On periodic solutions of nonautonomous second order Hamiltonian systems with (q, p)-Laplacian

The first author dedicates this paper to Professor George Dincă on the occasion of his 75th birthday with deep esteem and respect

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Abstract. A new existence result is obtained for nonautonomous second order Hamiltonian systems with (q, p)-Laplacian by using the minimax methods.

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1 Introduction

Consider the second-order Hamiltonian systems with (q, p)-Laplacian

$$\begin{cases} -\dfrac{d}{dt}\big(|\dot{u}_1(t)|^{q-2}\dot{u}_1(t)\big) = \nabla_{u_1}F(t, u_1(t), u_2(t)), & \text{a.e. } t \in [0, T], \\[2mm] -\dfrac{d}{dt}\big(|\dot{u}_2(t)|^{p-2}\dot{u}_2(t)\big) = \nabla_{u_2}F(t, u_1(t), u_2(t)), & \text{a.e. } t \in [0, T], \\[2mm] u_1(0) - u_1(T) = \dot{u}_1(0) - \dot{u}_1(T) = 0, \\[1mm] u_2(0) - u_2(T) = \dot{u}_2(0) - \dot{u}_2(T) = 0, \end{cases} \tag{1.1}$$

where $1 < p, q < +\infty$, $T > 0$, and $F : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfy the following assumption (A):

- F is measurable in t for each $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$;

- F is continuously differentiable in (x_1, x_2) for a.e. $t \in [0, T]$;

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- there exist $a_1, a_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F(t, x_1, x_2)|, \ |\nabla_{x_1} F(t, x_1, x_2)|, \ |\nabla_{x_2} F(t, x_1, x_2)| \leq \left[a_1(|x_1|) + a_2(|x_2|)\right] b(t)$$

for all $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $t \in [0, T]$.

When $p = q$ and $F(t, x_1, x_2) = F_1(t, x_1)$, problem (1.1) reduces to the following second order Hamiltonian system:

$$\begin{cases} -\dfrac{d}{dt}\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right) = \nabla F_1(t, u(t)) & \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases} \tag{1.2}$$

In the past decades, there are many papers concerning the existence of periodic solutions for problem (1.2) with $p = 2$ or more general with $p > 1$ via critical point theory, we refer the reader to [2, 4, 12–17] and the references therein. Specially, in [14], Tang and Wu established the existence of periodic solutions for problem (1.2) with $p = 2$ when potential F was subquadratic. Concretely speaking, they obtained the following theorems.

Theorem 1.1 (Tang and Wu [14]). *Suppose that F_1 satisfies assumption (A) and the following conditions:*

(S_1) *There exists $0 < \mu < 2, R > 0$ such that*

$$(\nabla F_1(t, x), x) \leq \mu F_1(t, x)$$

for all $|x| \geq R$ and a.e. $t \in [0, T]$;

(S_2) $F_1(t, x) \to +\infty$ *as $|x| \to +\infty$ uniformly for a.e. $t \in [0, T]$.*

Then problem (1.2) with $p = 2$ has at least one solution.

Theorem 1.2 (Tang and Wu [14]). *Suppose that F_1 satisfies assumption (A), (S_1) and the following conditions:*

(S_3) $\int_0^T F_1(t, x)dt \to +\infty$ *as $|x| \to +\infty$.*

(S_4) $F_1(t, \cdot)$ *is (β, γ)-subconvex with $\gamma > 0$ for a.e. $t \in [0, T]$, that is,*

$$F_1(t, \beta(x + y)) \leq \gamma(F_1(t, x) + F_1(t, y))$$

for all $x, y \in \mathbb{R}^N$.

Then problem (1.1) with $p = 2$ has at least one solution.

Inspired by some of our early papers in [6–10], the aim of this paper is to obtain new existence result for system (1.1) by imposing a more general growth conditions on the potential F.

For the sake of convenience, in the sequel, $\mathcal{H}$ will denote the space of continuous function space such that, for any $\theta \in \mathcal{H}$, there exists constant $M > 0$ such that

(i) $\theta(t) > 0 \quad \forall t \in \mathbb{R}^+$,

(ii) $\int_M^t \frac{1}{s\theta(s)} ds \to +\infty \quad$ as $t \to +\infty$.

The main result is the following theorem.

