Existence of standing wave solutions for coupled quasilinear Schrödinger systems with critical exponents in $\mathbb{R}^N$

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Abstract. This paper is concerned with the following quasilinear Schrödinger system in $\mathbb{R}^N$:

$$\begin{cases} -\varepsilon^2 \Delta u + V_1(x)u - \varepsilon^2 \Delta(u^2)u = K_1(x)|u|^{22^*-2}u + h_1(x,u,v)u, \\ -\varepsilon^2 \Delta v + V_2(x)v - \varepsilon^2 \Delta(v^2)v = K_2(x)|v|^{22^*-2}v + h_2(x,u,v)v, \end{cases}$$

where $N \geq 3$, $V_i(x)$ is a nonnegative potential, $K_i(x)$ is a bounded positive function, $i = 1, 2$. $h_1(x, u, v)u$ and $h_2(x, u, v)v$ are superlinear but subcritical functions. Under some proper conditions, minimax methods are employed to establish the existence of standing wave solutions for this system provided that ε is small enough, more precisely, for any $m \in \mathbb{N}$, it has m pairs of solutions if ε is small enough. And these solutions $(u_\varepsilon, v_\varepsilon) \to (0, 0)$ in some Sobolev space as $\varepsilon \to 0$. Moreover, we establish the existence of positive solutions when $\varepsilon = 1$. The system studied here can model some interaction phenomena in plasma physics.

Keywords: quasilinear Schrödinger system, critical growth, standing wave solutions, mountain pass theorem, $(PS)_c$ sequence.

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1 Introduction

In this article we discuss the following coupled quasilinear Schrödinger system with critical exponents in $\mathbb{R}^N$

$$\begin{cases} -\varepsilon^2 \Delta u + V_1(x)u - \varepsilon^2 \Delta(u^2)u = K_1(x)|u|^{22^*-2}u + h_1(x,u,v)u, \\ -\varepsilon^2 \Delta v + V_2(x)v - \varepsilon^2 \Delta(v^2)v = K_2(x)|v|^{22^*-2}v + h_2(x,u,v)v. \end{cases} \tag{1.1}$$

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In recent years, much attention has been devoted to the quasilinear Schrödinger equation of the form:

$$- \varepsilon^2 \Delta u + V(x)u - \varepsilon^2 \Delta(u^2)u = h(x,u), \tag{1.2}$$

where $\varepsilon > 0$ is a small parameter (e.g. see [28,31]). Part of the interest is due to the fact that the solution of (1.2) is closely related to the existence of solitary wave solutions for the following equation:

$$i\varepsilon \partial_t w = -\varepsilon^2 \Delta w + V(x)w - f(|w|^2)w - \varepsilon^2 k \Delta h(|w|^2)h'(|w|^2)w, \tag{1.3}$$

where $w : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $V(x)$ is a given potential, k is a real constant, f, h are suitable functions. In fact, the quasilinear equation (1.3) has been derived as models of several physical phenomena. For example, it models the superfluid film equation in plasma physics [20], in self-channeling of a high-power ultra short laser in matter [3,6,24], in condensed matter theory [22] etc. It is worth pointing out that the related semilinear Schrödinger equation arises in many mathematical physics problems and has been extensively studied. We only mention [9,11,19,23] and the references therein. Also, there are more and more papers being concerned with semilinear Schrödinger system involving two condensate amplitudes w_1, w_2. For example, Chen and Zhou [7] proved the uniqueness of positive solutions under some conditions for a coupled Schrödinger system. Tang [27] was concerned with multi-peak solutions to coupled Schrödinger systems with Neumann boundary conditions in a bounded domain of $\mathbb{R}^N$ for $N = 2, 3$ and proved that all peaks locate either near the local maxima or near the local minima of the mean curvature at the boundary of the domain. Yang, Wei and Ding [30] studied a Schrödinger system with nonlocal nonlineatities of Hartree type. Ye and Peng [32] considered a coupled Schrödinger system with doubly critical exponents on $\mathbb{R}^N$, which can be seen as a counterpart of the Brezis–Nirenberg problem.

Recently quasilinear systems also have been the focus for some researchers (e.g. [16,17,25]). But compared with semilinear systems, only a few papers are known for them. Guo and Tang [17] proved the existence of a ground state solution by using Nehari manifold and concentration compactness principle in a Orlicz space. Severo and Silva [25] established the existence of standing wave solutions for quasilinear Schrödinger systems involving subcritical nonlinearities in Orlicz spaces. By referring to some arguments and methods in [11,25,30,31], we consider the quasilinear Schrödinger systems (1.1) with critical nonlinearities and discuss the existence of a positive solution and multiple solutions as ε is small. Of particular interest to our paper is the results in [31], where the authors investigated the quasilinear Schrödinger equation (1.2) with critical exponent $h(x,u) = K(x)|u|^{22^*-2} + H_u(x,u)$ and proved it has at least one positive solution and multiple solutions when ε is small,where $H_u(x,u)$ is a superlinear but subcritical function and satisfies some suitable conditions. The difficulty is caused by the usual lack of compactness since these problems involve critical exponents and are dealt with in the whole $\mathbb{R}^N$. We remark that most papers above use the Cerami condition. But in this paper we prove that $(PS)_c$ condition also holds. We suppose that the following assumptions are satisfied, where $i = 1, 2$:

(V_1) $V_i \in C(\mathbb{R}^N, \mathbb{R})$ and there is a constant $b > 0$ such that $\mathrm{m}\{x \in \mathbb{R}^N : V_i(x) < b\} < \infty$, where m denotes the Lebesgue measure;

(V_2) $0 = V_i(0) \leq V_i(x) \leq \max V_i < +\infty$;

(K) $0 < C \leq K_i \in C(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N)$.

