Electronic Journal of Qualitative Theory of Differential Equations
2010, No. **41**, 1-12; **http://www.math.u-szeged.hu/ejqtde/**

Existence Theorems for Second Order Multi-Point Boundary Value Problems

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ABSTRACT

We are interested in the existence of nontrivial solutions for the second order nonlinear differential equation (E): $y''(t) = f\big(t, y(t)\big) = 0, 0 < t < 1$ subject to multi-point boundary conditions at $t = 1$ and either Dirichlet or Neumann conditions at $t = 0$. Assume that $f(t, y)$ satisfies $|f(t, y)| \leq k(t)|y| + h(t)$ for non-negative functions $k, h \in L^1(0, 1)$ for all $(t, y) \in (0, 1) \times \mathbb{R}$ and $f(t, 0) \not\equiv 0$ for $t \in (0, 1)$. We show without any additional assumption on $h(t)$ that if $\|k\|_1$ is sufficiently small where $\| \cdot \|_1$ denotes the norm of $L^1(0, 1)$ then there exists at least one non-trivial solution for such boundary value problems. Our results reduce to that of Sun and Liu [11] and Sun [10] for the three point problem with Neumann boundary condition at $t = 0$.

Key Words: Second Order nonlinear differential equations, Multi-point boundary value problem, Sign-changing nonlinearities

1. Introduction

We are interested in the existence of non-trivial solutions to the second order nonlinear differential equation:

$$y'' + f(t,y) = 0, \qquad 0 < t < 1, \tag{1.1}$$

where $f(t,y) \in C\big((0,1)\times\mathbb{R},\mathbb{R}\big)$ satisfies

$$|f(t,y)| \le k(t)|y| + h(t) \tag{1.2}$$

with $k, h \in L^1(0,1)$, subject to the following non-resonant boundary conditions:

$$(BC1) \qquad y'(0) = 0,\ y(1) = \langle \alpha, y(\eta)\rangle + \langle \beta, y'(\eta)\rangle$$

$$(BC2) \qquad y(0) = 0,\ y'(1) = \langle \alpha, y(\eta)\rangle + \langle \beta, y'(\eta)\rangle$$

$$(BC3) \qquad y(0) = 0,\ y(1) = \langle \alpha, y(\eta)\rangle + \langle \beta, y'(\eta)\rangle$$

where

$$\langle \alpha, y(\eta)\rangle = \sum_{i=1}^{m} \alpha_i y(\eta_i); \langle \beta, y'(\eta)\rangle = \sum_{i=1}^{m} \beta_i y'(\eta_i).$$

Here $\eta = (\eta_1, \eta_2, \cdots, \eta_m); 0 < \eta_1 < \eta_2 < \cdots < \eta_m < 1$ and $\alpha_i \ge 0, \beta_i \ge 0$ for all $i = 0, 1, \cdots, m$. Also, $y(\eta) = \big(y(\eta_1), \cdots, y(\eta_m)\big)$, $y'(\eta) = \big(y'(\eta_1), \cdots, y'(\eta_m)\big)$ are m-vectors and $\langle \alpha, y(\eta)\rangle$ denotes usual scalar product between two vectors α and $y(\eta)$ in R^m.

Solvability of boundary value problems (1.1) subject to boundary conditions (BC1), (BC2), (BC3) with $m = 1$ has been studied by Gupta [3], Ma [7], Marano [8], Ren and Ge [9] where $f(t,y)$ is allowed to change signs subject to condition (1.2). We refer to (1.1), (BC1); (1.1), (BC2); (1.1), (BC3) as (BVP1), (BVP2), (BVP3) respectively. In recent papers by Sun and Liu [11] and Sun [10], the three-point boundary value problem subject to special cases of (BC1), was studied where they applied the Leray-Schauder nonlinear alternative theorem to prove existence of non-trivial solutions.

