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Multiplicity of Positive Solutions for a Fourth-Order Quasilinear Singular Differential Equation*

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Abstract

This paper is concerned with the multiplicity of positive solutions of boundary value problem for the fourth-order quasilinear singular differential equation

$$(|u''|^{p-2}u'')'' = \lambda g(t)f(u), \quad 0 < t < 1,$$

where $p > 1$, $\lambda > 0$. We apply the fixed point index theory and the upper and lower solutions method to investigate the multiplicity of positive solutions. We have found a threshold $\lambda^* < +\infty$, such that if $0 < \lambda \leq \lambda^*$, then the problem admits at least one positive solution; while if $\lambda > \lambda^*$, then the problem has no positive solution. In particular, there exist at least two positive solutions for $0 < \lambda < \lambda^*$.

Key words: Multiplicity; Positive solutions; Fourth-order; Quasilinear; Singular.

1 Introduction

In this paper, we consider the multiplicity of positive solutions for the following fourth-order quasilinear singular differential equation

$$(|u''|^{p-2}u'')'' = \lambda g(t)f(u), \quad 0 < t < 1, \tag{1.1}$$

subject to the boundary value conditions

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{1.2}$$

where $p > 1$ and λ is a positive parameter.

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In the past few years, some fourth-order nonlinear equations have been proposed for image (signal) processing (i.e., edge detection, image denoising, etc.). And a number of authors hoped that these methods might perform better than some second-order equations [15–19]. Indeed, there are good reasons to consider fourth-order nonlinear equations. First, fourth-order differential damps oscillations at high frequencies (e.g., noise) much faster than second-order differential. Second, there is the possibility of having schemes that include effects of curvature, i.e., the second derivatives of the image (signal), in the dynamics. The equation (1.1) can be regarded as the analogue of the Euler-Lagrange equations from the variation problem in [15]. Similarly, the solution of the equation (1.1) can be regarded as the steady-state case of the fourth-order anisotropic diffusion equation in [16–19].

Equations of the form (1.1), especially the special case $p = 2$, have been the subject of intensive study during the last thirty years, see for example [1,2,5,9–14]. In particular, in a recent paper [5], under some structure conditions which permit some singularities for $g(t)$, the authors discussed the special case $p = 2$ subject to the boundary value conditions (1.2), and revealed the relation between the existence of positive solutions and the parameter λ. While for the case $p > 1$ with $g(t) \equiv 1$ and $f(s)$ being of power type, in [6], P. Dráek, M. Ôani considered the corresponding initial value problem, and obtain the local existence and uniqueness of solutions, see also [7] for some extension of the results.

In this paper, we mainly discuss the boundary value problem for the fourth-order quasilinear differential equation (1.1), namely, the problem (1.1), (1.2). Throughout this paper, we assume the following basic conditions:

(H1) $f \in C([0, +\infty), (0, +\infty))$ and is nondecreasing on $[0, +\infty)$. Furthermore, there exist $\overline{\delta} > 0, m > p - 1$ such that $f(u) > \overline{\delta} u^m, u \in [0, +\infty)$;

(H2) $g \in C((0, 1), (0, +\infty))$ and $g(t) \not\equiv 0$ on any subinterval of $(0, 1)$.

The main purpose of this paper is to investigate the existence, nonexistence and multiplicity of positive solutions of the problem (1.1), (1.2). Different from the known works, the equation we consider is quasilinear, which might have degeneracy or singularities. In fact, if $p > 2$, then the equation is degenerate at the points where $u'' = 0$; while if $1 < p < 2$, then the equation has singularity at the points where $u'' = 0$. It should be noticed that there is no Green's function compared with the special case $p = 2$.

The main result of this paper is the following

Theorem 1.1 *Let (H1) and (H2) be satisfied. If*

$$\int_0^1 s(1-s)g(s)ds < +\infty, \tag{1.3}$$

then there exists a threshold $0 < \lambda^* < +\infty$ *such that the problem (1.1), (1.2) has no positive solution for* $\lambda > \lambda^*$*, has at least one positive solution for* $\lambda = \lambda^*$*, and has at least two positive solutions for* $0 < \lambda < \lambda^*$.

Our method can also be applied to the discussion of the equation subject to another boundary value condition

$$u(0) = u'(1) = u''(0) = (|u''(t)|^{p-2}u''(t))'|_{t=1} = 0. \tag{1.4}$$

The corresponding extension of the result is as follows.

Theorem 1.2 *Let (H1) and (H2) be satisfied. If*

$$\int_0^1 sg(s)ds < +\infty, \tag{1.5}$$

then there exists a threshold $0 < \lambda^* < +\infty$ *such that the problem (1.1), (1.4) has no positive solution for* $\lambda > \lambda^*$*, has at least one positive solution for* $\lambda = \lambda^*$*, and has at least two positive solutions for* $0 < \lambda < \lambda^*$.

This paper is organized as follows. As preliminaries, we state some necessary lemmas in Section 2. Subsequently, in Section 3, we apply the fixed point index theory and the upper and lower solutions method to the proof of Theorem 1.1. In the last section, we are concerned with the proof of Theorem 1.2. Owing to the similarity with the proof of Theorem 1.1, we only give a sketch and omit the details of the proof of Theorem 1.2.

