+T]} \|v(s')\|_{k,\delta} \leq C_0, \quad \text{for some } k \in \left(\frac{N+1}{N-1} - \varepsilon_0, \infty\right), \tag{3.42}$$

where $\varepsilon_0 > 0$ is sufficiently small. In Lemma 3.11 we proved (3.42) for $k \in \left[1, \frac{N+1}{N-1}\right)$. For the whole proof we choose $M = \frac{k}{p+r}$, $M' = \min\left\{k, \frac{k}{q+s}\right\}$. For k chosen in (3.42), it holds $M, M' > 1$, since $\max\{p+r, q+s\} < \frac{N+1}{N-1}$. This is true, since

$$q + s < \frac{(1-r)\left(1 - \frac{N-1}{N+1}s\right)}{p - \frac{2}{N+1}} + s \leq \frac{N+1}{N-1}\left(1 - \frac{N-1}{N+1}s\right) + s = \frac{N+1}{N-1}.$$

If $r > 0$, then we use Hölder's inequality and (3.42) to obtain

$$\left(\int_\Omega u^{Mr} v^{Mp}(s') \varphi_1 \, \mathrm{d}x\right)^{\frac{1}{M}} \leq \|u(s')\|_{k,\delta}^r \|v(s')\|_{k,\delta}^p \leq C_0.$$

for $s' \in [\tau, \tau+T]$. Let $r > 0$. Denote $\beta := \frac{N+1}{2}\left(\frac{1}{M} - \frac{1}{K}\right)$. For $1 < K \leq \infty$ satisfying $\beta < 1$ and $t \in [0, T]$ we estimate

$$\|u(\tau+t)\|_{K,\delta} \leq C\left[\|u(\tau)\|_{K,\delta} + \int_\tau^{\tau+t} f\left(\int_\Omega u^{Mr} v^{Mp}(s')\varphi_1 \, \mathrm{d}x\right)^{\frac{1}{M}} \mathrm{d}s'\right],$$

where $f = f(s') := e^{-\frac{\lambda_1}{2}(\tau+t-s')}(\tau+t-s')^{-\beta}$. In particular, we can take

$$K < k_1(M) := \begin{cases} \dfrac{N+1}{\frac{N+1}{M} - 2}, & M \in \left[1, \frac{N+1}{2}\right), \\ \infty, & M \geq \frac{N+1}{2} \end{cases}$$

if $M \leq \frac{N+1}{2}$ and $K = \infty$ for $M > \frac{N+1}{2}$. Hence we have

$$\sup_{s' \in [\tau, \tau+T]} \|u(s')\|_{K,\delta} \leq C_0(1 + \|u(\tau)\|_{K,\delta}).$$

For $k_1(M) > K > M$, $t \in (\tau, \tau+T]$ we estimate

$$\int_\tau^{\tau+T} \|u(t)\|_{K,\delta} \, \mathrm{d}t \leq C(T)\left[\|u(\tau)\|_{M,\delta} + \int_\tau^{\tau+T}\left(\int_\Omega u^{Mr} v^{Mp}(s')\varphi_1 \, \mathrm{d}x\right)^{\frac{1}{M}} \mathrm{d}t\right].$$

Hence we obtain

$$\int_{\tau}^{\tau+T} \|u(t)\|_{K,\delta}\,\mathrm{d}t \le C(T,C_0)\left(1+\|u(\tau)\|_{M,\delta}\right) \le C(T,C_0)\left(1+\|u(\tau)\|_{k,\delta}\right),$$

since $k > M$. Finally, we get $\sup_{s'\in[\tau,\tau+T]} \|u(s')\|_{K,\delta} \le C_0$ for $k_1(M) > K > M$. Similarly, we get $\sup_{s'\in[\tau,\tau+T]} \|v(s')\|_{K,\delta} \le C_0$ for $k_1(M') > K > M'$. From these estimates we deduce that

$$\sup_{s'\in[\tau,\tau+T]} \|u(s')\|_{K,\delta} + \sup_{s'\in[\tau,\tau+T]} \|v(s')\|_{K,\delta} \le C_0$$

for all $K < k_1\left(\frac{k}{\max\{p+r,q+s\}}\right) =: k_2(k)$. Note that $k_2(k) = \infty$ for $k \ge \frac{(\max\{p+r,q+s\})(N+1)}{2}$ and we can take $K = \infty$ for $k > \frac{(\max\{p+r,q+s\})(N+1)}{2}$. To finish the proof we use bootstrap argument. □