In [11], Sun and Liu studied the three point boundary value problem, equation (1.1) subject to the boundary condition

$$y'(0) = 0, \quad y(1) = \alpha y(\eta), \tag{1.3}$$

where $0 < \eta < 1$ and $\alpha \neq 1$. Their main result is

Theorem A (Sun and Liu [14]) Suppose that $f(t, 0) \not\equiv 0$ in $[0, 1]$ and there exists nonnegative functions $k, h \in L^1(0, 1)$ such that (1.2) holds. If $\alpha \neq 1$, and

$$\left(1 + \left|\frac{1}{1-\alpha}\right|\right) \int_0^1 (1-s)k(s)ds + \left|\frac{\alpha}{1-\alpha}\right| \int_0^\eta (\eta - s)k(s)ds < 1, \tag{1.4}$$

then the boundary value problem (1.1), (1.3) has a non-trivial solution.

In [10], Sun considered a similar boundary value problem also with Neumann boundary condition at $t = 0$, i.e. equation (1.1) subject to

$$y'(0) = 0, \quad y(1) = \beta y'(\eta), \tag{1.5}$$

and proved

Theorem B (Sun [10]) Suppose that $f(t, y)$ satisfies the same assumptions as in Theorem A. If $k(t)$ satisfies

$$2\int_0^1 (1-s)k(s)ds + |\beta| \int_0^\eta k(s)ds < 1, \tag{1.6}$$

then the boundary value problem (1.1), (1.5) has a nontrivial solution.

The boundary condition at $t = 1$ which includes both (1.3) and (1.5) can be written as

$$y'(0) = 0, \; y(1) = \alpha y(\eta) + \beta y'(\eta). \tag{1.7}$$

Condition (1.7) is a special case of (BC1) with $m = 1$. In this note, we prove similar results for the more general m-point problems with boundary conditions (BC1), (BC2), (BC3). We show that the methodology given in [10], [11] is equally applicable to (BVP1), (BVP2), (BVP3). The fixed point theorem required is the following (See [2; p.61], [1; p27]) :

Theorem (Schauder Fixed Point Theorem) Let $T : X \to X$ be a completely continuous mapping on a Banach space X. Suppose that there exists $r > 0$ such that for all $x \in X$ with $\|x\| = r, Tx \neq \lambda x$ if $\lambda > 1$, then T has a fixed point in X.

2. Integral operators with Hammerstein kernels.

We shall represent the solutions of (BVP1), (BVP2), (BVP3) as fixed point of integral equations with kernel functions incorporating the three boundary conditions (BC1), (BC2), (BC3). We define the mapping

$$A_j y(t) = G_j[y](t) + C_j t + D_j, \quad j = 1, 2, 3 \tag{2.1}$$

where

$$G_j[y](t) = \int_0^1 g_j(t,s) f\big(s, y(s)\big) ds \tag{2.2}$$

and

$$g_1(t,s) = \begin{cases} 1-s, & 0 \le t \le s \le 1, \\ 1-t, & 0 \le s \le t \le 1; \end{cases} \tag{2.3}$$

$$g_2(t,s) = \begin{cases} s, & 0 \le s \le t \le 1 \\ t, & 0 \le t \le s \le 1; \end{cases} \tag{2.4}$$

$$g_3(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1 \\ s(1-t) & 0 \le s \le t \le 1. \end{cases} \tag{2.4}$$

The Green′s functions $g_j(t,s), j = 1, 2, 3$, given in (2.3), (2.4), (2.5), arise from two point homogeneous boundary conditions, i.e. associated with boundary conditions (BC1), (BC2), (BC3) with $\alpha = \beta = 0$. Thus we have $G_1'(0) = G_1(1) = 0, G_2(0) = G_2'(1) = 0$ and $G_3(0) = G_3(1) = 0$ upon evaluating (2.2) at $t = 0$ and $t = 1$. The constants $C_j, D_j, j = 1, 2, 3$ are determined from the boundary conditions (BC1), (BC2), (BC3) to be

$$C_1 = 0, D_1 = \frac{1}{1-\overline{\alpha}} \left\{ \langle \alpha, G_1(\eta) \rangle + \langle \beta, G_1'(\eta) \rangle \right\} \tag{2.6}$$

$$C_2 = \frac{1}{\triangle} \left\{ \langle \alpha, G_2(\eta) \rangle + \langle \beta, G_2'(\eta) \rangle \right\}, D_2 = 0 \tag{2.7}$$

$$C_3 = \frac{1}{\triangle} \left\{ \langle \alpha, G_3(\eta) \rangle + \langle \beta, G_3'(\eta) \rangle \right\}, D_3 = 0 \tag{2.8}$$

where $\triangle = 1 - \langle \alpha, \eta \rangle - \overline{\beta}$, and $\overline{\alpha} = \sum_{i=1}^m \alpha_i, \overline{\beta} = \sum_{i=1}^m \beta_i$.