2 Preliminaries

In this section, we first present necessary definitions and introduce some auxiliary lemmas, including those from the fixed point index theory and the theory based on the upper and lower solutions method. Subsequently, we construct some key integral operators, which is closely related to the problem (1.1), (1.2). As an important preparation to the proof of our main result, we devote the remaining part of this section to the proof of the complete continuity and monotonicity of the operator. We also give the a priori estimates on the positive solutions of the problem (1.1), (1.2).

First, because of the possible degeneracy and singularities, the exact meaning of solutions should be clarified.

Definition 2.1 *A function* $u(t)$ *is said to be a positive solution of the problem (1.1), (1.2), if* $u(t) > 0$*, for* $t \in (0,1)$*,* $u(t) \in C^2[0,1] \cap C^4(0,1)$*,* $|u''|^{p-2}u'' \in C^2(0,1)$*, and* $u(t)$ *satisfies (1.1) and (1.2).*

The following two lemmas from the fixed point index theory and the theory based on upper and lower solutions method will be used to obtain the multiplicity of positive solutions, see [4].

Lemma 2.1 *Let* D *be a cone in a real Banach space* X*,* Ω *be a bounded open subset of* X *with* $\theta \in \Omega$*, and* $A : D \cap \overline{\Omega} \to D$ *is a completely continuous operator. If*

$$Ax = \mu x, x \in D \cap \partial\Omega \Rightarrow \mu < 1,$$

then $i(A, D \cap \Omega, D) = 1$*; While if*

(1) $\inf\limits_{x \in D \cap \partial\Omega} \|Ax\| > 0$*;*

(2) $Ax = \mu x, x \in D \cap \partial\Omega \Rightarrow \mu \notin (0,1]$*,*

then $i(A, D \cap \Omega, D) = 0$.

Lemma 2.2 *Suppose X is a partially ordered Banach space, D is a normal cone of X, and $A : D \to X$ is an increasing and completely continuous operator. If there exist x_0, $y_0 \in D$ such that $x_0 \leq y_0$, $\langle x_0, y_0 \rangle \subset D$, and x_0, y_0 are a lower solution and an upper solution of the equation*

$$x - Ax = 0,$$

respectively, then the above equation has a minimal solution x^* and a maximal solution y^* in the ordered interval $\langle x_0, y_0 \rangle$, and $x^* \leq y^*$.

The following technical lemma, see [8], will be used to characterize some properties of a related operator.

Lemma 2.3 *If $y \in C^2(0,1) \cap C[0,1]$ with $y(t) \geq 0$ for $0 \leq t \leq 1$, and $y''(t) \leq 0$ for $0 < t < 1$, then*

$$y(t) \geq \frac{\xi}{4} \max_{0 \leq s \leq 1} |y(s)|, \quad \frac{\xi}{4} \leq t \leq 1 - \frac{\xi}{4},$$

where $0 \leq \xi \leq 1$.

We also need the following technical lemma on the property of the function f, see [3].

Lemma 2.4 *Suppose that $f : [0, +\infty) \to (0, +\infty)$ is continuous. For $s > 0$ and $M > 0$, there exists $\overline{s} > s$ and $h_0 > 0$ such that*

$$sf(u + h) < \overline{s} f(u), u \in [0, M], h \in (0, h_0).$$

We are now in the position to make preparation to prove our main result. We use the notation $\varphi_p(s) = |s|^{p-2}s$, for $p > 1$. Let $q = \frac{p}{p-1}$. Then φ_q is the inverse function of φ_p. Let $E = C[0,1]$ with the norm $\|u\| = \max\limits_{0 \leq t \leq 1} |u(t)|$ and

$$P = \{u \in E; u(t) \geq 0, t \in [0,1]\}.$$

It is clear that P is a normal cone of E. Define a set

S=$\{\lambda > 0$; such that the problem (1.1), (1.2) has at least one positive solution $\}$.

To show the existence, it is necessary to construct an appropriate operator and solve the corresponding operator equation. For this purpose, we first notice that $u(t)$ is a solution of the problem (1.1), (1.2), if and only if $u(t)$ is a solution of the following problem

$$v'' = \lambda g(t) f(u), \tag{2.1}$$

$$u'' = \varphi_q(v), \tag{2.2}$$

$$v(0) = v(1) = 0, \tag{2.3}$$

$$u(0) = u(1) = 0. \tag{2.4}$$

Because of (2.1) and (2.3), $v(t)$ can be expressed by

$$v(t) = -\lambda \int_0^1 k(t,s) g(s) f(u(s)) ds,$$

where $k(t,s)$ is the Green function of the equation (2.1) with the boundary value (2.3), and

$$k(t,s)=\begin{cases} t(1-s), & t\le s,\\ s(1-t), & t\ge s.\end{cases}$$

Then $u(t)$ can be expressed by

$$u(t)=\lambda^{\frac{1}{p-1}}\int_0^1 k(t,\tau)\varphi_q\left(\int_0^1 k(\tau,s)g(s)f(u(s))ds\right)d\tau. \tag{2.5}$$

It is easy to prove that the Green function $k(t,s)$ has the following properties.

