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On solutions of some fractional m-point boundary value problems at resonance *

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Abstract

In this paper, the following fractional order ordinary differential equation boundary value problem:

$$D_{0+}^{\alpha}u(t) = f(t, u(t), D_{0+}^{\alpha-1}u(t)) + e(t), \quad 0 < t < 1,$$
$$I_{0+}^{2-\alpha}u(t)\mid_{t=0} = 0, \quad D_{0+}^{\alpha-1}u(1) = \sum_{i=1}^{m-2} \beta_i D_{0+}^{\alpha-1}u(\eta_i),$$

is considered, where $1 < \alpha \le 2$, is a real number, D_{0+}^{α} and I_{0+}^{α} are the standard Riemann-Liouville differentiation and integration, and $f : [0,1] \times R^2 \to R$ is continuous and $e \in L^1[0,1]$, and $\eta_i \in (0,1)$, $\beta_i \in R$, $i = 1, 2, \cdots, m-2$, are given constants such that $\sum_{i=1}^{m-2} \beta_i = 1$. By using the coincidence degree theory, some existence results of solutions are established.

Key Words: Fractional differential equation; m-point boundary value problem; At resonance; Coincidence degree

MR (2000) Subject Classification: 34B15

1 Introduction

The subject of fractional calculus has gained considerable popularity and importance during the past decades or so, due mainly to its demonstrated applications in numerous seemingly and widespread fields of science and engineering. It does indeed provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables. For details, see [1-9, 13-18, 21-25] and the references therein.

Recently, m-point integer-order differential equation boundary value problems have been studied by many authors, see [4, 12, 13, 14]. However, there are few papers

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consider the nonlocal boundary value problem at resonance for nonlinear ordinary differential equations of fractional order. In [6] we investigated the nonlinear nonlocal non-resonance problem

$$D_{0+}^{\alpha}u(t) = f(t, u(t)), \quad 0 < t < 1,$$
$$u(0) = 0, \beta u(\eta) = u(1),$$

where $1 < \alpha \leq 2, 0 < \beta\eta^{\alpha-1} < 1$. In [7], we investigated the boundary value problem at resonance

$$D_{0+}^{\alpha}u(t) = f(t, u(t), D_{0+}^{\alpha-1}u(t)) + e(t), \quad 0 < t < 1,$$
$$I_{0+}^{2-\alpha}u(t) \mid_{t=0} = 0, \quad u(1) = \sum_{i=1}^{m-2} \beta_i u(\eta_i),$$

where $\beta_i \in R, \ \ i = 1, 2, \cdots, m-2, \ \ 0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$ are given constants such that $\sum_{i=1}^{m-2} \beta_i \eta_i^{\alpha-1} = 1$.

In this paper, the following fractional order ordinary differential equation boundary value problem:

$$D_{0+}^{\alpha}u(t) = f(t, u(t), D_{0+}^{\alpha-1}u(t)) + e(t), \quad 0 < t < 1, \tag{1.1}$$
$$I_{0+}^{2-\alpha}u(t) \mid_{t=0} = 0, \quad D_{0+}^{\alpha-1}u(1) = \sum_{i=1}^{m-2} \beta_i D_{0+}^{\alpha-1}u(\eta_i), \tag{1.2}$$

is considered, where $1 < \alpha \leq 2$ is a real number, D_{0+}^{α} and I_{0+}^{α} are the standard Riemann-Liouville differentiation and integration, and $f : [0, 1] \times R^2 \to R$ is continuous and $e \in L^1[0, 1]$, $\eta_i \in (0, 1), \ \ \beta_i \in R, \ \ i = 1, 2, \cdots, m-2$, are given constants such that $\sum_{i=1}^{m-2} \beta_i = 1$.

The m-point boundary value problem (1.1), (1.2) happens to be at resonance in the sense that its associated linear homogeneous boundary value problem

$$D_{0+}^{\alpha}u(t) = 0, \quad 0 < t < 1,$$
$$I_{0+}^{2-\alpha}u(t) \mid_{t=0} = 0, \quad D_{0+}^{\alpha-1}u(1) = \sum_{i=1}^{m-2} \beta_i D_{0+}^{\alpha-1}u(\eta_i),$$

has $u(t) = ct^{\alpha-1}, c \in R$ as a nontrivial solution.

The purpose of this paper is to study the existence of solution for boundary value problem (1.1), (1.2) at resonance case, and establish an existence theorem under nonlinear growth restrictions of f. Our method is based upon the coincidence degree theory of Mawhin [22]. Finally, we also give an example to demonstrate our result.

Now, we briefly recall some notation and an abstract existence result.