Theorem 1.3. *Suppose that F satisfies assumption (A) and the following conditions:*

(H_1) *there exist $\theta(|(x_1,x_2)|) \in \mathcal{H}$ with $0 < \frac{1}{\theta(|(x_1,x_2)|)} < r$, $r := \min(q,p)$, $M_1 > 0$ such that*

$$(\nabla_{(x_1,x_2)}F(t,x_1,x_2),(x_1,x_2)) \le \left(r - \frac{1}{\theta(|(x_1,x_2)|)}\right)F(t,x_1,x_2)$$

for all $|(x_1,x_2)| \ge M_1$ and a.e. $t \in [0,T]$;

(H_2) $F(t,x_1,x_2) \ge 0$ *as $|(x_1,x_2)| \to +\infty$ uniformly for a.e. $t \in [0,T]$;*

(H_3) $\int_0^T \frac{F(t,x_1,x_2)}{\theta(|(x_1,x_2)|)}dt \to +\infty$ *as $|(x_1,x_2)| \to +\infty$.*

Then problem (1.1) has at least one solution.

Remark 1.4. Let $\inf_{|(x_1,x_2)|\ge M} \frac{1}{\theta(|(x_1,x_2)|)} := k$, where k is a constant. We point out that

(a) It is clear that the set of hypotheses assumed in Theorem 1.3 is weaker than Theorem 1.1 even if $p = q = 2, F(t,x_1,x_2) = F_1(t,x_1)$. Therefore, Theorem 1.3 generalizes Theorem 1.1 completely.

(b) Theorem 1.3 also can be viewed as a partial extension of Theorem 1.2. In fact, on one hand, condition (S_4) in Theorem 1.2 is completely removed, on the other hand, under assumption (H_2), we can easily see that (H_1), (H_3) with $p = q = 2, F(t,x_1,x_2) = F_1(t,x_1)$ are equivalent to (S_1), (S_3) respectively when $k > 0$, however, (H_1) is much weaker than (S_1) when $k = 0$.

(c) There are functions F satisfying our Theorem 1.3 and not satisfying the results in [14]. For example, let

$$F(t,x_1,x_2) = d(t)\frac{2 + |x_1|^q + |x|^p}{\ln(2 + |x_1|^2 + |x_2|^2)}, \qquad \forall(t,x) \in \mathbb{R} \times \mathbb{R}^N,$$

where

$$d(t) := \begin{cases} \sin\frac{2\pi t}{T}, & t \in [0,T/2], \\ 0, & t \in [T/2,T]. \end{cases}$$

Setting $\theta(|(x_1,x_2)|) = \ln(2 + |x_1|^2 + |x_2|^2)$, a straightforward computation shows that F satisfies the conditions (H_1)–(H_3) of Theorem 1.3, but it does not satisfy the corresponding conditions of Theorem 1.1, Theorem 1.2.

The rest of this paper is organized as follows. In Section 2, we recall some important notations and present some preliminary results which will be used for the proofs of Theorem 1.3. In Section 3, we prove our main result.

2 Preliminaries

For the sake of convenience, in the following we will denote various positive constants as c_i, $i = 1,2,3,\ldots$ Firstly, we introduce some functional spaces. Let $T > 0$, $1 < q,p < +\infty$ and use $|\cdot|$ to denote the Euclidean norm in $\mathbb{R}^N$. We denote by $W_T^{1,p}$ the Sobolev space of

functions $u \in L^p(0, T; \mathbb{R}^N)$ having a weak derivative $\dot{u} \in L^p(0, T; \mathbb{R}^N)$. The norm in $W_T^{1,p}$ is defined by

$$\|u\|_{W_T^{1,p}} := \left(\int_0^T (|u(t)|^p + |\dot{u}(t)|^p) dt \right)^{\frac{1}{p}}.$$

Furthermore, we use the space W defined by

$$W := W_T^{1,q} \times W_T^{1,p}$$

with the norm $\|(u_1, u_2)\|_W := \|u_1\|_{W_T^{1,q}} + \|u_2\|_{W_T^{1,p}}$. It is clear that W is a reflexive Banach space.