The functions $h_1, h_2 \in C(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^+)$ and satisfy the following conditions.

(H_1) There is a constant $4 < \mu < 22^*$ satisfying $\mu H(x,u,v) \le h_1(x,u,v)u^2 + h_2(x,u,v)v^2$ for all $(x,u,v) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$, where $H(x,u,v) = \int_0^u h_1(x,t,v)t dt = \int_0^v h_2(x,u,t)t dt$.

(H_2) $h_1(x,u,v)u = o(|(u,v)|)$ and $h_2(x,u,v)v = o(|(u,v)|)$ uniformly in $x \in \mathbb{R}^N$ as $(u,v) \to (0,0)$.

(H_3) There exist constants $C_1, C_2 > 0$ and $p \in [3, 22^* - 1)$ such that $|h_1(x,u,v)u| + |h_2(x,u,v)v| \le C_1 + C_2|(u,v)|^{p-1}$ for all $(x,u,v) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$.

(H_4) $H(x,u,v) \ge C[|(u,v)|^2 + |(u,v)|]^q$, where $q \in (2, 2^*)$ is a constant.

(H_5) $h_1(x,-u,v) = h_1(x,u,v)$ and $h_2(x,u,-v) = h_2(x,u,v)$ for all $(x,u,v) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$.

Notations. We collect below a list of the main notation used throughout this paper.

- C will denote various positive constants whose value may change from line to line.

- If the functions f and g satisfy $\left|\frac{f(x)}{g(x)}\right| \le C$, $x \in U^0(x_0)$, then we define $f(x) = O(g(x))$ as $x \to x_0$.

- $|\mathbf{u}|$ denote the Euclidean norm of $\mathbf{u} \in \mathbb{R}^2$.

- The domain of integration is $\mathbb{R}^N$ by default.

- $\int f(x)dx$ will be represented by $\int f(x)$.

- We use $L^s(\mathbb{R}^N), 1 \le s \le \infty$, to denote the usual Lebesgue spaces with the norms

$$|u|_s := \left(\int |u|^s\right)^{\frac{1}{s}}, \qquad 1 \le s < \infty,$$

$$\|u\|_\infty := \inf\{C > 0 : |u(x)| \le C \text{ almost everywhere in } \mathbb{R}^N\}.$$

- S denotes the best Sobolev constant for $H^1(\mathbb{R}^N)$.

Theorem 1.1. *Assume that* (V_1)–(V_2), (K) *and* (H_1)–(H_5) *are satisfied. Then for any* $\sigma > 0$, *there is* $\tau_\sigma > 0$ *such that if* $\varepsilon \le \tau_\sigma$, *system* (1.1) *has at least one positive solution* $\mathbf{u}_\varepsilon = (u_\varepsilon, v_\varepsilon)$. *Moreover, for any* $m \in \mathbb{N}$ *and* $\sigma > 0$, *there is* $\tau_{\sigma m} > 0$ *such that if* $\varepsilon \le \tau_{\sigma m}$, *system* (1.1) *has at least m pairs of solutions* $\mathbf{u}_\varepsilon = (u_\varepsilon, v_\varepsilon) \to (0,0)$ *in* $\mathbb{E}$ *as* $\varepsilon \to 0$, *where* $\mathbb{E}$ *is stated later, satisfying*

$$\frac{\mu - 4}{2\mu} \int [\varepsilon^2(1 + 2u_\varepsilon^2)|\nabla u_\varepsilon|^2 + V_1(x)u_\varepsilon^2 + \varepsilon^2(1 + 2v_\varepsilon^2)|\nabla v_\varepsilon|^2 + V_2(x)v_\varepsilon^2] \le \sigma\varepsilon^N$$

and

$$\frac{1}{2N} \int [K_1(x)|u_\varepsilon|^{22^*} + K_2(x)|v_\varepsilon|^{22^*}] + \frac{\mu - 4}{4} \int H(x, u_\varepsilon, v_\varepsilon) \le \sigma\varepsilon^N.$$

The existence and multiplicity of solutions for system (1.1) depends on the small parameter ε. If the parameter ε is not small enough, such as $\varepsilon \equiv 1$, we cannot get the similar results as Theorem 1.1 unless we add some suitable conditions, where $i = 1, 2$:

(V_3) $V_i \in C(\mathbb{R}^N, \mathbb{R})$ is 1-periodic in x_j, $1 \le j \le N$, and there is a constant $a_0 > 0$ such that $V_i(x) \ge a_0 > 0$, $\forall x \in \mathbb{R}^N$.

(K') $K_i \in C(\mathbb{R}^N, \mathbb{R})$ is 1-periodic in x_j, $1 \le j \le N$, and there is a point $x_0 \in \mathbb{R}^N$ such that

(i) $K_i(x_0) = \sup_{x \in \mathbb{R}^N} K_i(x) > 0$.

(ii) $K_i(x) = K_i(x_0) + O(|x - x_0|^2)$, as $x \to x_0$.

The functions $h_1, h_2 \in C(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^+)$ and satisfy (H_1)–(H_4) and

(H_6) h_1, h_2 is 1-periodic in x_j, $1 \le j \le N$.