To apply Schauder fixed point theorem, we need to show that the operators A_j defined by (2.1) are completely continuous operators. Let $U = \{\varphi \in C[0,1] : \|\varphi\| \le 1\}$. We need to show that the set $A_j(U) \subset C[0,1]$ is uniformly bounded and equicontinuous.

Note firstly that $\sup\limits_{0\le t\le 1}\left|\int_0^1 g_j(t,s)\varphi(s)ds\right| \le \|\varphi\| \le 1$, and likewise constants C_j, D_j are bounded, so $\sup\limits_{\varphi\in U}|A_j(\varphi)|$ is bounded by a constant independent of φ. To show that $A_j\varphi(t)$ is equicontinuous, we observe

$$\begin{aligned}|A_j\varphi(t_1) - A_j\varphi(t_2)| &\le \sup_{0\le s\le 1}|g_j(t_1,s) - g_j(t_2,s)| + C_j\,|t_1 - t_2| \\ &\le (1 + C_j)\,|t_1 - t_2|\,.\end{aligned}$$

This proves that $A_j's$ are completely continuous for $j = 1, 2, 3$.

Remark 2 The boundary conditions involving the derivative of a solution at some interior points in general give rise to kernels associated with the operators A_j in (2.1) which are discontinuous in two variables t, s. However, they are shown above to be completely continuous operators.

In [10], [11], the authors used the more customary integral operator $I(t)$ defined by

$$I[y](t) = \int_0^t (t-s)f\big(s, y(s)\big)ds \tag{2.9}$$

instead of the Green's operator $G_j[y](t)$ given in (2.1).

Writing $I(t) = I[y](t), G_j(t) = G_j[y](t)$ for short, we can relate $G_j(t)$ with $I(t)$ as follows:

$$G_1(t) = -I(t) + I(1) \tag{2.10}$$

$$G_2(t) = -I(t) + I'(1)t \tag{2.11}$$

$$G_3(t) = -I(t) + I(1)t \tag{2.12}$$

Using (2.10), (2.11), (2.12), we can rewrite the operator equations in (2.1) as follows:

$$A_1y(t) = -I(t) + \frac{1}{1-\overline{\alpha}}\left\{I(1) - \langle\alpha, I(\eta)\rangle - \langle\beta, I'(\eta)\rangle\right\} \tag{2.13}$$

$$A_2y(t) = -I(t) + \frac{t}{\triangle}\left\{I'(1) - \langle\alpha, I(\eta)\rangle - \langle\beta, I'(\eta)\rangle\right\} \tag{2.14}$$

$$A_3y(t) = -I(t) + \frac{t}{\triangle}\left\{I(1) - \langle\alpha, I(\eta)\rangle - \langle\beta, I'(\eta)\rangle\right\} \tag{2.15}$$

Results in [10], [11] can then be proved using the operator equation (2.13) for the (BVP1), i.e. (1.1), (BC1).

3. Boundary value problem (1.1), (BC1)

We now prove a result generalizing both Theorems A and B for the boundary value problem (BVP1).

Theorem 1 Suppose that $f(t,0) \not\equiv 0$ in $[0,1]$ and condition (1.2) holds with $k, h \in L^1(0,1)$. If $k(t)$ satisfies for $\overline{\alpha} \neq 1$ that

$$\Lambda_1(k) = \mathrm{Max}_{0\leq t\leq 1} G_1[k](t) + \frac{1}{|1-\overline{\alpha}|}\left\{\langle |\alpha|, G_1[k](\eta)\rangle + \langle |\beta|, G_1'[k](\eta)\rangle\right\} < 1 \tag{3.1}$$

where

$$\overline{\alpha} = \sum_{i=1}^{m} \alpha_i, \ \ |\alpha| = (|\alpha_1|, \cdots, |\alpha_m|)\,, |\beta| = (|\beta_1|, \cdots, |\beta_m|) \tag{3.2}$$

and

$$G_j(t) = G_j[k](t) = \int_0^1 g_j(t,s)k(s)ds, j = 1,2,3 \tag{3.3}$$

with $g_j(t,s)$ as given in (2.3) (2.4), (2.5) then the (BVP1) has at least one non-trivial solution.