For $(u_1, u_2) \in W$, let

$$(\bar{u}_1, \bar{u}_2) := \frac{1}{T} \left(\int_0^T u_1(t) dt, \int_0^T u_2(t) dt \right) \quad \text{and} \quad (\tilde{u}_1(t), \tilde{u}_2(t)) := (u_1(t), u_2(t)) - (\bar{u}_1, \bar{u}_2),$$

then one has

$$\|\tilde{u}_1\|_\infty \le c_1 \|\dot{u}_1\|_q, \qquad \|\tilde{u}_2\|_\infty \le c_1 \|\dot{u}_2\|_p, \qquad \text{(Sobolev's inequality)}$$
$$\|\tilde{u}_1\|_q \le c_2 \|\dot{u}_1\|_q, \qquad \|\tilde{u}_2\|_p \le c_2 \|\dot{u}_2\|_p \qquad \text{(Wirtinger's inequality)}$$

for each $(u_1, u_2) \in W$, where $\|u_1\|_q := \left(\int_0^T |u_1(t)|^q dt \right)^{\frac{1}{q}}$, $\|u_2\|_p := \left(\int_0^T |u_2(t)|^p dt \right)^{\frac{1}{p}}$ and $\|\tilde{u}_i\|_\infty := \max_{0 \le t \le T} |\tilde{u}_i(t)|$ for $i = 1, 2$. Since the embedding of W into $C(0, T; \mathbb{R}^N) \times C(0, T; \mathbb{R}^N)$ is compact, there exists a constant $d > 0$ such that

$$\|(u_1, u_2)\|_\infty \le d\|(u_1, u_2)\|_W \tag{2.1}$$

for all $(u_1, u_2) \in W$.

It follows from assumption (A) that functional φ on W given by

$$\varphi(u_1, u_2) = \frac{1}{q} \int_0^T |\dot{u}_1(t)|^q dt + \frac{1}{p} \int_0^T |\dot{u}_2(t)|^p dt - \int_0^T F(t, u_1(t), u_2(t)) dt$$

is continuously differentiable and weakly lower semicontinuous on W (see [7]). Moreover, one has

$$(\varphi'(u_1, u_2), (v_1, v_2)) = \int_0^T (|\dot{u}_1|^{q-2} \dot{u}_1, \dot{v}_1) dt + \int_0^T (|\dot{u}_2|^{p-2} \dot{u}_2, \dot{v}_2) dt$$
$$- \int_0^T (\nabla_{(u_1, u_2)} F(t, u_1, u_2), (v_1, v_2)) dt$$

for all $u_i \in W_T^{1,q}, v_i \in W_T^{1,p}, i = 1, 2$. It is well known that the solutions of problem (1.1) correspond to the critical points of the functional φ.

To prove our main theorem, we need the following auxiliary result.

Proposition 2.1. *Suppose that $F(t, x_1, x_2)$ satisfies assumption (A), (H_1) and (H_2), then we have*

$$F(t, x_1, x_2) \le \frac{h(t)}{M^r} |(x_1, x_2)|^r G(|(x_1, x_2)|) + h(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$, where

$$h(t) := \max_{|(x_1, x_2)| \le M} \left[a_1(|x_1|) + a_2(|x_2|) \right] b(t),$$

$$G(|(x_1, x_2)|) := \exp \left(- \int_M^{|(x_1, x_2)|} \frac{1}{t\theta(t)} dt \right).$$

Proof. Take $f(s) := F(t, sx_1, sx_2)$. By (H_2), we know that there exists $M_2 > 0$ such that

$$f(s) \geq 0 \quad \text{for all } s \geq \frac{M_2}{|(x_1, x_2)|}. \tag{2.2}$$

In light of (H_1), one may prove that

$$\begin{aligned}
f'(s) &= \frac{1}{s} \left(\nabla_{(x_1, x_2)} F(t, s(x_1, x_2)), s(x_1, x_2) \right) \\
&\leq \frac{1}{s} \left(r - \frac{1}{\theta(s|(x_1, x_2)|)} \right) F(t, s(x_1, x_2)) \\
&= \frac{1}{s} \left(r - \frac{1}{\theta(s|(x_1, x_2)|)} \right) f(s)
\end{aligned} \tag{2.3}$$

for all $s \geq \frac{M_1}{|(x_1, x_2)|}$. Then, by (2.2), integrating the inequality (2.3), we derive

$$f(s) \leq \frac{f\left(\frac{M}{|(x_1, x_2)|} \right) |(x_1, x_2)|^r}{M^r} s^r G(s|(x_1, x_2)|)$$

for all $s \geq \frac{M}{|(x_1, x_2)|}$, where $M := \max\{M_1, M_2\}$. Therefore, for $|(x_1, x_2)| \geq M$, we obtain

$$F(t, x_1, x_2) = f(1) \leq \frac{F\left(t, \frac{M}{|(x_1, x_2)|}(x_1, x_2) \right) |(x_1, x_2)|^r}{M^r} G(|(x_1, x_2)|). \tag{2.4}$$