Let Y, Z be real Banach spaces, $L : dom(L) \subset Y \to Z$ be a Fredholm map of index zero and $P : Y \to Y,\ \ Q : Z \to Z$ be continuous projectors such that $Im(P) = Ker(L),\ \ Ker(Q) = Im(L)$ and $Y = Ker(L) \oplus Ker(P),\ \ Z = Im(L) \oplus Im(Q)$. It follows that $L|_{dom(L) \cap Ker(P)} :\ dom(L) \cap Ker(P) \to Im(L)$ is invertible. We denote the inverse of the map by K_P. If Ω is an open bounded subset of Y such that $dom(L) \cap \Omega \neq \emptyset$, the map $N : Y \to Z$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to Y$ is compact.

The main tool we use is the Theorem 2.4 of [22].

Theorem 1.1 *Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) $Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(dom(L) \backslash Ker(L)) \cap \partial\Omega] \times (0, 1)$;

(ii) $Nx \notin Im(L)$ for every $x \in Ker(L) \cap \partial\Omega$;

(iii) $deg\left(QN|_{Ker(L)}, \Omega \cap Ker(L), 0\right) \neq 0$, where $Q : Z \to Z$ is a projection as above with $Im(L) = Ker(Q)$.

Then the equation $Lx = Nx$ has at least one solution in $dom(L) \cap \overline{\Omega}$.

The rest of this paper is organized as follows. In section 2, we give some notations and lemmas. In section 3, we establish a theorem of existence of a solution for the problem (1.1), (1.2). In section 4, we give an example to demonstrate our result.

2 Background materials and preliminaries

For the convenience of the reader, we present here some necessary basic knowledge and definitions about fractional calculus theory. Which can be found in [6, 16, 24].

We use the classical Banach spaces $C[0, 1]$ with the norm $\|u\|_\infty = \max_{t \in [0,1]} |u(t)|$, $L^1[0, 1]$ with the norm $\|u\|_1 = \int_0^1 |u(t)| dt$. For $n \in N$, we denote by $AC^n[0, 1]$ the space of functions $u(t)$ which have continuous derivatives up to order $n - 1$ on $[0, 1]$ such that $u^{(n-1)}(t)$ is absolutely continuous:

$$AC^n[0, 1] = \left\{u \mid [0, 1] \to R \text{ and } (D^{n-1}u)(t) \text{ is absolutely continuous in } [0, 1]\right\}.$$

Definition 2.1 *The fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to R$ is given by*

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) \mathrm{d}s$$

provided the right side is pointwise defined on $(0, \infty)$.

Definition 2.2 *The fractional derivative of order $\alpha > 0$ of a function $y : (0, \infty) \to R$ is given by*

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s,$$

where $n = [\alpha] + 1$, provided the right side is pointwise defined on $(0, \infty)$.

It can be directly verified that the Riemann-Liouvell fractional integration and fractional differentiation operators of the power functions t^{μ} yield power functions of the same form. For $\alpha \geq 0, \mu > -1$, there are

$$I_{0+}^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \qquad D_{0+}^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}.$$

Lemma 2.1 *[17](Page 74, Lemma 2.5) Let $\alpha > 0, n = [\alpha] + 1$. Assume that $u \in L^1(0, 1)$ with a fractional integration of order $n - \alpha$ that belongs to $AC^n[0, 1]$. Then the equality*

$$(I_{0+}^{\alpha} D_{0+}^{\alpha} u)(t) = u(t) - \sum_{i=1}^{n} \frac{((I_{0+}^{n-\alpha} u)(t))^{(n-i)} \mid_{t=0}}{\Gamma(\alpha-i+1)} t^{\alpha-i},$$

holds almost everywhere on $[0, 1]$.

Now, we define another spaces which are fundamental in our work.

Definition 2.3 *Given $\mu > 0$ and $N = [\mu] + 1$ we can define a linear space*

$$C^{\mu}[0, 1] = \{u(t) | u(t) = I_{0+}^{\mu} x(t) + c_1 t^{\mu-1} + \cdots + c_{N-1} t^{\mu-(N-1)}, x \in C[0, 1], t \in [0, 1]\},$$

where $c_i \in R, i = 1, \ldots, N-1$.

Remark 2.1 *By means of the linear functional analysis theory, we can prove that with the norm*

$$\|u\|_{C^{\mu}} = \|D_{0+}^{\mu} u\|_{\infty} + \cdots + \|D_{0+}^{\mu-(N-1)} u\|_{\infty} + \|u\|_{\infty}$$

$C^{\mu}[0, 1]$ is a Banach space.

Remark 2.2 *If μ is a natural number, then $C^{\mu}[0, 1]$ is in accordance with the classical Banach space $C^n[0, 1]$.*

Definition 2.4 *Let $I^{\alpha}_{0+}(L^1(0,1)), \alpha > 0$, denote the space of functions $u(t)$, represented by fractional integral of order α of a summable function: $u = I^{\alpha}_{0+}v, v \in L^1(0,1)$.*

In the following Lemma, we use the unified notation of both for fractional integrals and fractional derivatives assuming that $I^{\alpha}_{0+} = D^{\alpha}_{0+}$ for $\alpha < 0$.