Lemma 2.5 *For all $t,s\in[0,1]$, we have*

$$\begin{aligned} &k(t,s)>0, \qquad \textit{for}\quad (t,s)\in(0,1)\times(0,1);\\ &k(t,s)\le k(s,s)=s(1-s), \qquad \textit{for}\quad t,s\in[0,1];\\ &0\le k(t,s)\le\frac{1}{4}, \qquad \textit{for}\quad t,s\in[0,1].\end{aligned}$$

Lemma 2.6 *For all $t\in[\theta,1-\theta]$, we have*

$$k(t,s)\ge\theta k(s,s), \qquad \theta\in(0,\frac{1}{2}), s\in[0,1].$$

Proof. In fact, for all $t\in[\theta,1-\theta]$, we have

$$\begin{aligned}\frac{k(t,s)}{k(s,s)}&=\begin{cases}\dfrac{t}{s}, & 1\ge s\ge t\ge 0,\\ \dfrac{1-t}{1-s}, & 0\le s\le t\le 1,\end{cases}\\ &\ge\begin{cases} t\ge\theta, & s\ge t,\\ 1-t\ge\theta, & s\le t.\end{cases}\end{aligned}$$

Therefore, For all $t\in[\theta,1-\theta]$, we have

$$k(t,s)\ge\theta k(s,s), \qquad \theta\in(0,\frac{1}{2}), s\in[0,1].$$

The proof is complete. □

Next, we consider the following approximate problem

$$(|u''|^{p-2}u'')''=\lambda g(t)f(u), \quad 0<t<1, \tag{2.6}$$
$$u(0)=u(1)=h\ge 0, \qquad u''(0)=u''(1)=0. \tag{2.7}$$

Define a cone $K\subset P$ as follows

$$K=\left\{u\in E; u(t)\ge 0, t\in[0,1] \quad\text{and}\quad \min_{1/8\le t\le 7/8}u(t)\ge\frac{1}{8}\|u\|\right\}.$$

And we define an integral operator T_λ^h on K by

$$T_\lambda^h u(t)=h+\lambda^{\frac{1}{p-1}}\int_0^1 k(t,\tau)\varphi_q\left(\int_0^1 k(\tau,s)g(s)f(u(s))ds\right)d\tau. \tag{2.8}$$

Now, we discuss the properties of the operator T_λ^h. First, by the definition of T_λ^h and the assumption (H1), we have

Lemma 2.7 *Let (H1) and (H2) be satisfied. Then the operator T_λ^h defined by (2.8) is non-decreasing, that is, the inequality $u_1 \le u_2$ implies $T_\lambda^h u_1 \le T_\lambda^h u_2$, where " $\le$ " is the partial order defined on K.*

The complete continuity of T_λ^h can be obtained by the following lemma.

Lemma 2.8 *Let (H1), (H2) and* (1.3) *be satisfied. Then the operator T_λ^h defined by (2.8) is completely continuous, and $T_\lambda^h K \subset K$.*

Proof. Firstly, we testify the complete continuity of T_λ^h. Let $\{u_n\} \subset K$, $u \in K$ with $\|u_n - u\| \to 0$ as $n \to +\infty$. Then we have

$$\begin{aligned}
&\|(T_\lambda^h u_n)(t) - (T_\lambda^h u)(t)\| \\
&\le \lambda^{\frac{1}{p-1}} \int_0^1 k(t,\tau) \left|\varphi_q\left(\int_0^1 k(\tau,s)g(s)f(u_n(s))ds\right) - \varphi_q\left(\int_0^1 k(\tau,s)g(s)f(u(s))ds\right)\right| d\tau \\
&\le \lambda^{\frac{1}{p-1}} \sup_{t\in[0,1]} \left|\varphi_q\left(\int_0^1 k(t,s)g(s)f(u_n(s))ds\right) - \varphi_q\left(\int_0^1 k(t,s)g(s)f(u(s))ds\right)\right|.
\end{aligned}$$

Since $\|u_n - u\| \to 0$ as $n \to \infty$ and $u_n, u \in K$, $\{u_n(t)\}$ is bounded uniformly. Then there exists a constant $M_0 > 0$, such that $|u(t)| \le M_0, |u_n(t)| \le M_0$, for any $t \in [0,1], n = 1, 2, \cdots$. Due to the continuity of $f(s)$, $f(u_n)$ is bounded uniformly in $[0, M_0]$. Moreover, because of the continuity of $\varphi_q(s)$, (1.3) and (H2), by the Lebesgue dominated convergence theorem, we have

$$\|T_\lambda^h u_n - T_\lambda^h u\| \to 0 \qquad (n \to \infty).$$

Therefore, we see that T_λ^h is continuous. And the compactness of the operator T_λ^h is easily obtained from the Arzela-Ascoli theorem.

Next, we testify that $T_\lambda^h K \subset K$. For each $u \in K$, it is easy to check that $(T_\lambda^h u)''(t) \le 0$ for $0 < t < 1$ and $(T_\lambda^h u)(t) \ge 0$ for $0 \le t \le 1$. Then according to Lemma 2.3, the following inequality holds

$$(T_\lambda^h u)(t) \ge \frac{1}{8}\|T_\lambda^h u\|, \quad \frac{1}{8} \le t \le \frac{7}{8},$$

which implies that $(T_\lambda^h u)(t) \in K$. Hence we testify that $T_\lambda^h K \subset K$. The proof is complete. □

By the definition of T_λ^h, namely (2.8), and a direct computation, we obtain the following lemma.

Lemma 2.9 *Let (H1) and (H2) be satisfied. Then the problem (1.1), (1.2) has a positive solution u if and only if u is a fixed point of T_λ^0. And the problem (2.6), (2.7) has a positive solution u if and only if u is a fixed point of T_λ^h.*

In order to apply the fixed point index lemmas, we need the a priori estimates on positive solutions of the problem (1.1), (1.2).