Furthermore, by assumption (A), we also have

$$F\left(t, \frac{M}{|(x_1, x_2)|}(x_1, x_2) \right) \leq h(t) \tag{2.5}$$

for all $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $t \in [0, T]$. From (2.4), (2.5) and assumption (A), we obtain

$$F(t, x_1, x_2) \leq \frac{h(t)}{M^r} |(x_1, x_2)|^r G(|(x_1, x_2)|) + h(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$. □

Remark 2.2. Making use of property (ii) of θ, we know that $G(|(x_1, x_2)|) \to 0$ as $|(x_1, x_2)| \to +\infty$. It should be noted that function $t^r G(t)$ is increasing on t. This fact follows easily from the range of $\frac{1}{\theta}$, and $(t^r G(t))' = t^{r-1} G(t) \left(r - \frac{1}{\theta(t)} \right) > 0$.

3   Proof of the main result

We start with a compactness condition, which plays a crucial role in establishing our result. Recall that a sequence $\{(u_{1n}, u_{2n})\} \subset W$ is said to be a (C) sequence of φ if $\varphi(u_{1n}, u_{2n})$ is bounded and $(1 + \|(u_{1n}, u_{2n})\|) \|\varphi'(u_{1n}, u_{2n})\| \to 0$ as $n \to \infty$. The functional φ satisfies condition (C) if every (C) sequence of φ has a convergent subsequence. This condition is due to G. Cerami [3].

Lemma 3.1. *Assume that (A), (H_1) and (H_3) hold, then the functional φ satisfies condition (C).*

Proof. Suppose that $\{(u_{1n}, u_{2n})\} \subset W$ is a (C) sequence of φ, that is, $\varphi(u_{1n}, u_{2n})$ is bounded and $(1 + \|(u_{1n}, u_{2n})\|)\|\varphi'(u_{1n}, u_{2n})\| \to 0$ as $n \to +\infty$. Then there exists a constant $L > 0$ such that

$$|\varphi(u_{1n}, u_{2n})| \leq L, \qquad (1 + \|(u_{1n}, u_{2n})\|)\|\varphi'(u_{1n}, u_{2n})\| \leq L \tag{3.1}$$

for all $n \in \mathbb{N}$. In a similar way to the proof of Lemma 8 in [8], we only need to prove $\{(u_{1n}, u_{2n})\}$ is bounded.

Combining assumption (A) with (H_1), we have

$$-\tilde{h}(t) + \left(\nabla_{(x_1, x_2)} F(t, x_1, x_2), (x_1, x_2) \right) \leq \left(r - \frac{1}{\theta(|(x_1, x_2)|)} \right) F(t, x_1, x_2) \tag{3.2}$$

for all $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $t \in [0, T]$, where $\tilde{h}(t) = (r + M)h(t) \geq 0$. Taking into account of (3.2) and assumption (A), we conclude from (3.1) that

$$
\begin{aligned}
(r+1)L &\geq (1 + \|(u_{1n}, u_{2n})\|)\|\varphi'(u_{1n}, u_{2n})\| - r\varphi(u_{1n}, u_{2n}) \\
&\geq (\varphi'(u_{1n}, u_{2n}), (u_{1n}, u_{2n})) - r\varphi(u_{1n}, u_{2n}) \\
&= \left(1 - \frac{r}{q} \right) \int_0^T |\dot{u}_{1n}(t)|^q dt + \left(1 - \frac{r}{p} \right) \int_0^T |\dot{u}_{2n}(t)|^p dt \\
&\quad - \int_0^T (\nabla_{(u_{1n}, u_{2n})} F(t, u_{1n}(t), u_{2n}(t)), (u_{1n}(t), u_{2n}(t))) dt \\
&\quad + r \int_0^T F(t, u_{1n}(t), u_{2n}(t)) dt \\
&\geq \left(1 - \frac{r}{q} \right) \int_0^T |\dot{u}_{1n}(t)|^q dt + \left(1 - \frac{r}{p} \right) \int_0^T |\dot{u}_{2n}(t)|^p dt \\
&\quad + \int_0^T \frac{F(t, u_{1n}(t), u_{2n}(t))}{\theta(|(u_{1n}(t), u_{2n}(t))|)} dt - \int_0^T \tilde{h}(t) dt \\
&\geq \int_0^T \frac{F(t, u_{1n}(t), u_{2n}(t))}{\theta(|(u_{1n}(t), u_{2n}(t))|)} dt - \int_0^T \tilde{h}(t) dt,
\end{aligned} \tag{3.3}
$$