Lemma 2.2 *[16]The relation*

$$I^{\alpha}_{0+}I^{\beta}_{0+}\varphi = I^{\alpha+\beta}_{0+}\varphi$$

is valid in any of the following cases:

1) $\beta \geq 0, \alpha+\beta \geq 0, \quad \varphi(t) \in L^1(0,1)$;

2) $\beta \leq 0, \alpha \geq 0, \quad \varphi(t) \in I^{-\beta}_{0+}(L^1(0,1))$;

3) $\alpha \leq 0, \alpha+\beta \leq 0, \quad \varphi(t) \in I^{-\alpha-\beta}_{0+}(L^1(0,1))$.

Lemma 2.3 *[11] (Page 74, Property 2.3) Denote by $D = \frac{d}{dt}$. If $(D_{0+}u^{\alpha})(t)$ and $(D_{0+}u^{\alpha+m})(t)$ all exist, then there holds $(D^m D^{\alpha}_{0+}u)(t) = (D^{\alpha+m}_{0+})u(t)$.*

Lemma 2.4 *[7] $F \subset C^{\mu}[0,1]$ is a sequentially compact set if and only if F is uniformly bounded and equicontinuous. Here uniformly bounded means there exists $M > 0$ such that for every $u \in F$*

$$\|u\|_{C^{\mu}} = \|D^{\mu}_{0+}u\|_{\infty} + \cdots + \|D^{\mu-(N-1)}_{0+}u\|_{\infty} + \|u\|_{\infty} < M,$$

and equicontinuous means that $\forall\varepsilon > 0$, $\exists\delta > 0$, for all $t_1, t_2 \in [0,1]$, $|t_1 - t_2| < \delta$, $u \in F$, $i \in \{0, \cdots, N-1\}$, there hold

$$|u(t_1) - u(t_2)| < \varepsilon, \qquad |D^{\mu-i}_{0+}u(t_1) - D^{\mu-i}_{0+}u(t_2)| < \varepsilon.$$

3 Existence result

In this section, we always suppose that $1 < \alpha \leq 2$ is a real number and $\sum_{i=1}^{m-2}\beta_i = 1$. Let $Z = L^1[0,1]$. $Y = C^{\alpha-1}[0,1] = \{u(t)|u(t) = I^{\alpha-1}_{0+}x(t), x \in C[0,1], t \in [0,1]\}$ with the norm $\|u\|_{C^{\alpha-1}} = \|D^{\alpha-1}_{0+}u\|_{\infty} + \|u\|_{\infty}$. Then Y is a Banach space.

Given a function u such that $D^{\alpha}_{0+}u = f(t) \in L^1(0,1)$ and $I^{2-\alpha}_{0+}u(t) \mid_{t=0}= 0$, there holds $u \in C^{\alpha-1}[0,1]$. In fact, with Lemma 2.1, one has

$$u(t) = I^{\alpha}_{0+}f(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}.$$

Combine with $I_{0+}^{2-\alpha}u(t)\mid_{t=0}=0$, there is $c_2=0$. So,

$$u(t)=I_{0+}^{\alpha}f(t)+c_1t^{\alpha-1}=I_{0+}^{\alpha-1}\left[I_{0+}^{1}f(t)+c_1\Gamma(\alpha)\right],$$

Thus $u\in C^{\alpha-1}[0,1]$. Define L to be the linear operator from $dom(L)\cap Y$ to Z with

$$dom(L)=\Big\{u\in C^{\alpha-1}[0,1] \\ \left| D_{0+}^{\alpha}u\in L^1(0,1), I_{0+}^{2-\alpha}u(0)=0, D_{0+}^{\alpha-1}u(1)=\sum_{i=1}^{m-2}\beta_iD_{0+}^{\alpha-1}u(\eta_i)\right.\Bigg\},$$

and

$$Lu=D_{0+}^{\alpha}u,\quad u\in dom(L). \tag{3.1}$$

Define $N:Y\to Z$ by

$$Nu(t)=f\left(t,u(t),D_{0+}^{\alpha-1}u(t)\right)+e(t),\quad t\in[0,1].$$

Then boundary value problem (1.1), (1.2) can be written as

$$Lu=Nu.$$

Lemma 3.1 *Let L be defined as* (3.1), *then*

$$Ker(L)=\{ct^{\alpha-1}|c\in R\} \tag{3.2}$$

and

$$Im(L)=\left\{y\in Z\left|\sum_{i=1}^{m-2}\beta_i\int_{\eta_i}^{1}y(s)ds=0\right.\right\}. \tag{3.3}$$

Proof. By Lemma 2.1, Lemma 2.2, $D_{0+}^{\alpha}u(t)=0$ has solution

$$\begin{aligned}u(t)&=\frac{(I_{0+}^{2-\alpha}u(t))'\mid_{t=0}}{\Gamma(\alpha)}t^{\alpha-1}+\frac{I_{0+}^{2-\alpha}u(t)\mid_{t=0}}{\Gamma(\alpha-1)}t^{\alpha-2}\\&=\frac{D_{0+}^{\alpha-1}u(t)\mid_{t=0}}{\Gamma(\alpha)}t^{\alpha-1}+\frac{I_{0+}^{2-\alpha}u(t)\mid_{t=0}}{\Gamma(\alpha-1)}t^{\alpha-2}\end{aligned}$$

Combine with (1.2), one has (3.2) holds.