Proof. Since $f(t,0) \not\equiv 0$, we note from (1.2) and (2.10) that

$$\begin{aligned}\sup_{0\leq t\leq 1} G_1[h](t) &= \int_0^1 (1-s)h(s)ds \\ &\geq \int_0^1 (1-s)|f(s,0)|ds > 0,\end{aligned}$$

so by (3.1), we have $\Lambda(h) > 0$. Condition (3.1) now permits us to define $r > 0$ by

$$r = \Lambda_1(h)\big(1 - \Lambda_1(k)\big)^{-1} \text{ and } \quad \Omega_r = \{y(t) \in C[0,1] : \|y\| < r\}.$$

Now suppose that there exists $y_0 \in \partial\Omega_r$, i.e. $\|y_0\| = r$, and $A_1 y_0 = \lambda y_0$ for some $\lambda > 1$. Using (3.1), we obtain from (2.1), (2.2), (2.6) and (1.2) that the operator A_1 satisfies

$$\|A_1 y_0\| \leq \Lambda_1(k)\|y_0\| + \Lambda_1(h),$$

or

$$\lambda\|y_0\| \leq \Lambda_1(k)\|y_0\| + \Lambda_1(h). \tag{3.4}$$

Substituting $r = \Lambda_1(h)\big(1-\Lambda_1(k)\big)^{-1}$ for $\|y_0\|$ in (3.4), we find $\lambda r \leq r$ which contradicts the assumption that $\lambda > 1$. Thus by Schauder's Fixed point theorem, A_1 has a fixed point in $\overline{\Omega}_r$ which is not the identically zero function because of $f(t,0) \not\equiv 0$. This completes the proof.

Theorem 2 Under the same assumptions as in Theorem 1, if $k(t)$ satisfies for $\overline{\alpha} \neq 1$ that

$$\Gamma_1(k) = \left(+\frac{1}{|1-\overline{\alpha}|}\right) I[k](1) + \frac{1}{|1-\overline{\alpha}|}\left\{\langle|\alpha|, I[k](\eta)\rangle + \langle|\beta|, I'[k](\eta)\rangle\right\} < 1 \tag{3.5}$$

then the (BVP1) has at least one non-trivial solution where $I[k](t)$ and $I'[k](t)$ are defined like (2.9) by

$$I[k](t) = \int_0^1 (t-s)k(s)ds, \;\; I'[k](t) = \int_0^t k(s)ds. \tag{3.6}$$

Proof. We use the integral representation (2.13) for the operator A_1. Since $f(t,0) \not\equiv 0$ in $[0,1]$, we also have $\Gamma_1(h) > 0$ by (1.2). Using (3.5) we define the positive constant $r_1 > 0$ by

$$r_1 = \Gamma_1(h)\big(1-\Gamma_1(k)\big)^{-1}, \Omega_{r_1} = \{y \in C[0,1] : \|y\| < r_1\} \tag{3.7}$$

To apply the Schauder Fixed Point Theorem, we suppose that there exists $\overline{y} \in \partial\Omega_{r_1} = \big\{y \in \overline{\Omega}_{r_1} : \|y\| = r_1\big\}$ such that $A_1\overline{y} = \overline{\lambda}\overline{y}$ for some $\overline{\lambda} > 1$. Now apply (1.2), (3.5) to the integral representation given by (2.13), and obtain by (3.7)

$$\lambda r_1 = \|A_1\overline{y}\| \leq \Gamma_1(k)\|\overline{y}\| + \Gamma_1(h) \leq \Gamma_1(h)\big(1-\Gamma_1(k)\big)^{-1} = r_1$$

which contradicts the assumption that $\lambda > 1$. Now Schauder$'$s Fixed point theroem shows that there exists $\widehat{y} \in \overline{\Omega}_{r_1}$ such $A_1\widehat{y} = \widehat{y}$. Since $f(t,0) \not\equiv 0$, so $\widehat{y}$ cannot be the identically zero solution. This complets of the proof.