for all $n \in \mathbb{N}$, taking into account the fact that $r = \min(q, p)$. Hence, we get

$$\int_0^T \frac{F(t, u_{1n}(t), u_{2n}(t))}{\theta(|(u_{1n}(t), u_{2n}(t))|)} dt \leq c_3 \tag{3.4}$$

for all $n \in \mathbb{N}$. In addition, by using the relation (3.1), (2.1), Proposition 2.1, Remark 2.2 and Wirtinger's inequality, one has

$$
\begin{aligned}
L &\geq \varphi(u_{1n}, u_{2n}) = \frac{1}{q} \int_0^T |\dot{u}_{1n}(t)|^q dt + \frac{1}{p} \int_0^T |\dot{u}_{2n}(t)|^p dt - \int_0^T F(t, u_{1n}(t), u_{2n}(t)) dt \\
&\geq c_4 \left(\|\dot{u}_{1n}\|_{L^q}^r + \|\dot{u}_{2n}\|_{L^p}^r \right) - \int_0^T \left(\frac{h(t)}{M^r} |(u_{1n}(t), u_{2n}(t))|^r G(|(u_{1n}(t), u_{2n}(t))|) + h(t) \right) \\
&\geq c_5 \left(\|\tilde{u}_{1n}\|_{W_T^{1,q}}^r + \|\tilde{u}_{2n}\|_{W_T^{1,p}}^r \right) - c_6 \|(u_{1n}, u_{2n})\|_\infty^r G(\|(u_{1n}, u_{2n})\|_\infty) - c_7 \\
&\geq c_8 \left(\|\tilde{u}_{1n}\|_{W_T^{1,q}} + \|\tilde{u}_{2n}\|_{W_T^{1,p}} \right)^r - c_9 \|(u_{1n}, u_{2n})\|_W^r G(d\|(u_{1n}, u_{2n})\|_W) - c_7 \\
&= c_8 \|(\tilde{u}_{1n}, \tilde{u}_{2n})\|_W^r - c_9 \|(u_{1n}, u_{2n})\|_W^r G(d\|(u_{1n}, u_{2n})\|_W) - c_7
\end{aligned} \tag{3.5}
$$

for all $n \in \mathbb{N}$.

Finally, we claim (u_{1n}, u_{2n}) is bounded, otherwise, going if necessary to a subsequence, we can assume that $\|(u_{1n}, u_{2n})\| \to +\infty$ as $n \to +\infty$. Put

$$
\begin{aligned}
(v_{1n}, v_{2n}) &= \frac{(u_{1n}, u_{2n})}{\|(u_{1n}, u_{2n})\|_W} = \frac{(\bar{u}_{1n}, \bar{u}_{2n})}{\|(u_{1n}, u_{2n})\|_W} + \frac{(\tilde{u}_{1n}, \tilde{u}_{2n})}{\|(u_{1n}, u_{2n})\|_W} \\
&= (\bar{v}_{1n}, \bar{v}_{2n}) + (\tilde{v}_{1n}, \tilde{v}_{2n}).
\end{aligned}
\tag{3.6}
$$

Then, $\{(v_{1n}, v_{2n})\}$ is bounded in W and by the compactness of the embedding $W = W_T^{1,q} \times W_T^{1,p} \subset C(0, T; \mathbb{R}^N) \times C(0, T; \mathbb{R}^N)$, there is a subsequence, again denoted by $\{(v_{1n}, v_{2n})\}$, such that

$$(v_{1n}, v_{2n}) \rightharpoonup (v_1, v_2) \quad \text{weakly in } W, \tag{3.7}$$

$$(v_{1n}, v_{2n}) \to (v_1, v_2) \quad \text{strongly in } C(0, T; \mathbb{R}^N) \times C(0, T; \mathbb{R}^N). \tag{3.8}$$

Dividing both sides of (3.5) by $\|(u_{1n}, u_{2n})\|_W^r$, by Remark 2.2 and (3.6), we find that

$$\|(\tilde{v}_{1n}, \tilde{v}_{2n})\|_W \to 0 \quad \text{as } n \to +\infty. \tag{3.9}$$