If $y\in Im(L)$, then there exists a function $u\in dom(L)$ such that $y(t)=D_{0+}^{\alpha}u(t)$. By Lemma 2.1,

$$I_{0+}^{\alpha}y(t)=u(t)-c_1t^{\alpha-1}-c_2t^{\alpha-2}.$$

where

$$c_1=\frac{D_{0+}^{\alpha-1}u(t)\mid_{t=0}}{\Gamma(\alpha)},\quad c_2=\frac{I_{0+}^{2-\alpha}u(t)\mid_{t=0}}{\Gamma(\alpha-1)}.$$

By the boundary condition $I_{0+}^{2-\alpha}u(t)\mid_{t=0}=0$, one has $c_2=0$. So,

$$u(t)=I_{0+}^{\alpha}y(t)+c_1t^{\alpha-1}$$

and by Lemma 2.2,

$$D_{0+}^{\alpha-1}u(t)=D_{0+}^{\alpha-1}I_{0+}y(t)+D_{0+}^{\alpha-1}(c_1t^{\alpha-1})=I_{0+}^{1}y(t)+c_1\Gamma(\alpha)$$

In view of the condition $D_{0+}^{\alpha-1}u(1)=\sum_{i=1}^{m-2}\beta_iD_{0+}^{\alpha-1}u(\eta_i)$, we have

$$\sum_{i=1}^{m-2}\beta_i\int_{\eta_i}^{1}y(s)ds=0,$$

thus, we obtain (3.3).

On the other hand, suppose $y\in Z$ and satisfies:

$$\sum_{i=1}^{m-2}\beta_i\int_{\eta_i}^{1}y(s)ds=0.$$

Let $u(t)=I_{0+}^{\alpha}y(t)$, then $u\in dom(L)$ and $D_{0+}^{\alpha}u(t)=y(t)$. So, $y\in Im(L)$. ¶

Lemma 3.2 *There exists $k\in\{0,1,\cdots,m-2\}$ satisfies $\sum_{i=1}^{m-2}\beta_i\eta_i^{k+1}\neq 1$.*

Proof. Suppose it is not true, we have

$$\begin{pmatrix} \eta_1 & \eta_2 & \cdots & \eta_{m-2}\\ \eta_1^1 & \eta_2^1 & \cdots & \eta_{m-2}^1\\ \vdots & \vdots & \ddots & \vdots\\ \eta_1^{m-2} & \eta_2^{m-2} & \cdots & \eta_{m-2}^{m-2} \end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_{m-2}\end{pmatrix}=\begin{pmatrix}1\\1\\ \vdots\\ 1\end{pmatrix}.$$

It is equal to

$$\begin{pmatrix} \eta_1 & \eta_2 & \cdots & \eta_{m-2} & 1\\ \eta_1^1 & \eta_2^1 & \cdots & \eta_{m-2}^1 & 1\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \eta_1^{m-3} & \eta_2^{m-3} & \cdots & \eta_{m-2}^{m-3} & 1\\ \eta_1^{m-2} & \eta_2^{m-2} & \cdots & \eta_{m-2}^{m-2} & 1 \end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_{m-2}\\ -1\end{pmatrix}=\begin{pmatrix}0\\0\\ \vdots\\ 0\\ 0\end{pmatrix}.$$

However, it is well known that the Vandermonde Determinant is not equal to zero, so there is a contradiction. ¶

Lemma 3.3 *$L: dom(L)\cap Y\to Z$ is a Fredholm operator of index zero. Furthermore, the linear continuous projector operators $Q:Z\to Z$ and $P:Y\to Y$ can be defined by*

$$Qu=C_ut^k,\quad \textit{for every } u\in Z,$$

$$Pu(t) = D_{0+}^{\alpha-1}u(t) \mid_{t=0} t^{\alpha-1}, \quad \textit{for every } u \in Y,$$

where

$$C_u = \frac{\sum_{i=1}^{m-2} \beta_i \int_{\eta_i}^1 u(s)ds}{(k+1)(1 - \sum_{i=1}^{m-2} \beta_i \eta_i^{k+1})}$$

Here $k \in \{0, 1, \cdots, m-2\}$ *satisfies* $\sum_{i=1}^{m-2} \beta_i \eta_i^{k+1} \neq 1$. *And the linear operator* K_P : $Im(L) \to dom(L) \cap Ker(P)$ *can be written by*

$$K_P(y) = I_{0+}^{\alpha} y(t).$$

Furthermore

$$\|K_P y\|_{C^{\alpha-1}} \leq \left(1 + \frac{1}{\Gamma(\alpha)}\right) \|y\|_1, \textit{ for all } y \in Im(L).$$

Proof. For $y \in Z$, let $y_1 = y - Qy$, then $y_1 \in Im(L)$ (since $\sum_{i=1}^{m-2} \beta_i \int_{\eta_i}^1 y_1(s)ds = 0$). Hence $Z = Im(L) + \{ct^k \mid c \in R\}$. Since $Im(L) \cap \{ct^k \mid c \in R\} = \{0\}$, we have $Z = Im(L) \oplus \{ct^k \mid c \in R\}$. Thus

$$\dim\ Ker(L) = \dim\ \{ct^k \mid c \in R\} = \text{co}\ \dim\ Im(L) = 1.$$

So, L is a Fredholm operator of index zero.