Lemma 2.10 *Let (H1), (H2) and* (1.3) *be satisfied. And suppose that $\lambda \in S, S_1 = (\lambda, +\infty) \cap S \not\equiv \emptyset$. Then there exists a constant $R(\lambda) > 0$, such that $\|u_{\lambda'}\| \le R(\lambda)$, where $\lambda' \in S_1$, and the function $u_{\lambda'} \in K$ is a positive solution of the problem (1.1), (1.2) with λ replaced by λ'.*

Proof. For any fixed $\lambda' \in S$, let $u_{\lambda'}$ be a positive solution of the problem (1.1), (1.2). Then, by Lemma 2.9, we have

$$u_{\lambda'}(t) = T^0_{\lambda'} u_{\lambda'}(t) = (\lambda')^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\varphi_q \left(\int_0^1 k(\tau,s)g(s)f(u_{\lambda'}(s))ds\right) d\tau.$$

Let

$$R(\lambda) = \max\left\{\left(\lambda^{\frac{1}{p-1}}\left(\frac{1}{8}\right)^{\frac{m+p}{p-1}} \int_{1/8}^{7/8} k(\tau,\tau)\varphi_q\left(\int_{1/8}^{7/8} k(s,s)g(s)\overline{\delta}ds\right)d\tau\right)^{-(p-1)/(m-p+1)}, 1\right\}.$$

We conclude that $\|u_{\lambda'}\| \le R(\lambda)$. Indeed, if $\|u_{\lambda'}\| < 1$, the result is easily obtained; while if $\|u_{\lambda'}\| \ge 1$, by (H1), Lemma 2.3 and Lemma 2.6, we have

$$\begin{aligned}
\|u_{\lambda'}\| &\ge (\lambda')^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\varphi_q\left(\int_0^1 k(\tau,s)g(s)f(u_{\lambda'}(s))ds\right)d\tau \\
&\ge \lambda^{\frac{1}{p-1}} \int_{1/8}^{7/8} k(t,\tau)\varphi_q\left(\left(\frac{1}{8}\right)^{m+1}\|u_{\lambda'}\|^m \int_{1/8}^{7/8} k(s,s)g(s)\overline{\delta}ds\right)d\tau \\
&\ge \lambda^{\frac{1}{p-1}}\left(\frac{1}{8}\right)^{(m+p)/(p-1)} \|u_{\lambda'}\|^{m/(p-1)} \int_{1/8}^{7/8} k(\tau,\tau)\varphi_q\left(\int_{1/8}^{7/8} k(s,s)g(s)\overline{\delta}ds\right)d\tau.
\end{aligned}$$

Consequently,

$$\|u_{\lambda'}\|^{m/(p-1)-1} \le \left(\lambda^{\frac{1}{p-1}}\left(\frac{1}{8}\right)^{(m+p)/(p-1)} \int_{1/8}^{7/8} k(\tau,\tau)\varphi_q\left(\int_{1/8}^{7/8} k(s,s)g(s)\overline{\delta}ds\right)d\tau\right)^{-1},$$

$$\|u_{\lambda'}\| \le \left(\lambda^{\frac{1}{p-1}}\left(\frac{1}{8}\right)^{(m+p)/(p-1)} \int_{1/8}^{7/8} k(\tau,\tau)\varphi_q\left(\int_{1/8}^{7/8} k(s,s)g(s)\overline{\delta}ds\right)d\tau\right)^{-(p-1)/(m-p+1)}.$$

Therefore, $\|u_{\lambda'}\| \le R(\lambda)$, which completes the proof of the lemma. □

3 Proofs of the Main Result

In this section, we give the proof of the main result, that is, Theorem 1.1. The proof will be divided into two parts. Firstly, by the upper and lower solutions method, we investigate the basic existence of positive solutions of the problem (1.1), (1.2). Exactly, we will determine the threshold λ^* of the parameter λ, such that the problem is solvable if and only if $0 < \lambda \le \lambda^*$. Furthermore, by the fixed point index theory, we establish the multiplicity of positive solutions for $0 < \lambda < \lambda^*$.

We first present and prove the basic existence result of positive solutions of the problem (1.1), (1.2).

Proposition 3.1 *Let (H1), (H2) and* (1.3) *be satisfied. Then there exists a threshold λ^* with $0 < \lambda^* < +\infty$, such that the problem (1.1), (1.2) admits at least one positive solution for any $\lambda \in (0, \lambda^*]$, and has no positive solution for any $\lambda > \lambda^*$.*

Proof. Let S be the set defined in the previous section, namely

$S = \{\lambda > 0; \text{such that the problem (1.1), (1.2) has at least one positive solution}\}.$

We first show that $S \neq \emptyset$. Let $\beta(t)$ be a solution of the boundary value problem

$$\begin{aligned} &(|u''|^{p-2}u'')'' = g(t), \quad 0 < t < 1, \\ &u(0) = u(1) = u''(0) = u''(1) = 0. \end{aligned} \tag{3.1}$$

Then, by (2.5) we have

$$\beta(t) = \int_0^1 k(t,\tau)\varphi_q \left(\int_0^1 k(\tau,s)g(s)ds \right) d\tau.$$