Corollary 1 Suppose that $f(t,y)$ satisfies the assumptions of Theorem 1. If $k \in L^1(0,1)$ satisfies either

$$\widehat{\Lambda}_1(k) = \left(1 + \left|\frac{\alpha}{1-\alpha}\right|\right)\int_0^1 (1-s)k(s)ds + \frac{|\beta|}{|1-\alpha|}\int_0^\eta (s)ds < 1 \tag{3.8}$$

or

$$\widehat{\Gamma}_1(k) = \left(1 + \left|\frac{1}{1-\alpha}\right|\right) \int_0^1 (1-s)k(s)ds + \frac{|\alpha|}{|1-\alpha|} \int_0^\eta (\eta - s)(s)ds + \frac{|\beta|}{|1-\alpha|} \int_0^\eta k(s)ds < 1, \tag{3.9}$$

then the three-point boundary value problem (1.1), (1.7) has at least one non-trivial solution.

Proof. From (2.10), we have $G_1[k](\eta) = -I[k](\eta) + I[k](1)$ so $|\alpha|G_1[k](\eta) \leq |\alpha|I[k](1)$. Using this in (3.1), we obtain (3.8). Next we note that (3.9) is simply (3.5) with $m = 1$. This completes the proof.

REMARK 3 Condition (3.9) reduces to (1.4) when $\beta = 0$ and it becomes (1.6) when $\alpha = 0$. Thus Corollary 1 includes both Theorem A and B. Condition (3.8) is sharper than condition (1.6) when $\alpha = 0$ where the "2" can be replaced by "1", so Corollary 1 improves upon Theorem B. When $\beta = 0$, conditions (3.8) and (3.9) are not strictly comparable because their values depend on α and η.

4. Boundary value problem (BVP2), (BVP3)

We now use the integral representations (2.1), (2.2), (2.4) and (2.1), (2.2), (2.5) and state analogues of Theorems 1 and 2 for (BVP2),(BVP3).

Theorem 3 Suppose that $f(t, 0) \not\equiv 0$ in $[0, 1]$ and condition (1.2) holds with $k, h \in L^1(0, 1)$. If $k(t)$ satisfies for $\triangle = 1 - \langle\alpha, \eta\rangle + \overline{\beta} \neq 0$

$$\Lambda_2(k) = \underset{0 \leq t \leq 1}{\mathrm{Max}}\, G_2[k](t) + \frac{1}{\triangle}\{\langle|\alpha|, G_2[k](\eta)\rangle + \langle|\beta|, G_2'[k](\eta)\rangle\} < 1, \tag{4.1}$$

where $G_2[k](t)$ is given by (3.3), then the (BVP2) has at least one non-trivial solution.

Theorem 4 Under the same assumptions as in Theorem 3, if $k(t)$ satisfies

$$\Lambda_3(k) = \underset{0 \leq t \leq 1}{\mathrm{Max}}\, G_3[k](t) + \frac{1}{\triangle}\{\langle|\alpha|, G_3[k](\eta)\rangle + \langle|\beta|, G_3'[k](\eta)|\rangle\} < 1, \tag{4.2}$$

where $G_3[k](t)$ is defined by (3.3), then the (BVP3) has at least one non-trivial solution.

Likewise we use representations (2.14), (2.15) for operators A_2, A_3 in terms of $I[y](t)$ as defined by (2.9) and can prove the following results for (BVP2), (BVP3).

Theorem 5 Under the same assumptions of Theorem 3, if $k(t)$ satisfies

$$\Gamma_2(k) = I[k](1) + \frac{1}{\triangle}\{I'[k](1) + \langle|\alpha|, I[k](\eta)\rangle + \langle|\beta|, I'[k](\eta)\rangle\} < 1, \tag{4.3}$$

then the boundary value problem (BVP2) has at least one non-trivial solution.

Theorem 6 Under the same assumptions of Theorem 5, if $k(t)$ satisfies

$$\Gamma_3(k) = I[k]\left(1 + \frac{1}{\triangle}\right) + \frac{1}{\triangle}\{\langle|\alpha|, I[k](\eta)\rangle + \langle|\beta|, I'[k](\eta)\rangle\} < 1, \tag{4.4}$$

then the boundary value problem (BVP3) has at least one non-trivial solution.