With definitions of P, K_P, it is easy to show that the generalized inverse of L : $Im(L) \to dom(L) \cap Ker(P)$ is K_P. In fact, for $y \in Im(L)$, we have

$$(LK_P)y = D_{0+}^{\alpha} I_{0+}^{\alpha} y = y,$$

and for $u \in dom(L) \cap Ker(P)$, we know

$$(K_P L)u(t) = I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2},$$

where

$$c_1 = \frac{D_{0+}^{\alpha-1}u(t) \mid_{t=0}}{\Gamma(\alpha)}, \quad c_2 = \frac{I_{0+}^{2-\alpha}u(t) \mid_{t=0}}{\Gamma(\alpha-1)}.$$

In view of $u \in dom(L) \cap Ker(P)$, $D_{0+}^{\alpha-1}u(t) \mid_{t=0} = 0, I_{0+}^{2-\alpha}u(t) \mid_{t=0} = 0$, we have $c_1 = c_2 = 0$, thus

$$(K_P L)u(t) = u(t).$$

This shows that $K_P = (L|_{dom(L) \cap Ker(P)})^{-1}$.

Again since

$$\begin{aligned}
\|K_P y\|_{C^{\alpha-1}} &= \|I_{0+}^{\alpha} y\|_{C^{\alpha-1}} \\
&= \|D_{0+}^{\alpha-1} I_{0+}^{\alpha} y\|_\infty + \|I_{0+}^{\alpha} y\|_\infty \\
&= \|I_{0+}^{1} y\|_\infty + \|I_{0+}^{\alpha} y\|_\infty \\
&= \left\| \int_0^t y(s)\mathrm{d}s \right\|_\infty + \frac{1}{\Gamma(\alpha)} \left\| \int_0^t (t-s)^{\alpha-1} y(s)\mathrm{d}s \right\|_\infty \\
&\le \|y\|_1 + \frac{1}{\Gamma(\alpha)} \|y\|_1 \\
&= \left(1 + \frac{1}{\Gamma(\alpha)}\right) \|y\|_1.
\end{aligned}$$

The proof is complete. ¶

Lemma 3.4 *[7] For given $e \in L^1[0,1]$, $K_P(I-Q)N : Y \to Y$ is completely continuous.*

Theorem 3.1 *Let $f : [0,1] \times R^2 \to R$ be continuous. Assume that*

(A_1) There exist functions $a, b, c, r \in L^1[0,1]$, and constant $\theta \in [0,1)$ such that for all $(x,y) \in R^2$, $t \in [0,1]$ either

$$|f(t,x,y)| \le a(t)|x| + b(t)|y| + c(t)|y|^\theta + r(t) \tag{3.4}$$

or else

$$|f(t,x,y)| \le a(t)|x| + b(t)|y| + c(t)|x|^\theta + r(t). \tag{3.5}$$

(A_2) There exists constant $M > 0$ such that for $u \in dom(L)$, if $|D_{0+}^{\alpha-1} u(t)| > M$ for all $t \in [0,1]$, then

$$\sum_{i=1}^{m-2} \beta_i \int_{\eta_i}^1 \left[f(s, u(s), D_{0+}^{\alpha-1} u(s)) + e(s) \right] ds \ne 0.$$

(A_3) There exists $M^* > 0$ such that for any $c \in R$, if $|c| > M^*$, then either

$$c \left(\sum_{i=1}^{m-2} \beta_i \int_{\eta_i}^1 \left[f(s, cs^{\alpha-1}, c\Gamma(\alpha)) + e(s) \right] ds \right) < 0.$$

or else

$$c \left(\sum_{i=1}^{m-2} \beta_i \int_{\eta_i}^1 \left[f(s, cs^{\alpha-1}, c\Gamma(\alpha)) + e(s) \right] ds \right) > 0.$$

Then, for every $e \in L^1[0,1]$, the boundary value problem (1.1), (1.2) has at least one solution in $C^{\alpha-1}[0,1]$ provided that

$$\|a\|_1 + \|b\|_1 < \frac{1}{\overline{C}},$$

where $\overline{C} = \Gamma(\alpha) + 2 + \frac{1}{\Gamma(\alpha)}$.