Theorem 1.2. *Let $\varepsilon = 1$. Assume that (V_3), (K'), (H_1)–(H_4) and (H_6) are satisfied. Then system (1.1) has at least one positive solution $\mathbf{u} = (u, v)$ if N and q satisfy one of the following two conditions:*

(N_1) $3 \le N < 6$ and $\frac{N+2}{N-2} < q < 2^*$;

(N_2) $N \ge 6$ and $2 < q < 2^*$.

Remark 1.1. *Guo and Li in [18] discussed a class of modified nonlinear Schrödinger systems*

$$\begin{cases} \Sigma_{i,j=1}^N D_j(a_{ij}(u)D_i u) - \frac{1}{2}\Sigma_{i,j=1}^N D_s a_{ij}(u)D_i u D_j u - a(x)u + F_u(u,v) = 0, \\ \Sigma_{i,j=1}^N D_j(a_{ij}(v)D_i v) - \frac{1}{2}\Sigma_{i,j=1}^N D_s a_{ij}(v)D_i v D_j v - a(x)v + F_v(u,v) = 0, \end{cases} \quad (1.4)$$

where $F(u,v) = |u|^\alpha |v|^\beta + |u|^p |v|^q$, $\alpha, \beta, p, q > 1$, $\alpha + \beta = 22^*$ and $4 < p + q < 22^*$, and they proved the existence of a ground state positive solution by using a perturbation method. For the special case of $a_{ij}(s) = (1 + 2s^2)\delta_{ij}$, system (1.4) can be rewritten as

$$-\Delta u_j + V_j(x)u_j - \Delta(u_j^2)u_j = \Sigma_{i \ne j}^2 \beta_{ij}(|u_i|^{\alpha_i}|u_j|^{\beta_j} + |u_i|^{p_i}|u_j|^{q_j}), \qquad j = 1, 2. \quad (1.5)$$

Comparing with (1.5), the coupling term in the present paper is not critical growth, but is more general than the coupling subcritical term of (1.5). The subcritical nonlinearities of (1.5) do not satisfy our condition (H_4). Hence, the proof in this paper is different from the one in [18].

The organization of this paper is as follows. In Section 2, we introduce the variational framework and restate the problem in a equivalent form by replacing ε^{-2} with λ. Furthermore, we reduce the quasilinear problem into a semilinear one by making change of variables and show some preliminary results. In Section 3, we prove the behaviors of the bounded $(PS)_c$ sequences and then show that the energy functional satisfies the $(PS)_c$ condition under some suitable conditions. In Section 4, we verify the geometry of the mountain pass theorem and estimate the minimax values. In Section 5, we complete the proof of Theorem 1.1. In the final section, we prove Theorem 1.2.

2   An equivalent variational problem

To prove the existence of standing wave solutions of system (1.1) for small ε, we rewrite (1.1) in a equivalent form. Let $\lambda = \varepsilon^{-2}$. Then system (1.1) can be rewritten as

$$\begin{cases} -\Delta u + \lambda V_1(x)u - \Delta(u^2)u = \lambda K_1(x)|u|^{22^*-2}u + \lambda h_1(x, u, v)u, \\ -\Delta v + \lambda V_2(x)v - \Delta(v^2)v = \lambda K_2(x)|v|^{22^*-2}v + \lambda h_2(x, u, v)v, \end{cases} \quad (2.1)$$

for $\lambda \to +\infty$.

We introduce the Hilbert spaces

$$E_i := \left\{ u \in H^1(\mathbb{R}^N) : \int V_i(x)u^2 < \infty \right\}$$

with inner products

$$(u, v)_i := \int \nabla u \nabla v + V_i(x) u v$$

and the associated norms

$$\|u\|_i^2 = (u, u)_i, \qquad i = 1, 2.$$

We shall work in the product space $\mathbb{E} = E_1 \times E_2$ with elements $\mathbf{u} = (u, v)$. Thus, the norm in $\mathbb{E}$ can be defined as $\|\mathbf{u}\|^2 = \|u\|_1^2 + \|v\|_2^2$. It follows from (V_1) and (V_2) that E_i embeds continuously in $H^1(\mathbb{R}^N)$ (e.g. see [12]) and consequently $\mathbb{E}$ embeds continuously in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Notice that the norm $\| \cdot \|_i$ is equivalent to $\| \cdot \|_{i,\lambda}$ induced by the inner product

$$(u, v)_{i,\lambda} := \int \nabla u \nabla v + \lambda V_i(x) u v$$

for each $\lambda > 0$. Hence $\| \cdot \|$ is equivalent to the norm $\| \cdot \|_\lambda$ induced by

$$(\mathbf{u}, \mathbf{v})_\lambda := \int \nabla u_1 \nabla v_1 + \lambda V_1(x) u_1 v_1 + \int \nabla u_2 \nabla v_2 + \lambda V_2(x) u_2 v_2.$$

It is thus clear that, for each $s \in [2, 2^*]$, there is a $\nu_s > 0$ being independent of λ such that if $\lambda \geq 1$

$$|\mathbf{u}|_s \leq \nu_s \|\mathbf{u}\| \leq \nu_s \|\mathbf{u}\|_\lambda, \qquad \forall \mathbf{u} \in \mathbb{E},$$

where $| \cdot |_s$ denotes the standard norm in $L^s(\mathbb{R}^N) \times L^s(\mathbb{R}^N)$.