The proofs of Theorems 3, 4, 5, 6 are similar to those given for Theorem 1 and 2 and we shall not repeat them here.

Remark 4 Denote $K_1 = I[k](1), K_2 = I'[k](1)$. We can give upper bounds of $\Gamma_1(k), \Gamma_2(k), \Gamma_3(k)$ in terms of K_1, K_2 as follows

$$\begin{aligned}
\Gamma_1(k) &\le K_1\left\{1 + \frac{1}{|1-\alpha|}(1 + |\widehat{\alpha}|)\right\} + K_2\frac{|\widehat{\beta}|}{|1-\alpha|},\\
\Gamma_2(k) &\le K_1\left(1 + \frac{1}{\triangle}|\widehat{\alpha}|\right) + K_2\left(1 + \frac{1}{\triangle}|\widehat{\beta}|\right),\\
\Gamma_3(k) &\le K_1\left\{1 + \frac{1}{\triangle}(1 + |\widehat{\alpha}|)\right\} + K_2\frac{|\widehat{\beta}|}{\triangle},
\end{aligned}$$

where $|\hat{\alpha}| = \sum_{i=1}^{m} |\alpha_i|, |\hat{\beta}| = \sum_{i=1}^{m} |\beta_i|$. This provides a convenient method to establish existence of a non-trivial solution for (BVP1), (BVP2), (BVP3).

5. Discussion

We illustrate our results with examples in three point boundary value problems and begin with two examples discussed in [10], [11].

Example 1 Consider the boundary value problem

$$(E1)\begin{cases} y'' + c\sqrt{t}(1+y^4)^{-1}y^3 - \sin t = 0, & 0 < t < 1\\ y'(0) = 0, \quad y(1) = 2y(\tfrac{1}{2}), \quad c > 0, \end{cases}$$

which was discussed in [11, Example 3] with $c = 1$ and was shown to possess at least one non-trivial solution. Here $f(t,y) = c\sqrt{t}y^3(1+y^4)^{-1}$ so $|f(t,y)| \le k(t)|x| + h(t)$ with $k(t) = \frac{c}{2}\sqrt{t}$ and $h(t) = \sin t$. Apply Corollary 1 with $\beta = 0, \alpha = 2$ and $\eta = \frac{1}{2}$, we find $c < 60(16+7\sqrt{2})^{-1}$, so in particular (E1) has a non-trivial solution for $c = 2$.

Example 2 Consider the boundary value problem

$$(E2)\begin{cases} y'' + (t-t^2)|y|\sin y - t^2 y + t^3 - 2\sin t = 0, \ 0 < t < 1, \\ y'(0) = 0, \qquad y(1) = \alpha y(\frac{1}{2}), + \beta y'(\frac{1}{2}). \end{cases}$$

This example was studied in [10; Example 4.1] with $\alpha = 0$, $\beta = 4$. Here $f(t,y)$ satisfies (1.2) with $k(t) = t, h(t) = t^3 + 2\sin t$. Apply (3.9) in Corollary 1, we find $|\beta| < 16/3$ the same as from Theorem B. However, using (3.8) in Corollary 1 with $\alpha = 0$, we obtain $|\beta| < 20/3$ which ensures the existence of a nontrivial solution of (E2). When $\beta = 0$, (3.8) requires

$$\frac{1}{6}\left(1 + \frac{|\alpha|}{|1-\alpha|}\right) + \frac{1}{2}\frac{1}{|1-\alpha|} < 1.$$

Solving the above inequality, we require $\alpha \notin [\frac{1}{3}, \frac{1}{2}]$ for the existence of a non-trivial solution of (E2).