Proof. Set

$$\Omega_1 = \{u \in dom(L)\backslash Ker(L) | Lu = \lambda Nu \text{ for some } \lambda \in (0,1)\}.$$

Then for $u \in \Omega_1, Lu = \lambda Nu$, and $Nu \in Im(L)$, hence

$$\sum_{i=1}^{m-2} \beta_i \int_{\eta_i}^{1} \left[f(s, u(s), D_{0+}^{\alpha-1} u(s)) + e(s) \right] ds = 0.$$

Thus, from (A_2), there exists $t_0 \in [0,1]$ such that $|D_{0+}^{\alpha-1} u(t) \mid_{t=t_0} | \leq M$. For $u \in \Omega_1$, there holds $D_{0+}^{\alpha-1} u \in C^{\alpha-1}[0,1], D_{0+}^{\alpha} u \in (L^1(0,1))$. By Lemma 2.3,

$$DD_{0+}^{\alpha-1} u(t) = D_{0+}^{\alpha} u(t).$$

So,

$$D_{0+}^{\alpha-1} u(t) \mid_{t=0} = D_{0+}^{\alpha-1} u(t) \mid_{t=t_0} - I_{0+}^{1} D_{0+}^{\alpha} u(t) \mid_{t=t_0} .$$

Thus,

$$\left| D_{0+}^{\alpha-1} u(t) \mid_{t=0} \right| \leq M + \|D_{0+}^{\alpha} u(t)\|_1 = M + \|Lu\|_1 \leq M + \|Nu\|_1. \tag{3.6}$$

Again for $u \in \Omega_1$, $u \in dom(L)\backslash Ker(L)$, then $(I-P)u \in dom(L) \cap Ker(P)$ and $LPu = 0$. Thus from Lemma 3.3, we have

$$\begin{aligned} \|(I-P)u\|_{C^{\alpha-1}} &= \|K_P L(I-P)u\|_{C^{\alpha-1}} \\ &\leq \left(1 + \frac{1}{\Gamma(\alpha)}\right) \|L(I-P)u\|_1 \\ &= \left(1 + \frac{1}{\Gamma(\alpha)}\right) \|Lu\|_1 \\ &\leq \left(1 + \frac{1}{\Gamma(\alpha)}\right) \|Nu\|_1. \end{aligned} \tag{3.7}$$

From (3.6), (3.7), we have

$$\begin{aligned} & \|u\|_{C^{\alpha-1}} \leq \|Pu\|_{C^{\alpha-1}} + \|(I-P)u\|_{C^{\alpha-1}} \\ = \ & (\Gamma(\alpha)+1) \left| D_{0+}^{\alpha-1} u(t) \mid_{t=0} \right| + \|(I-P)u\|_{C^{\alpha-1}} \\ \leq \ & (\Gamma(\alpha)+1)(M + \|Nu\|_1) + \left(1 + \frac{1}{\Gamma(\alpha)}\right) \|Nu\|_1 \\ = \ & (\Gamma(\alpha)+1)M + \left(\Gamma(\alpha) + 2 + \frac{1}{\Gamma(\alpha)}\right) \|Nu\|_1 \\ = \ & (\Gamma(\alpha)+1)M + \overline{C}\|Nu\|_1. \end{aligned} \tag{3.8}$$

If (3.4) holds, then from (3.8), we get

$$\begin{aligned} \|u\|_{C^{\alpha-1}} \leq \ & \overline{C} \left[\|a\|_1 \|u\|_\infty + \|b\|_1 \|D_{0+}^{\alpha-1} u\|_\infty \right. \\ & \left. + \|c\|_1 \|D_{0+}^{\alpha-1} u\|_\infty^{\theta} + \|r\|_1 + \|e\|_1 \right] + (\Gamma(\alpha)+1)M. \end{aligned} \tag{3.9}$$

Thus, from $\|u\|_\infty \le \|u\|_{C^{\alpha-1}}$ and (3.9), we obtain

$$\begin{aligned}\|u\|_\infty \;\le\; & \frac{\overline{C}}{1-\overline{C}\|a\|_1}\Big[\|b\|_1\|D_{0+}^{\alpha-1}u\|_\infty \\ & + \|c\|_1\|D_{0+}^{\alpha-1}u\|_\infty^\theta + \|r\|_1 + \|e\|_1 + \frac{(\Gamma(\alpha)+1)M}{\overline{C}}\Big]. \qquad (3.10)\end{aligned}$$

Again, from (3.9), (3.10), one has

$$\begin{aligned}\|D_{0+}^{\alpha-1}u\|_\infty \;& \le \frac{\overline{C}\|c\|_1}{1-\overline{C}(\|a\|_1+\|b\|_1)}\|D_{0+}^{\alpha-1}u\|_\infty^\theta \\ & + \frac{\overline{C}}{1-\overline{C}(\|a\|_1+\|b\|_1)}\left[\|r\|_1 + \|e\|_1 + \frac{(\Gamma(\alpha)+1)M}{\overline{C}}\right]. \qquad (3.11)\end{aligned}$$

Since $\theta \in [0,1)$, from the above last inequality, there exists $M_1 > 0$ such that

$$\|D_{0+}^{\alpha-1}u\|_\infty \le M_1,$$

thus from (3.10) and (3.11), there exists $M_2 > 0$ such that

$$\|u\|_\infty \le M_2,$$

hence $\|u\|_{C^{\alpha-1}} = \|u\|_\infty + \|D_{0+}^{\alpha-1}u\|_\infty \le M_1 + M_2$. Therefore $\Omega_1 \subset Y$ is bounded.