Associated to system (2.1), the energy functional is

$$J(\mathbf{u}_1) := \frac{1}{2} \int (1 + 2u_1^2)|\nabla u_1|^2 + \lambda V_1(x) u_1^2 + (1 + 2v_1^2)|\nabla v_1|^2 + \lambda V_2(x) v_1^2$$
$$- \frac{\lambda}{22^*} \int K_1(x)|u_1|^{22^*} + K_2(x)|v_1|^{22^*} - \lambda \int H(x, u_1, v_1),$$

which is not well defined in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. To save from this trouble, we make use of a change of variables $u := f^{-1}(u_1)$, $v := f^{-1}(v_1)$ (see [8,10,13,21]), where f is defined by

$$f'(t) = \frac{1}{\sqrt{1 + 2f^2(t)}} \quad \text{on } [0, +\infty) \quad \text{and} \quad f(t) = -f(-t) \quad \text{on } (-\infty, 0].$$

We list some properties of f. Their proofs may be found in the above references.

Lemma 2.1. *The function f satisfies the following properties:*

(i) *f is uniquely defined, C^∞ and invertible;*

(ii) *$|f'(t)| \leq 1$ for all $t \in \mathbb{R}$;*

(iii) *$|f(t)| \leq |t|$ for all $t \in \mathbb{R}$;*

(iv) *$f(t)/t \to 1$ as $t \to 0$;*

(v) *$f(t)/\sqrt{t} \to 2^{1/4}$ as $t \to +\infty$;*

(vi) *$f(t)/2 \leq t f'(t) \leq f(t)$ for $t \geq 0$;*

(vii) *$|f(t)| \leq 2^{1/4}|t|^{1/2}$ for all $t \in \mathbb{R}$;*

(viii) *there exists a positive constant C such that $|f(t)| \geq C|t|$ for $|t| \leq 1$ and $|f(t)| \geq C|t|^{1/2}$ for $|t| \geq 1$;*

(ix) *$|f(t)f'(t)| < 1/\sqrt{2}$ for all $t \in \mathbb{R}$;*

(x) *there exists a positive constant A such that*

$$f^{22^*}(t) = 2^{\frac{N}{N-2}} t^{2^*} - A t^{2^*-1} \ln t + O(t^{2^*-1}), \quad \text{as } t \to +\infty.$$

After the change of variables, we obtain the following functional

$$\Phi_\lambda(\mathbf{u}) := \frac{1}{2} \int |\nabla u|^2 + \lambda V_1(x) f^2(u) + |\nabla v|^2 + \lambda V_2(x) f^2(v)$$
$$- \frac{\lambda}{22^*} \int K_1(x)|f(u)|^{22^*} + K_2(x)|f(v)|^{22^*} - \lambda \int H(x, f(u), f(v)).$$

Then Φ_λ is well-defined on $\mathbb{E}$ and belongs to C^1 under hypotheses (V_1), (V_2), (K) and (H_3). Furthermore, we can check that

$$\langle \Phi'_\lambda(\mathbf{u}), \mathbf{w} \rangle = \langle \Phi'_\lambda(u, v), (\varphi, \psi) \rangle$$
$$= \int \nabla u \nabla \varphi + \lambda V_1(x) f(u) f'(u) \varphi + \nabla v \nabla \psi + \lambda V_2(x) f(v) f'(v) \psi$$
$$- \lambda \int K_1(x)|f(u)|^{22^*-2} f(u) f'(u) \varphi + K_2(x)|f(v)|^{22^*-2} f(v) f'(v) \psi$$
$$- \lambda \int h_1(x, f(u), f(v)) f(u) f'(u) \varphi + h_2(x, f(u), f(v)) f(v) f'(v) \psi,$$

for all $\mathbf{u}, \mathbf{w} \in \mathbb{E}$. We observe that if $\mathbf{u} = (u, v) \in \mathbb{E}$ is a critical point of the functional Φ_λ, then it is a weak solution of the following system associated with the functional Φ_λ

$$\begin{cases} -\Delta u + \lambda V_1(x) f(u) f'(u) = \lambda K_1(x)|f(u)|^{22^*-2} f(u) f'(u) + \lambda h_1(x, f(u), f(v)) f(u) f'(u), \\ -\Delta v + \lambda V_2(x) f(v) f'(v) = \lambda K_2(x)|f(v)|^{22^*-2} f(v) f'(v) + \lambda h_2(x, f(u), f(v)) f(v) f'(v). \end{cases}$$
$$(2.2)$$

Hence $(f(u), f(v))$ is a weak solution of system (2.1) (cf. [8]). Theorem 1.1 can be restated as

Theorem 2.1. *Assume that (V_1)–(V_2), (K) and (H_1)–(H_5) are satisfied. Then for any $\sigma > 0$, there is $\Lambda_\sigma > 0$ such that if $\lambda \geq \Lambda_\sigma$, system (2.2) has at least one positive solution $\mathbf{u}_\lambda = (u_\lambda, v_\lambda)$. Moreover, for any $m \in \mathbb{N}$ and $\sigma > 0$, there is $\Lambda_{\sigma m} > 0$ such that if $\lambda \geq \Lambda_{\sigma m}$, system (2.2) has at least m pairs of solutions $\mathbf{u}_\lambda = (u_\lambda, v_\lambda)$, converging to $(0,0)$ in $\mathbb{E}$ as $\lambda \to \infty$ and satisfying*

$$\frac{1}{2N} \int K_1(x)|f(u_\lambda)|^{22^*} + K_2(x)|f(v_\lambda)|^{22^*} + \frac{\mu - 4}{4} \int H(x, f(u_\lambda), f(v_\lambda)) \leq \sigma \lambda^{-\frac{N}{2}}$$

and

$$\frac{\mu - 4}{2\mu} \int |\nabla u_\lambda|^2 + \lambda V_1(x) f^2(u_\lambda) + |\nabla v_\lambda|^2 + \lambda V_2(x) f^2(v_\lambda) \leq \sigma \lambda^{1-\frac{N}{2}}.$$