Corollary 3.13. *Let p,q,r,s be as in Theorem 1.2 with $r = s = 0$ and $p,q \ge 1$, $pq > 1$. Then for every $\tau > 0$, there exists $C = C(\Omega,p,q,\tau)$ such that*

$$\|u(t)\|_\infty + \|v(t)\|_\infty \le C, \quad t \ge \tau$$

for every global nonnegative solution (u,v) of problem (1.1). The constant C may explode if $\tau \to 0^+$.

Proof. It suffices to prove

$$\|u(t)\|_{1,\delta} + \|v(t)\|_{1,\delta} \le C, \quad t \ge 0, \tag{3.43}$$

since we can then use Theorem 1.2. Estimate (3.43) was for $p,q > 1$ proven in [6, Proposition 4.1]. Let $p = 1$ and $q > 1$. From (2.2) we obtain $\|u(1)\|_{1,\delta} \ge \int_0^1 e^{-\lambda_1(1-s)}\|v(s)\|_{1,\delta}\,\mathrm{d}s$. This inequality along with (2.3) imply

$$\|u(1)\|_{1,\delta} \ge e^{-\lambda_1}\|v_0\|_{1,\delta}. \tag{3.44}$$

On the other hand, Lemma 2.2 with (2.9) imply

$$u(2) \ge C\delta\|u(1)\|_{1,\delta}, \quad v(2) \ge C\delta\|v(1)\|_{1,\delta}.$$

Let $a \in A$. The estimate (2.9) yields $\|u(1)\|_{1,\delta}^{a}\|v(1)\|_{1,\delta}^{1-a} \le C$. Finally, using (2.3) and (3.44) we have $\|v_0\|_{1,\delta} \le C$. Using (2.2), Jensen's inequality and (2.3) we estimate

$$C \ge \|v(1)\|_{1,\delta} \ge \int_0^1 e^{-\lambda_1(1-s)}\|u^q(s)\|_{1,\delta}\,\mathrm{d}s \ge C\|u_0\|_{1,\delta}^{q}.$$

This proves the estimate (3.43) for $p = 1$, $q > 1$. □

Proof of Theorem 1.3. We prove Theorem 1.3 for $b_1 = 0$. One can easily modify this proof (and also the proof of Theorem 1.2) for system (1.6) with $b_1 > 0$. The constants in this proof may depend on Ω, b_2, however we will not emphasize this dependence. Observe that for problem (1.6), it holds $A = (0,1)$, since $r = p = q = 1$ and $s = 0$ (in sense of the problem (1.1)). Let $s' \ge 0$. Lemma 2.3 implies

$$\int_\Omega u^a v^{1-a}(s')\varphi_1\,\mathrm{d}x \le C, \quad a \in (0,1). \tag{3.45}$$

Thus there holds

$$\int_0^t e^{-\lambda_1(t-s')} \int_\Omega \left(u^a v^{2-a}(s') + u^{1+a}v^{-a}(s')\right)\varphi_1\,\mathrm{d}x\,\mathrm{d}s' \le C \tag{3.46}$$

for $a \in (0,1)$. We use inequality $u^{\frac{2+a}{2}} = \left[u^{\frac{a^2}{2}} v^{\frac{(2-a)a}{2}}\right] \left[u^{\frac{2+a-a^2}{2}} v^{-\frac{(2-a)a}{2}}\right] \leq \frac{a}{2} u^a v^{2-a} + \frac{2-a}{2} u^{1+a} v^{-a}$ to deduce

$$\int_0^1 e^{-\lambda_1(1-s')} \int_\Omega u^{\frac{2+a}{2}}(s')\varphi_1 \, dx \, ds' \leq C_1, \tag{3.47}$$

where C_1 is independent of u. The constants C_i, $i \in \mathbb{N}$ will be fixed during the proof (where C_i, $i > 1$ will appear below).