Let $\beta_0 = \max_{t\in[0,1]} \beta(t)$. Then by (H1) and (2.8), we have

$$\begin{aligned} T_\lambda^0 \beta(t) &\leq T_\lambda^0 \beta_0 \\ &\leq \lambda^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\varphi_q \left(\int_0^1 k(\tau,s)g(s)f(\beta_0)ds \right) d\tau. \\ &\leq \lambda^{\frac{1}{p-1}} f(\beta_0)^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\varphi_q \left(\int_0^1 k(\tau,s)g(s)ds \right) d\tau. \\ &\leq \beta(t), \qquad \forall\, 0 < \lambda < \frac{1}{f(\beta_0)}, \end{aligned}$$

which implies that $\beta(t)$ is an upper solution of T_λ^0. On the other hand, let $\alpha(t) \equiv 0, t \in [0,1]$. Then $\alpha(t)$ is a lower solution of T_λ^0, and $\alpha(t) \leq \beta(t), t \in (0,1)$. Hence, by Lemma 2.2, Lemma 2.7 and Lemma 2.8, T_λ^0 has a fixed point $u_\lambda \in [\alpha, \beta]$, for $0 < \lambda < \frac{1}{f(\beta_0)}$. So u_λ is a positive solution of the problem (1.1), (1.2). Therefore, for any $0 < \lambda < \frac{1}{f(\beta_0)}$, we have $\lambda \in S$, which implies that $S \neq \emptyset$.

Next, we show that if $\lambda_1 \in S$, then $(0, \lambda_1) \subset S$. In fact, if u_{λ_1} be a positive solution of the problem (1.1), (1.2), then, by Lemma 2.9, we have

$$u_{\lambda_1}(t) = T_{\lambda_1}^0 u_{\lambda_1}(t), \quad t \in [0,1].$$

Therefore, for any $\lambda \in (0, \lambda_1)$, by (2.8), we have

$$\begin{aligned} T_\lambda^0 u_{\lambda_1}(t) &= \lambda^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\varphi_q \left(\int_0^1 k(\tau,s)g(s)f(u_{\lambda_1}(s))ds \right) d\tau \\ &\leq \lambda_1^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\varphi_q \left(\int_0^1 k(\tau,s)g(s)f(u_{\lambda_1}(s))ds \right) d\tau \\ &= T_{\lambda_1}^0 u_{\lambda_1}(t) \\ &= u_{\lambda_1}(t), \end{aligned}$$

which implies that u_{λ_1} is an upper solution of T_λ^0. Taking this into account, noticing the fact that the function $\alpha(t) \equiv 0$ is a lower solution of T_λ^0, and using Lemma 2.2, Lemma 2.7 and Lemma 2.8, we see that the problem (1.1), (1.2) has a positive solution, and therefore $\lambda \in S$, which implies that $(0, \lambda_1) \subset S$.

Now, we conclude that $\sup S < +\infty$. If this were not true, then we would have $\mathbb{N} \subset S$, where $\mathbb{N}$ denotes the set of natural numbers. Therefore, for any $n \in \mathbb{N}$, by Lemma 2.8, there exists $u_n \in K$ satisfying

$$u_n = T_n^0 u_n = n^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\varphi_q\left(\int_0^1 k(\tau,s)g(s)f(u_n(s))ds\right)d\tau.$$

If $\|u_n\| \geq 1$, then we have

$$\begin{aligned}\|u_n\| &\geq n^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\varphi_q\left(\int_0^1 k(\tau,s)g(s)f(u_n(s))ds\right)d\tau\\ &\geq n^{\frac{1}{p-1}} \int_{1/8}^{7/8} k(t,\tau)\varphi_q\left(\left(\frac{1}{8}\right)^{m+1}\|u_n\|^m \int_{1/8}^{7/8} k(s,s)g(s)\overline{\delta}ds\right)d\tau\\ &\geq n^{\frac{1}{p-1}}\left(\frac{1}{8}\right)^{(m+p)/(p-1)}\|u_n\|^{m/(p-1)}\int_{1/8}^{7/8} k(\tau,\tau)\varphi_q\left(\int_{1/8}^{7/8} k(s,s)g(s)\overline{\delta}ds\right)d\tau.\end{aligned}$$

Consequently,

$$1 \geq n^{\frac{1}{p-1}}\left(\frac{1}{8}\right)^{(m+p)/(p-1)}\int_{1/8}^{7/8} k(\tau,\tau)\varphi_q\left(\int_{1/8}^{7/8} k(s,s)g(s)\overline{\delta}ds\right)d\tau. \tag{3.2}$$

If $\|u_n\| \leq 1$, then

$$1 \geq \|u_n\| \geq n^{\frac{1}{p-1}}\left(\frac{1}{8}\right)^{p/(p-1)}\int_{1/8}^{7/8} k(\tau,\tau)\varphi_q\left(\int_{1/8}^{7/8} k(s,s)g(s)f(0)ds\right)d\tau > 0. \tag{3.3}$$

Letting $n \to +\infty$ in (3.2) and (3.3), we get a contradiction. Therefore, $\sup S < +\infty$.