Remark 2.1. In order to get the positive solution, we introduce

$$\Phi_\lambda^+(\mathbf{u}) := \frac{1}{2} \int |\nabla u|^2 + \lambda V_1(x) f^2(u) + |\nabla v|^2 + \lambda V_2(x) f^2(v)$$
$$- \frac{\lambda}{22^*} \int K_1(x)|f(u^+)|^{22^*} + K_2(x)|f(v^+)|^{22^*} - \lambda \int H(x, f(u^+), f(v^+)),$$

where $u^+ := \max\{u, 0\}$, $v^+ := \max\{v, 0\}$. Then $\Phi_\lambda^+ \in C^1$ and the critical points of Φ_λ^+ are the positive solutions of system (2.2).

3 Behavior of $(PS)_c$ sequences

At this point, we recall that a sequence $(\mathbf{u}_n) \subset \mathbb{E}$ is a $(PS)_c$ sequence at level c ($(PS)_c$ sequence for short), if $\Phi_\lambda(\mathbf{u}_n) \to c$ and $\Phi'_\lambda(\mathbf{u}_n) \to 0$. Φ_λ is said to satisfy the $(PS)_c$ condition if any $(PS)_c$ sequence contains a convergent subsequence. However, due to the unboundedness of the domain and the critical term, we can not prove the $(PS)_c$ condition holds in general. By establishing several lemmas, we will discuss the behaviors of $(PS)_c$ sequences.

Lemma 3.1. *Suppose that (V_2), (K) and (H_1) hold. Let $(\mathbf{u}_n) \subset \mathbb{E}$ be a $(PS)_c$ sequence for Φ_λ. Then $c \geq 0$ and $(\mathbf{u}_n)$ is bounded in $\mathbb{E}$.*

Proof. Set $(\mathbf{u}_n)$ to be a $(PS)_c$ sequence:

$$\Phi_\lambda(\mathbf{u}_n) \to c, \qquad \Phi'_\lambda(\mathbf{u}_n) \to 0, \qquad n \to \infty.$$

By Lemma 2.1 (vi) and (H_1), one sees that

$$c + o(1) + o(1)\|\mathbf{u}_n\|_\lambda \geq \Phi_\lambda(\mathbf{u}_n) - \frac{2}{\mu}\langle \Phi'_\lambda(\mathbf{u}_n), \mathbf{u}_n\rangle$$

$$\geq \left(\frac{1}{2} - \frac{2}{\mu}\right) \int |\nabla u_n|^2 + \lambda V_1(x)f^2(u_n) + |\nabla v_n|^2 + \lambda V_2(x)f^2(v_n)$$

$$- \left(\frac{1}{22^*} - \frac{1}{\mu}\right) \lambda \int K_1(x)|f(u_n)|^{22^*} + K_2(x)|f(v_n)|^{22^*}. \tag{3.1}$$

Hence

$$\begin{cases} \int |\nabla u_n|^2 + \lambda V_1(x)f^2(u_n) + |\nabla v_n|^2 + \lambda V_2(x)f^2(v_n) \leq c + o(1) + o(1)\|\mathbf{u}_n\|_\lambda, \\ \int K_1(x)|f(u_n)|^{22^*} + K_2(x)|f(v_n)|^{22^*} \leq c + o(1) + o(1)\|\mathbf{u}_n\|_\lambda. \end{cases} \tag{3.2}$$

From (3.2), we only need to prove that $\lambda \int V_1(x)|u_n|^2 + V_2(x)|v_n|^2 \leq c + o(1) + o(1)\|\mathbf{u}_n\|_\lambda$. We write that

$$\lambda \int V_1(x)|u_n|^2 = \lambda \int_{|u_n|\geq 1} V_1(x)|u_n|^2 dx + \lambda \int_{|u_n|\leq 1} V_1(x)|u_n|^2 dx.$$

Combining (V_2), (K), (3.2) and Lemma 2.1 (viii), we have

$$\lambda \int_{|u_n|\geq 1} V_1(x)|u_n|^2 dx \leq C\lambda \max V_1 \int_{|u_n|\geq 1} |f(u_n)|^{22^*} dx$$

$$\leq C \int_{|u_n|\geq 1} K_1(x)|f(u_n)|^{22^*} dx$$

$$\leq c + o(1) + o(1)\|\mathbf{u}_n\|_\lambda$$

and

$$\lambda \int_{|u_n|\leq 1} V_1(x)|u_n|^2 dx \leq \frac{\lambda}{C^2} \int_{|u_n|\leq 1} V_1(x)f^2(u_n)dx$$

$$\leq c + o(1) + o(1)\|\mathbf{u}_n\|_\lambda.$$

Thus $\lambda \int V_1(x)|u_n|^2 \leq c + o(1) + o(1)\|\mathbf{u}_n\|_\lambda$. Similarly, we can get $\lambda \int V_2(x)|v_n|^2 \leq c + o(1) + o(1)\|\mathbf{u}_n\|_\lambda$. Then $\|\mathbf{u}_n\|_\lambda^2 \leq c + o(1) + o(1)\|\mathbf{u}_n\|_\lambda$. Thus $(\mathbf{u}_n)$ is bounded in $\mathbb{E}$. Taking the limit in (3.1) we shows that $c \geq 0$. $\square$

By the above lemma, we know that every $(PS)_c$ sequence $(\mathbf{u}_n)$ is bounded. We may assume up to a subsequence that $\mathbf{u}_n \rightharpoonup \mathbf{u}$ in $\mathbb{E}$ and in $L^s \times L^s$, $2 \le s \le 2^*$, $\mathbf{u}_n \to \mathbf{u}$ in $L^s_{\text{loc}} \times L^s_{\text{loc}}$, $1 \le s < 2^*$ and $\mathbf{u}_n(x) \to \mathbf{u}(x)$ a.e. in $\mathbb{R}^N$. Clearly $\mathbf{u}$ is a critical point of Φ_λ.