If (3.5) holds, similar to the above argument, we can prove that Ω_1 is bounded too.

Let

$$\Omega_2 = \{u \in Ker(L) | Nu \in Im(L)\}.$$

For $u \in \Omega_2$, there is $u \in Ker(L) = \{u \in dom(L) | u = ct^{\alpha-1}, c \in R, t \in [0,1]\}$, and $Nu \in Im(L)$, thus

$$\sum_{i=1}^{m-2}\beta_i \int_{\eta_i}^1 \big[f(s, cs^{\alpha-1}, c\Gamma(\alpha)) + e(s)\big]\, ds = 0.$$

From (A_2), we get $|c| \le \frac{M}{\Gamma(\alpha)}$, thus Ω_2 is bounded in Y.

Next, according to the condition (A_3), for any $c \in R$, if $|c| > M^*$, then either

$$c\left(\sum_{i=1}^{m-2}\beta_i \int_{\eta_i}^1 \big[f(s, cs^{\alpha-1}, c\Gamma(\alpha)) + e(s)\big]\, ds\right) < 0. \qquad (3.12)$$

or else

$$c\left(\sum_{i=1}^{m-2}\beta_i \int_{\eta_i}^1 \big[f(s, cs^{\alpha-1}, c\Gamma(\alpha)) + e(s)\big]\, ds\right) > 0. \qquad (3.13)$$

If (3.12) holds, set

$$\Omega_3 = \{u \in Ker(L) | -\lambda Vu + (1-\lambda)QNu = 0, \lambda \in [0,1]\},$$

here $V : Ker(L) \to Im(Q)$ is the linear isomorphism given by $V(ct^{\alpha-1}) = ct^k, \forall c \in R, t \in [0,1]$. For $u = c_0 t^{\alpha-1} \in \Omega_3$,

$$\lambda c_0 t^k = (1-\lambda)\left(\sum_{i=1}^{m-2} \beta_i \int_{\eta_i}^{1} \left[f(s, c_0 s^{\alpha-1}, c_0\Gamma(\alpha)) + e(s)\right] ds\right).$$

If $\lambda = 1$, then $c_0 = 0$. Otherwise, if $|c_0| > M^*$, in view of (3.12), one has

$$c_0(1-\lambda)\left(\sum_{i=1}^{m-2} \beta_i \int_{\eta_i}^{1} \left[f(s, c_0 s^{\alpha-1}, c_0\Gamma(\alpha)) + e(s)\right] ds\right) < 0,$$

which contradicts to $\lambda c_0^2 \geq 0$. Thus $\Omega_3 \subset \{u \in Ker(L) \mid u = ct^{\alpha-1}, |c| \leq M^*\}$ is bounded in Y.

If (3.13) holds, then define the set

$$\Omega_3 = \{u \in Ker(L) | \lambda V u + (1-\lambda) QNu = 0, \lambda \in [0,1]\},$$

here V as in above. Similar to above argument, we can show that Ω_3 is bounded too.

In the following, we shall prove that all conditions of Theorem 1.1 are satisfied. Set Ω be a bounded open set of Y such that $\bigcup_{i=1}^{3} \overline{\Omega}_i \subset \Omega$. By Lemma 3.4, $K_P(I-Q)N : \overline{\Omega} \to Y$ is compact, thus N is $L-$compact on $\overline{\Omega}$. Then by above arguments, we have

(i) $Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(dom(L) \backslash Ker(L)) \cap \partial\Omega] \times (0,1)$;

(ii) $Nx \notin Im(L)$ for every $x \in Ker(L) \cap \partial\Omega$.

Finally, we will prove that (iii) of Theorem 1.1 is satisfied. Let $H(u, \lambda) = \pm\lambda V u + (1-\lambda) QNu$. According to the above argument, we know

$$H(u, \lambda) \neq 0, \quad \text{for all } u \in Ker(L) \cap \partial\Omega.$$

Thus, by the homotopy property of degree

$$\begin{aligned} deg(QN|_{Ker(L)}, \Omega \cap Ker(L), 0) &= deg(H(\cdot, 0), \Omega \cap Ker(L), 0) \\ &= deg(H(\cdot, 1), \Omega \cap Ker(L), 0) \\ &= deg(\pm V, \Omega \cap Ker(L), 0) \neq 0. \end{aligned}$$