Moreover, it follows from (3.8) and (3.9) that

$$(v_{1n}, v_{2n}) \to (\bar{v}_1, \bar{v}_2) \quad \text{as } n \to +\infty,$$

which implies that

$$(v_1, v_2) = (\bar{v}_1, \bar{v}_2) \quad \text{and} \quad |(\bar{v}_1, \bar{v}_2)|^r \geq |\bar{v}_1|^r + |\bar{v}_2|^r \geq c_{10} \|(\bar{v}_1, \bar{v}_2)\|_W^r = c_{10}.$$

Consequently, $|(u_{1n}(t), u_{2n}(t))| \to +\infty$ uniformly for a.e. $t \in [0, T]$. From (H_3), we get

$$
\lim_{|(u_{1n}(t), u_{2n}(t))| \to +\infty} \int_0^T \frac{F(t, u_{1n}(t), u_{2n}(t))}{\theta(|(u_{1n}(t), u_{2n}(t))|)} dt \to +\infty,
$$

which contradicts (3.4). Therefore, $\{(u_{1n}, u_{2n})\}$ is bounded in W, then φ satisfies condition (C). $\qquad \square$

Now, we are ready to prove our main result.

Proof of Theorem 1.3. Let $\widetilde{W} = \widetilde{W}_T^{1,q} \times \widetilde{W}_T^{1,p}$ be the subspace of W given by $\widetilde{W} := \{(u_1, u_2) \in W \mid (\bar{u}_1, \bar{u}_2) = (0, 0)\}$. Then $W = \widetilde{W} \bigoplus (\mathbb{R}^N \times \mathbb{R}^N)$. From Lemma 3.1, we obtain that $\varphi \in C^1(W, R)$ satisfies condition (C). As shown in [1], a deformation lemma can be proved with the weaker condition (C) replacing the usual Palais–Smale condition, and it turns out that the saddle point theorem holds true under condition (C). By saddle point theorem (see Theorem 4.6 in [11]), we have only to verify the assertion:

(φ_1) $\varphi(u_1, u_2) \to +\infty$ as $\|(u_1, u_2)\| \to +\infty$ in $\widetilde{W}$ and

(φ_2) $\varphi(u_1, u_2) \to -\infty$ as $|(u_1, u_2)| \to +\infty$ in $\mathbb{R}^N \times \mathbb{R}^N$.

We first prove (φ_1). For $(u_1, u_2) \in \widetilde{W}$, by Proposition 2.1, Remark 2.2, (2.1) and Wirtinger's inequality, we obtain

$$
\begin{aligned}
\varphi(u_1, u_2) &= \frac{1}{q} \int_0^T |\dot{u}_1(t)|^q dt + \frac{1}{p} \int_0^T |\dot{u}_2(t)|^p dt - \int_0^T F(t, u_1(t), u_2(t)) dt \\
&\geq c_{11} \left(\|u_1\|_{W_T^{1,q}}^q + \|u_1\|_{W_T^{1,p}}^p \right) - \int_0^T \left(\frac{h(t)}{M^r} |(u_1, u_2)|^r G(|(u_1, u_2)|) + h(t) \right) dt \\
&\geq c_{12} \|(u_1, u_2)\|_W^r - c_{13} \|(u_1, u_2)\|_\infty^r G(\|(u_1, u_2)\|_\infty) - c_{14} \\
&\geq [c_{12} - c_{13} G(d \|(u_1, u_2)\|_W)] \|(u_1, u_2)\|_W^r - c_{14}.
\end{aligned}
\tag{3.10}
$$

Hence, (φ_1) holds.

On the other hand, since $0 < \frac{1}{\theta(|(x_1, x_2)|)} < r$, by (H_2) and (H_3), one then arrives at

$$
\begin{aligned}
\varphi(u_1, u_2) &= - \int_0^T F(t, u_1(t), u_2(t)) dt \\
&\leq -\frac{1}{r} \int_0^T \frac{F(t, u_1(t), u_2(t))}{\theta(|(u_1(t), u_2(t))|)} dt \\
&\to -\infty \quad \text{as } |(u_1(t), u_2(t))| \to +\infty \text{ in } \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}},
\end{aligned}
\tag{3.11}
$$

which implies (φ_2). It follows from the saddle point theorem that Theorem 1.3 holds. $\qquad\square$

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