Now we prove that there exists $t_0 \geq 0$ possibly depending on v, such that

$$\int_\Omega v(t_0)\varphi_1 \, dx \leq \frac{4}{\lambda_1} b_2 C_1. \tag{3.48}$$

To prove (3.48) we multiply the second equation in (1.6) by φ_1 and integrate on $\Omega \times (0,1)$. Thus using (3.47) we have

$$\begin{aligned} \int_\Omega v(1)\varphi_1 \, dx + \lambda_1 \int_0^1 \int_\Omega v(s')\varphi_1 \, dx \, ds' &= b_2 \int_0^1 \int_\Omega u(s')\varphi_1 \, dx \, ds' + \int_\Omega v_0 \varphi_1 \, dx \\ &\leq b_2 C_1 + \int_\Omega v_0 \varphi_1 \, dx. \end{aligned} \tag{3.49}$$

Denote $C_2 := \int_\Omega v(0)\varphi_1 \, dx$. If there holds $C_2 \leq \frac{4}{\lambda_1} b_2 C_1$, then (3.48) is true with $t_0 = 0$. If there holds $C_2 > \frac{4}{\lambda_1} b_2 C_1$, then necessarily $\int_\Omega v(t_1)\varphi_1 \, dx < \frac{1+\frac{\lambda_1}{2}}{\lambda_1+1} C_2$ for some $t_1 \in (0,1)$. Indeed, if $\int_\Omega v(t)\varphi_1 \, dx \geq \frac{1+\frac{\lambda_1}{2}}{\lambda_1+1} C_2$ for all $t \in (0,1)$, then (3.49) implies a contradiction. In n-th such step we obtain

$$\int_\Omega v(t_n)\varphi_1 \, dx < \frac{1+\frac{\lambda_1}{2}}{\lambda_1+1} C_{n+1} < \cdots < \left(\frac{1+\frac{\lambda_1}{2}}{\lambda_1+1}\right)^n C_2$$

for some $t_n \in (t_{n-1}, t_{n-1}+1)$ if

$$C_2, C_3, \ldots, C_{n+1} > \frac{4}{\lambda_1} b_2 C_1. \tag{3.50}$$

Hence there exists $n_0 = n_0(C_2)$ such that (3.48) is true with $t_0 = t_{n_0}$ provided there holds (3.50) with n replaced by n_0.

For $t \geq 0$, $a \in (0,1)$, $\varepsilon \in (0,a)$ $\gamma \in \left[1, \frac{N+1}{N-1}\right)$ and t_0 from (3.48) we estimate

$$\|v(1+t_0+t)\|_{\gamma,\delta} \leq C \left[(t+1)^{-\beta} \|v(t_0)\|_{1,\delta} + \int_{t_0}^{1+t_0+t} e^{-\lambda_1 f(\varepsilon+1-a)} f^{-\beta} \int_\Omega u(s')\varphi_1 \, dx \, ds'\right],$$

where $f = f(t,s') = 1 + t_0 + t - s'$ and $\beta = \frac{N+1}{2}\left(1 - \frac{1}{\gamma}\right)$. Hölder's inequality

$$\int_\Omega u(s')\varphi_1 \, dx \leq \left(\int_\Omega u^a v^{1-a}(s')\varphi_1 \, dx\right)^a \left(\int_\Omega u^{1+a} v^{-a}(s')\varphi_1 \, dx\right)^{1-a}$$

and (3.45) yield

$$\|v(1+t_0+t)\|_{\gamma,\delta} \leq C \left[\|v(t_0)\|_{1,\delta} + \int_{t_0}^{1+t_0+t} e^{-\lambda_1 \varepsilon f} f^{-\beta} \left(e^{-\lambda_1 f} \int_\Omega u^{1+a} v^{-a}(s')\varphi_1 \, dx\right)^{1-a} ds'\right].$$

For $a < 1$ close to 1 we have $\frac{\beta}{a} < 1$. Finally, using Hölder's inequality, (3.48) and (3.46) we have $\|v(t)\|_{\gamma,\delta} \leq C$, $t \geq T'$ for some $T' = T'(v)$ and $\gamma \in \left[1, \frac{N+1}{N-1}\right)$. The estimate (3.47) implies $\|u(t')\|_{1,\delta} \leq C$ for some $t' \in (T', T'+1)$. Finally, we obtain $\|u(t')\|_{1,\delta} + \|v(t')\|_{1,\delta} \leq C$ and we use Theorem 1.2 (where u, p, r is interchanged with v, q, s, respectively) to conclude the proof. $\square$

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