Then by Theorem 1.1, $Lu = Nu$ has at least one solution in $dom(L) \cap \overline{\Omega}$, so that the problem (1.1), (1.2) has one solution in $C^{\alpha-1}[0,1]$. The proof is complete. ¶

4 An example

Example 4.1 *Consider the boundary value problem*

$$D_{0+}^{\frac{3}{2}}u(t) = \frac{1}{10}\sin\left(u(t)\right) + \frac{1}{10}D_{0+}^{\frac{1}{2}}u(t) + 3\sin\left(D_{0+}^{\frac{1}{2}}u(t)\right)^{\frac{1}{3}} + 1 + \cos^2 t, \quad (4.1)$$

$$I_{0+}^{\frac{1}{2}}u(0) = 0, \quad D_{0+}^{\alpha-1}u(1) = 6D_{0+}^{\frac{1}{2}}u\left(\frac{1}{3}\right) - 5D_{0+}^{\frac{1}{2}}u\left(\frac{1}{2}\right). \quad (4.2)$$

Let $\beta_1 = 6, \beta_2 = -5, \eta_1 = \frac{1}{3}, \eta_2 = \frac{1}{2}$ and

$$f(t,x,y) = \frac{\sin x}{10} + \frac{y}{10} + 3\sin\left(y^{\frac{1}{3}}\right), \quad e(t) = 1 + \cos^2 t,$$

then

$$\beta_1 + \beta_2 = 1. \quad |f(t,x,y)| \le \frac{|x|}{10} + \frac{|y|}{10} + 3|y|^{\frac{1}{3}}.$$

Again, taking $a(t) = b(t) \equiv \frac{1}{10}$, then

$$\|a\|_1 + \|b\|_1 = \frac{1}{5} < \frac{1}{\Gamma\left(\frac{3}{2}\right) + 2 + \frac{1}{\Gamma\left(\frac{3}{2}\right)}} \approx \frac{1}{4}.$$

Finally, taking $M = 52$, for any $u \in C^{\frac{1}{2}} \bigcap I_{0+}^{\frac{3}{2}}(L^1[0,1])$, assume $|D_{0+}^{\frac{1}{2}}u(t)| > M$ holds for any $t \in [0,1]$. Since the continuity of $D_{0+}^{\frac{1}{2}}u$, then either $D_{0+}^{\frac{1}{2}}u(t) > M$ or $D_{0+}^{\frac{1}{2}}u(t) < -M$ holds for any $t \in [0,1]$. If $D_{0+}^{\frac{1}{2}}u(t) > M$ holds for any $t \in [0,1]$, then

$$f\left(t, u(t), D_{0+}^{\frac{1}{2}}u(t)\right) + e(t) \ge \frac{M-21}{10} > 0,$$

so

$$\begin{aligned} &6\int_{\frac{1}{36}}^{1}\left[f\left(s,u(s),D_{0+}^{\frac{1}{2}}u(s)\right) + e(s)\right]ds - 5\int_{\frac{1}{25}}^{1}\left[f\left(s,u(s),D_{0+}^{\frac{1}{2}}u(s)\right) + e(s)\right]ds \\ &> \int_{\frac{1}{36}}^{1}\left[f\left(s,u(s),D_{0+}^{\frac{1}{2}}u(s)\right) + e(s)\right]ds \\ &\ge \frac{35(M-21)}{360} > 0. \end{aligned}$$

If $D_{0+}^{\frac{1}{2}}u(t) < -M$ hold for any $t \in [0,1]$, then

$$f\left(t, u(t), D_{0+}^{\frac{1}{2}}u(t)\right) + e(t) \le \frac{51-M}{10} < 0,$$

so

$$\begin{aligned} &6\int_{\frac{1}{36}}^{1}\left[f\left(s,u(s),D_{0+}^{\frac{1}{2}}u(s)\right) + e(s)\right]ds - 5\int_{\frac{1}{25}}^{1}\left[f\left(s,u(s),D_{0+}^{\frac{1}{2}}u(s)\right) + e(s)\right]ds \\ &< \int_{\frac{1}{36}}^{1}\left[f\left(s,u(s),D_{0+}^{\frac{1}{2}}u(s)\right) + e(s)\right]ds \\ &\le \frac{35(51-M)}{360} < 0. \end{aligned}$$

Thus, the condition (A_2) holds. Again, taking $M^* = \frac{52}{\Gamma(3/2)}$, for any $c \in R$, if $|c| > M^*$, we have

$$c\left(6\int_{\frac{1}{36}}^{1}\left[f\left(s, cs^{\frac{1}{2}}, c\Gamma\left(\frac{3}{2}\right)\right) + e(s)\right]ds - 5\int_{\frac{1}{25}}^{1}\left[f\left(s, cs^{\frac{1}{2}}, c\Gamma\left(\frac{3}{2}\right)\right) + e(s)\right]ds\right) > 0.$$

So, the condition (A_3) holds. Thus, with Theorem 3.1, the boundary value problem (4.1), (4.2) has at least one solution in $C^{\frac{1}{2}}[0,1]$.

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