We are now in a position to determine the threshold λ^*. We conclude that

$$\lambda^* = \sup S.$$

It remains to show that $\lambda^* \in S$. Let $\{\lambda_n\}$ be an increasing sequence in $[\frac{\lambda^*}{2}, \lambda^*)$ with $\lambda_n \to \lambda^*(n \to +\infty)$, and let u_n be the corresponding solutions of the problem (1.1), (1.2) with λ replaced by λ_n. By Lemma 2.10, there exists $R(\frac{\lambda^*}{2}) > 0$ such that

$$\|u_n\| \leq R\left(\frac{\lambda^*}{2}\right), \qquad n = 1, 2, \cdots.$$

Therefore, $\{u_n\}$ is an equicontinuous and uniformly bounded subset of $C[0,1]$. By the Arzela-Ascoli theorem, $\{u_n\}$ has a convergent subsequence, and denoted also by itself, with $u_n \to u^*$ as $n \to +\infty$. Since $u_n = T^0_{\lambda_n} u_n$, due to the continuity of $f(s)$, we see that $f(u_n)$ is bounded uniformly in $[0, R(\frac{\lambda^*}{2})]$, which together with the continuity of $\varphi_q(s)$ and (H2), by the Lebesgue dominated convergence theorem, implies that $u^* = T^0_{\lambda^*}u^*$. Hence, by Lemma 2.9, u^* is a positive solution of the problem (1.1), (1.2) with λ replaced by λ^*. Summing up, we have completed the proof of the proposition. □

To complete the proof of our main result, that is Theorem 1.1, it remains to show the multiplicity of solutions for $0 < \lambda < \lambda^*$. To do this, we need also the following lemma.

Lemma 3.1 *Let (H1), (H2) and* (1.3) *be satisfied. Then there exists* $h_0 \in (0,1)$*, such that for any* $\bar{h} \in (0, h_0)$ *and* $\lambda \in (0, \lambda^*)$*, the problem (2.6), (2.7) admits at least one positive solution.*

Proof. Let $\alpha(t) \equiv \bar{h}$ for $t \in [0,1]$. It is obvious that, for any fixed $\lambda \in (0, \lambda^*)$, $\alpha(t)$ is a lower solution of the operator $T_\lambda^{\bar{h}}$. On the other hand, by Lemma 2.10, there exists $R(\lambda) > 0$ such that $\|u_{\lambda'}\| \leq R(\lambda)$, where $\lambda' \in [\lambda, \lambda^*]$ and $u_{\lambda'}$ is a positive solution of the problem (1.1), (1.2) with λ replaced by λ'. By Lemma 2.4, there exist $\overline{\lambda} \in (\lambda, \lambda^*)$ and $h_0 \in (0,1)$ satisfying

$$\lambda f(u + \bar{h}) < \overline{\lambda} f(u), \quad u \in [0, R(\lambda)], \quad \bar{h} \in (0, h_0).$$

Let $u_{\overline{\lambda}}$ be a positive solution of the problem (1.1), (1.2) with λ replaced by $\overline{\lambda}$, and $\overline{u}_\lambda(t) = u_{\overline{\lambda}} + \bar{h}$, $\bar{h} \in (0, h_0)$. Then

$$\begin{aligned}
\overline{u}_\lambda(t) &= u_{\overline{\lambda}} + \bar{h} \\
&= \bar{h} + \int_0^1 k(t,\tau)\varphi_q\left(\int_0^1 k(\tau,s)\overline{\lambda} g(s) f(u_{\overline{\lambda}}(s))ds\right)d\tau \\
&\geq \bar{h} + \lambda^{\frac{1}{p-1}} \int_0^1 k(t,\tau)\varphi_q\left(\int_0^1 k(\tau,s) g(s) f(u_{\overline{\lambda}}(s) + \bar{h})ds\right)d\tau \\
&= T_\lambda^{\bar{h}} \overline{u}_\lambda(t),
\end{aligned}$$

which implies that $\overline{u}_\lambda(t)$ is an upper solution of the operator $T_\lambda^{\bar{h}}$. Then, by Lemma 2.2, Lemma 2.7 and Lemma 2.8, the problem (2.6), (2.7) has a positive solution, which completes the proof. □

We can now establish the multiplicity of positive solutions by the fixed point index theory.

Proposition 3.2 *Let (H1), (H2) and* (1.3) *be satisfied. If* $0 < \lambda < \lambda^*$*, then there exist at least two positive solutions of the problem (1.1), (1.2).*

Proof. Just as mentioned above, the arguments are based on fixed point index theory. Exactly, we apply Lemma 2.1 to calculate the indexes of the corresponding operator in two different domains, and then complete the proof by the index theory.

Let K be the set defined in the previous section, namely

$$K = \left\{ u \in E; u(t) \geq 0, t \in [0,1] \quad \text{and} \quad \min_{1/8 \leq t \leq 7/8} u(t) \geq \frac{1}{8}\|u\| \right\}.$$