Lemma 3.2. *Let $(\mathbf{u}_n)$ be stated as in Lemma 3.1 and $s \in [2, 2^*)$. There is a subsequence $(\mathbf{u}_{n_j})$ such that for each $\epsilon > 0$, there exists $R_\epsilon > 0$ with*

$$\limsup_{j \to \infty} \int_{B_j \setminus B_R} |u_{n_j}|^s dx \le \epsilon$$

and

$$\limsup_{j \to \infty} \int_{B_j \setminus B_R} |v_{n_j}|^s dx \le \epsilon$$

for all $R \ge R_\epsilon$.

Proof. The proof is similar as that in [11]. We omit it here. □

For notational convenience, we can assume in the following that Lemma 3.2 holds for both $s = 2$ and $s = \frac{p+1}{2}$ with the same subsequence. Let $\eta : [0, \infty) \to [0, 1]$ be a smooth function satisfying $\eta(t) = 1$ if $t \le 1$, $\eta(t) = 0$ if $t \ge 2$. Define $\tilde{\mathbf{u}}_j(x) = \eta\left(\frac{2|x|}{j}\right)\mathbf{u}(x)$. It is known that

$$\|\mathbf{u} - \tilde{\mathbf{u}}_j\|_\lambda \to 0, \quad \text{as } j \to \infty. \tag{3.3}$$

Lemma 3.3. *Let $(\mathbf{u}_{n_j})$ be stated as in Lemma 3.2. Then*

$$\lim_{j \to \infty} \int |f(u_{n_j})|^p - |f(u_{n_j} - \tilde{u}_j)|^p - |f(\tilde{u}_j)|^p = 0$$

and

$$\lim_{j \to \infty} \int |f(v_{n_j})|^p - |f(v_{n_j} - \tilde{v}_j)|^p - |f(\tilde{v}_j)|^p = 0,$$

where $p \in [2, 22^*]$.

Proof. We only show that the first equality holds. As in [29], for any fixed $\epsilon > 0$, there exists $C_\epsilon > 0$ such that, for all $a, b \in \mathbb{R}$

$$||a + b|^q - |a|^q| \le \epsilon|a|^q + C_\epsilon|b|^q, \qquad 1 \le q < +\infty.$$

We deduce that, by Lemma 2.1 (ix), for any fixed $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$
\begin{aligned}
|f(u_{n_j})|^p - |f(u_{n_j} - \tilde{u}_j)|^p &= |f^2(u_{n_j})|^{\frac{p}{2}} - |f^2(u_{n_j} - \tilde{u}_j)|^{\frac{p}{2}} \\
&\le \epsilon|f^2(u_{n_j} - \tilde{u}_j)|^{\frac{p}{2}} + C_\epsilon|f^2(u_{n_j}) - f^2(u_{n_j} - \tilde{u}_j)|^{\frac{p}{2}} \\
&\le \epsilon|f(u_{n_j} - \tilde{u}_j)|^p + C_\epsilon|2f(u_{n_j} - \theta\tilde{u}_j)f'(u_{n_j} - \theta\tilde{u}_j)\tilde{u}_j|^{\frac{p}{2}} \\
&\le \epsilon|f(u_{n_j} - \tilde{u}_j)|^p + C_\epsilon|\tilde{u}_j|^{\frac{p}{2}},
\end{aligned}
$$

where and below $\theta \in (0, 1)$. Then by Lemma 2.1 (vii)

$$
\begin{aligned}
\Gamma^\epsilon_{n_j} &:= (|f(u_{n_j})|^p - |f(u_{n_j} - \tilde{u}_j)|^p - |f(\tilde{u}_j)|^p - \epsilon|f(u_{n_j} - \tilde{u}_j)|^p)^+ \\
&\le |f(\tilde{u}_j)|^p + C_\epsilon|\tilde{u}_j|^{\frac{p}{2}} \\
&\le C_\epsilon|u|^{\frac{p}{2}}.
\end{aligned}
$$

The Lebesgue Dominated Convergence Theorem implies that $\int \Gamma_{n_j}^\epsilon \to 0$ as $j \to \infty$. Hence

$$\int |f(u_{n_j})|^p - |f(u_{n_j} - \tilde{u}_j)|^p - |f(\tilde{u}_j)|^p \le \int \Gamma_{n_j}^\epsilon + \epsilon |f(u_{n_j} - \tilde{u}_j)|^p \le C\epsilon.$$