Example 3 Consider the boundary value problem

$$(E3)\begin{cases} y'' + \frac{\sigma}{\sqrt{t}}(|y|\sin y + t^2) + \frac{\cos t}{\sqrt{t}} = 0, \ 0 < t < 1, \\ y(0) = 0, \qquad y'(1) = \frac{3}{10}y(\frac{1}{3}) + \frac{1}{10}y'(\frac{1}{3}). \end{cases}$$

where $\sigma > 0$ and (1.2) is satisfied with $k(t) = \sigma t^{-\frac{1}{2}}$ and $h(t) = t^{-\frac{1}{2}} + \sigma$. The boundary value problem is a special case of (BVP2) and we can apply Theorem 3 to compute $\Lambda_2(k)$ as defined by (4.1). Here $\triangle = 4/5$.

$$\operatorname*{Max}_{0\le t\le 1} G_2[k](t) = \operatorname*{Max}_{0\le t\le 1} \int_0^1 g_2(t,s)k(s)ds \le \sigma \int_0^1 \sqrt{s}ds = \frac{2\sigma}{3},$$
$$G_2[k]\left(\frac{1}{3}\right) = \left(\frac{2}{3} - \frac{4}{9\sqrt{3}}\right)\sigma, \ G_2'[k]\left(\frac{1}{3}\right) = \left(2 - \frac{2}{\sqrt{3}}\right)\sigma,$$

hence $\Lambda_2(k) \le \left(\frac{7}{6} - \frac{5\sqrt{3}}{36}\right)\sigma = 0.9261\sigma < 1$. In particular when $\sigma = 1$, the boundary value problem (E3) has a non-trivial solution.

Example 4 Consider the three point BVP

$$(E4)\begin{cases} y'' + \frac{2ty^2e^{-y}}{t^2+y^2} + 3\sin^2 t - \cos e^t = 0, \ 0 < t < 1 \\ y(0) = 0, \qquad y(1) = 4y\left(\frac{1}{2}\right) + \beta y'\left(\frac{1}{2}\right). \end{cases}$$

A similar equation in (E4) was discussed in [10; Example 4.5] as a special case of (BVP1). The boundary value problem (E4) is a special case of (BVP3). Here (1.2) is satisfied with $k(t) \equiv 1$, $h(t) = 3\sin^2 t + \cos e^t$. We now apply Theorem 4 and compute $\Lambda_3(k)$ as given in (4.2). Observe that

$$\operatorname*{Max}_{0\le t\le 1} G_3[k](t) : \operatorname*{Max}_{0\le t\le 1} \int_0^1 g_3(t,s)k(s)ds = \int_0^1 s(1-s)ds = \frac{1}{6},$$

$G_3[k]\left(\frac{1}{2}\right) = \frac{5}{16}$, $G_3'[k]\left(\frac{1}{2}\right) = 0$. Using these in (4.2), we find $\left|\frac{1}{\beta+1}\right| < \frac{2}{3}$, alternatively $\beta \notin \left[-\frac{5}{2}, \frac{1}{2}\right]$, which shows that the boundary value problem (E4) has a non-trivial solution, when $\beta \ge \frac{1}{2}$.

We close our discussion with several additional remarks:

1. The condition that $\overline{\alpha} \neq 1$ for (BVP1) and $\langle \alpha, \eta\rangle + \overline{\beta} \neq 1$ for (BVP2), (BVP3) are known as non-resonance conditions. These conditions ensure that the constants C_j, D_j in (2.6), (2.7), (2.8) can be determined by requiring $A_j y(t)$, as defined by the operator equation (2.1) (2.2), to satisfy the boundary conditions (BC1), (BC2), (BC3).

2. Consider the simple three point boundary value problem

$$(E5) \qquad y'' + y = 1, \qquad y'(0) = 0, \ \ y(1) = \beta y'\left(\frac{1}{2}\right),$$

a special case of (BVP1). With $k(t) \equiv 1$ in (1.6), Theorem B is not applicable. We can use (3.8) in Corollary 1 and find $\widehat{\Lambda}_1(k) = \frac{1}{2}(1 + |\beta|) < 1$, or $|\beta| < 1$. However, (E5) admits an exact unique solution $y(t) = 1 - (\cos 1 + \beta \sin \frac{1}{2})^{-1}$ cost for all $\beta \neq \cos 1/\sin\frac{1}{2}$. This shows that conditions (3.8) and (3.9) are not the best possible.

3. We refer the reader to the papers by Liu and Yu [5] which discussed similar problem in resonant cases where $\|k\|$ is also required to be small as compared with the value 1. There are also recent papers by Han and Wu [4], and Liu, Liu and Wu [6] which dealt with sign-changing nonlinearities like condition (1.2) by comparison with the smallest positive eigenvalue of certain associated linear boundary value problem.

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(Received December 14, 2009)