To calculate the index of the operator T_λ^0 on some subset of K, we need to check the validity of the conditions in Lemma 2.1. By Lemma 3.1, there exists $h_0 \in (0,1)$, such that for any $\bar{h} \in (0, h_0)$ and $\lambda \in (0, \lambda^*)$, the problem (2.6), (2.7) admits at least one positive solution. Let $v_\lambda(t)$ be a positive solution of the problem (2.6), (2.7), and $\Omega = \{u \in K; u(t) < v_\lambda(t), t \in [0,1]\}$. It is clear that the set $\Omega \subset K$ is nonempty, bounded and open. If $u \in \partial\Omega$, then there exists $t_0 \in [0,1]$, such that $u(t_0) = v_\lambda(t_0)$. Therefore, for any $\mu \geq 1$, $\bar{h} \in (0, h_0)$ and $u \in \partial\Omega$, we have

$$T_\lambda^0 u(t_0) < \bar{h} + T_\lambda^0 u(t_0)$$

$$\le \bar{h} + T_\lambda^0 v_\lambda(t_0)$$
$$= T_\lambda^{\bar{h}} v_\lambda(t_0),$$

and by Lemma 2.9, we have

$$T_\lambda^{\bar{h}} v_\lambda(t_0) = v_\lambda(t_0) = u(t_0) \le \mu u(t_0).$$

Hence, for any $\mu \ge 1$, we see that $T_\lambda^0 u \ne \mu u$, $u \in \partial\Omega$. Therefore, by Lemma 2.1,

$$i(T_\lambda^0, \Omega, K) = 1. \tag{3.4}$$

Now, we calculate the index of the operator T_λ^0 on another relevant subset of K. For this purpose, we check the conditions of Lemma 2.1. Firstly, we check the validity of the condition (1) of the second part of Lemma 2.1. In fact, for any $u \in K$, we have

$$\begin{aligned} T_\lambda^0 u(\frac{1}{2}) &= \lambda^{\frac{1}{p-1}} \int_0^1 k(\frac{1}{2},\tau)\varphi_q\left(\int_0^1 k(\tau,s)g(s)f(u(s))ds\right)d\tau \\ &\ge \lambda^{\frac{1}{p-1}} \int_{1/8}^{7/8} k(\frac{1}{2},\tau)\varphi_q\left(\left(\frac{1}{8}\right)^{m+1}\|u\|^m \int_{1/8}^{7/8} k(s,s)g(s)\overline{\delta}ds\right)d\tau \\ &\ge \lambda^{\frac{1}{p-1}}\|u\|^{m/(p-1)} \int_{1/8}^{7/8} k(\frac{1}{2},\tau)\varphi_q\left(\left(\frac{1}{8}\right)^{m+1} \int_{1/8}^{7/8} k(s,s)g(s)\overline{\delta}ds\right)d\tau \\ &\ge \lambda^{\frac{1}{p-1}}\|u\|^{(m-p+1)/(p-1)} \int_{1/8}^{7/8} k(\frac{1}{2},\tau)\varphi_q\left(\left(\frac{1}{8}\right)^{m+1} \int_{1/8}^{7/8} k(s,s)g(s)\overline{\delta}ds\right)d\tau\|u\|. \end{aligned} \tag{3.5}$$

Choose $\overline{R} > 0$ such that

$$\lambda^{\frac{1}{p-1}}\overline{R}^{(m-p+1)/(p-1)} \int_{1/8}^{7/8} k(\frac{1}{2},\tau)\varphi_q\left(\left(\frac{1}{8}\right)^{m+1} \int_{1/8}^{7/8} k(s,s)g(s)\overline{\delta}ds\right)d\tau > 1. \tag{3.6}$$

Therefore, for any $R > \overline{R} > 0$ and $B_R \subset K$, by (3.5) and (3.6) we have

$$\|T_\lambda^0 u\| > \|u\| > 0, \qquad u \in \partial B_R, \tag{3.7}$$

where $B_R = \{u \in K; \|u\| < R\}$. Hence the condition (1) of the second part of Lemma 2.1 is fulfilled. It remains to check the validity of the condition (2) of the second part of Lemma 2.1. In fact, if this condition were not satisfied, then there would exist a function $u_1 \in K \cap \partial B_R$, $0 < \mu_1 \le 1$, such that $T_\lambda^0 u_1 = \mu_1 u_1$. Then, we have $\|T_\lambda^0 u_1\| \le \|u_1\|$, which conflicts with (3.7). Therefore by Lemma 2.1, we have

$$i(T_\lambda^0, B_R, K) = 0. \tag{3.8}$$

Finally, from (3.8), by the additivity of the fixed point index, we can now complete the proof of the proposition. In fact, we have

$$0 = i(T_\lambda^0, B_R, K) = i(T_\lambda^0, \Omega, K) + i(T_\lambda^0, B_R\backslash\overline{\Omega}, K),$$

which, together with (3.4), implies that

$$i(T_\lambda^0, B_R\backslash\overline{\Omega}, K) = -1. \tag{3.9}$$

Consequently, from (3.4) and (3.9), by the properties of the fixed point index, there are a fixed point of T_λ^0 in Ω and a fixed point of T_λ^0 in $B_R\backslash\overline{\Omega}$, respectively. Therefore by Lemma 2.9, the problem (1.1), (1.2) has at least two positive solutions. The proof is complete.□

Combining Proposition 3.1 with Proposition 3.2, we have completed the proof of Theorem 1.1.

4 The Extension

Just as mentioned in the introduction, our approach for treating the problem (1.1), (1.2) can also be applied to the problem (1.1), (1.4). Since several details are quite similar, we only give a sketch for the proof of Theorem 1.2.