$\square$

Lemma 3.4. *Let $(\mathbf{u}_{n_j})$ be stated as in Lemma 3.2. Denote by*

$$h_{n_j}(x) := h_1(x, f(u_{n_j}), f(v_{n_j})) f(u_{n_j}) f'(u_{n_j}) - h_1(x, f(\tilde{u}_{n_j}), f(\tilde{v}_{n_j})) f(\tilde{u}_{n_j}) f'(\tilde{u}_{n_j})$$
$$- h_1(x, f(u_{n_j} - \tilde{u}_j), f(v_{n_j} - \tilde{v}_j)) f(u_{n_j} - \tilde{u}_j) f'(u_{n_j} - \tilde{u}_j)$$

and

$$g_{n_j}(x) := h_2(x, f(u_{n_j}), f(v_{n_j})) f(v_{n_j}) f'(v_{n_j}) - h_2(x, f(\tilde{u}_{n_j}), f(\tilde{v}_{n_j})) f(\tilde{v}_{n_j}) f'(\tilde{v}_{n_j})$$
$$- h_2(x, f(u_{n_j} - \tilde{u}_j), f(v_{n_j} - \tilde{v}_j)) f(v_{n_j} - \tilde{v}_j) f'(v_{n_j} - \tilde{v}_j).$$

We have

$$\lim_{j \to \infty} \int h_{n_j}(x)\varphi = 0$$

and

$$\lim_{j \to \infty} \int g_{n_j}(x)\psi = 0$$

uniformly for $\|\mathbf{w}\|_\lambda = \|(\varphi, \psi)\|_\lambda \le 1$.

Proof. Note that (3.3) and the local compactness of the Sobolev embedding theorem imply that, for any $R > 0$

$$\lim_{j \to \infty} \left| \int_{B_R} h_{n_j}(x)\varphi dx \right| = 0 \tag{3.4}$$

uniformly for $\|\varphi\|_{1,\lambda} \le 1$. For any $\varepsilon > 0$, from (3.3) and the integrability of $|u|^s$ on $\mathbb{R}^N$, we can choose $R > 0$ such that

$$\limsup_{j \to \infty} \int_{B_j \setminus B_R} |\tilde{u}_j|^s dx \le \int_{B_R^c} |u|^s dx \le \varepsilon.$$

Combining (H_2), (H_3) and Lemma 2.1 (ii), (iii), (vii), we get that

$$|h_1(x, f(u), f(v)) f(u) f'(u)\varphi| \le C[|(f(u), f(v))| + |(f(u), f(v))|^{p-1}]|\varphi|$$
$$\le C(|u||\varphi| + |v||\varphi| + |u|^{\frac{p-1}{2}}|\varphi| + |v|^{\frac{p-1}{2}}|\varphi|). \tag{3.5}$$

Therefore, it follows from (3.4), (3.5), the Hölder inequality and Lemma 3.2 that

$$\limsup_{j \to \infty} \left| \int h_{n_j}(x)\varphi \right|$$

$$\le \limsup_{j \to \infty} \int_{B_j \setminus B_R} |h_{n_j}(x)\varphi| dx$$

$$\le C \limsup_{j \to \infty} \int_{B_j \setminus B_R} (|u_{n_j}| + |u_{n_j} - \tilde{u}_j| + |\tilde{u}_j|)|\varphi|$$

$$+ (|v_{n_j}| + |v_{n_j} - \tilde{v}_j| + |\tilde{v}_j|)|\varphi| + \left(|u_{n_j}|^{\frac{p-1}{2}} + |u_{n_j} - \tilde{u}_j|^{\frac{p-1}{2}} + |\tilde{u}_j|^{\frac{p-1}{2}} \right) |\varphi|$$

$$+ \left(|v_{n_j}|^{\frac{p-1}{2}} + |v_{n_j} - \tilde{v}_j|^{\frac{p-1}{2}} + |\tilde{v}_j|^{\frac{p-1}{2}} \right) |\varphi|$$

$$\leq C \limsup_{j \to \infty} \left[\left(\int_{B_j \setminus B_R} |u_{n_j}|^2 dx \right)^{\frac{1}{2}} + \left(\int_{B_j \setminus B_R} |\tilde{u}_j|^2 dx \right)^{\frac{1}{2}} \right.$$

$$\left. + \left(\int_{B_j \setminus B_R} |v_{n_j}|^2 dx \right)^{\frac{1}{2}} + \left(\int_{B_j \setminus B_R} |\tilde{v}_j|^2 dx \right)^{\frac{1}{2}} \right] \left(\int |\varphi|^2 \right)^{\frac{1}{2}}$$

$$+ C \limsup_{j \to \infty} \left[\left(\int_{B_j \setminus B_R} |u_{n_j}|^{\frac{p+1}{2}} dx \right)^{\frac{p-1}{p+1}} + \left(\int_{B_j \setminus B_R} |\tilde{u}_j|^{\frac{p+1}{2}} dx \right)^{\frac{p-1}{p+1}} \right.$$

$$\left. + \left(\int_{B_j \setminus B_R} |v_{n_j}|^{\frac{p+1}{2}} dx \right)^{\frac{p-1}{p+1}} + \left(\int_{B_j \setminus B_R} |\tilde{v}_j|^{\frac{p+1}{2}} dx \right)^{\frac{p-1}{p+1}} \right] \left(\int |\varphi|^{\frac{p+1}{2}} dx \right)^{\frac{2}{p+1}}$$

$$\leq C \epsilon^{\frac{1}{2}} + C \epsilon^{\frac{p-1}{p+1}}$$

uniformly for $\|\varphi\|_{1,\lambda} \leq 1$. Similarly, we can get that the other equality holds. $\qquad \square$

Lemma 3.5. *Let $(\mathbf{u}_{n_j})$ be stated as in Lemma 3.2. One has along a subsequence:*

(i) $\Phi_\lambda(\mathbf{u}_{n_j} - \tilde{\mathbf{u}}_j) \to c - \Phi_\lambda(\mathbf{u})$.