Similar to the proof of the Theorem 1.1, it is necessary to construct an appropriate operator and solve the corresponding operator equation. For this purpose, we first notice that $u(t)$ is a solution of the problem (1.1), (1.4), if and only if $u(t)$ is a solution of the following problem

$$v'' = \lambda g(t) f(u), \tag{4.1}$$

$$u'' = \varphi_q(v), \tag{4.2}$$

$$v(0) = v'(1) = 0, \tag{4.3}$$

$$u(0) = u'(1) = 0, \tag{4.4}$$

where φ_q is the inverse function of φ_p. Because of (4.1) and (4.3), $v(t)$ can be expressed by

$$v(t) = -\lambda \int_0^1 \hat{k}(t,s) g(s) f(u(s)) ds, \tag{4.5}$$

where $\hat{k}(t,s)$ is the Green function of the equation (4.1) with the boundary value (4.3), and

$$\hat{k}(t,s) = \begin{cases} t, & t \le s, \\ s, & t \ge s. \end{cases}$$

From (4.2), (4.4), and (4.5), $u(t)$ can be expressed by

$$u(t) = \lambda^{\frac{1}{p-1}} \int_0^1 \hat{k}(t,\tau) \varphi_q \left(\int_0^1 \hat{k}(\tau,s) g(s) f(u(s)) ds \right) d\tau.$$

Next, we consider the following approximate problem

$$(|u''|^{p-2}u'')'' = \lambda g(t) f(u), \quad 0 < t < 1, \tag{4.6}$$

$$u(0) = u'(1) = h \ge 0, \quad (|u''(t)|^{p-2}u''(t))|_{t=0} = (|u''(t)|^{p-2}u''(t))'|_{t=1} = 0. \tag{4.7}$$

Define an integral operator $\mathscr{T}_\lambda^h : E \to E$ related to the problem (4.6), (4.7) by

$$(\mathscr{T}_\lambda^h u)(t) = (1+t)h + \lambda^{\frac{1}{p-1}} \int_0^1 \hat{k}(t,\tau) \varphi_q \left(\int_0^1 \hat{k}(\tau,s) g(s) f(u(s)) ds \right) d\tau. \tag{4.8}$$

A direct computation shows that

Lemma 4.1 *Let (H1) and (H2) be satisfied. Then the problem (1.1), (1.4) has a positive solution u if and only if u is a fixed point of $\mathscr{T}_\lambda^0$. And the problem (4.6), (4.7) has a positive solution u if and only if u is a fixed point of $\mathscr{T}_\lambda^h$.*

Now, we discuss the properties of the operator $\mathscr{T}_\lambda^h$. From (H1), we can easily obtain the monotonicity of the operator $\mathscr{T}_\lambda^h$. While the complete continuity of $\mathscr{T}_\lambda^h$ can be obtained by the following lemma.

Lemma 4.2 *Let (H1), (H2) and* (1.5) *be satisfied. Then the operator $\mathscr{T}_\lambda^h$ is completely continuous, and $\mathscr{T}_\lambda^h K \subset K$.*

Proof. Firstly, we testify the complete continuity of $\mathscr{T}_\lambda^h$. The proof is parallel to that of the front part of Lemma 2.8, and so, we omit the details.

Now, we testify $\mathscr{T}_\lambda^h K \subset K$. For each $u \in K$, it is easy to check that $(\mathscr{T}_\lambda^0)''(t) \leq 0$ for $0 < t < 1$ and $(\mathscr{T}_\lambda^0)(t) \geq 0$ for $0 < t < 1$. Then according to Lemma 2.3, the following inequality holds

$$(\mathscr{T}_\lambda^0 u)(t) \geq \frac{1}{8}\|\mathscr{T}_\lambda^0 u\|, \quad \frac{1}{8} \leq t \leq \frac{7}{8}, \tag{4.9}$$

which implies that $(\mathscr{T}_\lambda^0 u)(t) \in K$. For any $u \in K$, by (4.8), we have

$$\|\mathscr{T}_\lambda^h u\| \leq \|\mathscr{T}_\lambda^0 u\| + 2h.$$

On the other hand, by (4.9), we have

$$\begin{aligned}
\min_{1/8 \leq t \leq 7/8} \mathscr{T}_\lambda^h u(t) &\geq \min_{1/8 \leq t \leq 7/8} \mathscr{T}_\lambda^0 + \min_{1/8 \leq t \leq 7/8}((1+t)h) \\
&\geq \min_{1/8 \leq t \leq 7/8} \mathscr{T}_\lambda^0 + \frac{9}{8}h \\
&\geq \frac{1}{8}\|\mathscr{T}_\lambda^0 u\| + \frac{9}{8}h \\
&\geq \frac{1}{8}\|\mathscr{T}_\lambda^h u\|,
\end{aligned}$$

which implies that $\mathscr{T}_\lambda^h K \subset K$. The proof is complete. □

Similar to Lemma 2.10, we obtain the following a priori estimates on positive solutions of the problem (1.1), (1.4).

Lemma 4.3 *Let (H1), (H2) and* (1.5) *be satisfied. Suppose that $\lambda \in S$ and $S_1 = (\lambda, +\infty) \cap S \not\equiv \emptyset$, where S=$\{\lambda > 0$; such that the problem (1.1), (1.4) has at least one positive solution $\}$. Then there exists $R(\lambda) > 0$, such that $\|u_{\lambda'}\| \leq R(\lambda)$, where $\lambda' \in S_1$, and $u_{\lambda'} \in K$ is a positive solution of the problem (1.1), (1.4) with λ replaced by λ'.*

The Proof of Theorem 1.2. The proof is parallel to that of Theorem 1.1, and so, we omit the details. □

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