(ii) $\Phi'_\lambda(\mathbf{u}_{n_j} - \tilde{\mathbf{u}}_j) \to 0$.

Proof. (*i*) Obviously, we can see

$$\Phi_\lambda(\mathbf{u}_{n_j} - \tilde{\mathbf{u}}_j) = \Phi_\lambda(\mathbf{u}_{n_j}) - \Phi_\lambda(\tilde{\mathbf{u}}_j) - \frac{\lambda}{2} \int V_1(x)[|f(u_{n_j})|^2 - |f(u_{n_j} - \tilde{u}_j)|^2 - |f(\tilde{u}_j)|^2]$$

$$- \frac{\lambda}{2} \int V_2(x)[|f(v_{n_j})|^2 - |f(v_{n_j} - \tilde{v}_j)|^2 - |f(\tilde{v}_j)|^2]$$

$$+ \frac{\lambda}{22^*} \int K_1(x)[|f(u_{n_j})|^{22^*} - |f(u_{n_j} - \tilde{u}_j)|^{22^*} - |f(\tilde{u}_j)|^{22^*}]$$

$$+ \frac{\lambda}{22^*} \int K_2(x)[|f(v_{n_j})|^{22^*} - |f(v_{n_j} - \tilde{v}_j)|^{22^*} - |f(\tilde{v}_j)|^{22^*}]$$

$$+ \lambda \int H(x, f(u_{n_j}), f(v_{n_j})) - H(x, f(u_{n_j} - \tilde{u}_j), f(v_{n_j} - \tilde{v}_j)) - H(x, f(\tilde{u}_j), f(\tilde{v}_j)).$$

We claim that

$$\lim_{j \to \infty} \int V_1(x)[|f(u_{n_j})|^2 - |f(u_{n_j} - \tilde{u}_j)|^2 - |f(\tilde{u}_j)|^2] = 0, \tag{3.6}$$

$$\lim_{j \to \infty} \int V_2(x)[|f(v_{n_j})|^2 - |f(v_{n_j} - \tilde{v}_j)|^2 - |f(\tilde{v}_j)|^2] = 0, \tag{3.7}$$

$$\lim_{j \to \infty} \int K_1(x)[|f(u_{n_j})|^{22^*} - |f(u_{n_j} - \tilde{u}_j)|^{22^*} - |f(\tilde{u}_j)|^{22^*}] = 0, \tag{3.8}$$

$$\lim_{j \to \infty} \int K_2(x)[|f(v_{n_j})|^{22^*} - |f(v_{n_j} - \tilde{v}_j)|^{22^*} - |f(\tilde{v}_j)|^{22^*}] = 0, \tag{3.9}$$

$$\lim_{j \to \infty} \int H(x, f(u_{n_j}), f(v_{n_j})) - H(x, f(u_{n_j} - \tilde{u}_j), f(v_{n_j} - \tilde{v}_j)) - H(x, f(\tilde{u}_j), f(\tilde{v}_j)) = 0. \tag{3.10}$$

By conditions (V_2), (K) and Lemma 3.3, we conclude that (3.6)–(3.9) hold. Similar to the proof of Lemma 3.4, it is easy to see that (3.10) holds. Using the fact $\Phi_\lambda(\mathbf{u}_{n_j}) \to c$ and $\Phi_\lambda(\tilde{\mathbf{u}}_j) \to \Phi_\lambda(\mathbf{u})$, we get conclusion 1.

(*ii*) We first notice that, for any given $\mathbf{w} = (\varphi, \psi) \in \mathbb{E}$ satisying $\|\mathbf{w}\|_\lambda \leq 1$,

$$
\begin{aligned}
(\Phi'_\lambda(\mathbf{u}_{n_j} - \tilde{\mathbf{u}}_j), \mathbf{w}) = {} & (\Phi'_\lambda(\mathbf{u}_{n_j}), \mathbf{w}) - (\Phi'_\lambda(\tilde{\mathbf{u}}_j), \mathbf{w}) \\
& - \lambda \int V_1(x) [f(u_{n_j}) f'(u_{n_j}) - f(u_{n_j} - \tilde{u}_j) f'(u_{n_j} - \tilde{u}_j) - f(\tilde{u}_j) f'(\tilde{u}_j)] \varphi \\
& - \lambda \int V_2(x) [f(v_{n_j}) f'(v_{n_j}) - f(v_{n_j} - \tilde{v}_j) f'(v_{n_j} - \tilde{v}_j) - f(\tilde{v}_j) f'(\tilde{v}_j)] \psi \\
& + \lambda \int K_1(x) \Big[|f(u_{n_j})|^{22^*-2} f(u_{n_j}) f'(u_{n_j}) \\
& \qquad - |f(u_{n_j} - \tilde{u}_j)|^{22^*-2} f(u_{n_j} - \tilde{u}_j) f'(u_{n_j} - \tilde{u}_j) - |f(\tilde{u}_j)|^{22^*-2} f(\tilde{u}_j) f'(\tilde{u}_j) \Big] \varphi \\
& + \lambda \int K_2(x) \Big[|f(v_{n_j})|^{22^*-2} f(v_{n_j}) f'(v_{n_j}) \\
& \qquad - |f(v_{n_j} - \tilde{v}_j)|^{22^*-2} f(v_{n_j} - \tilde{v}_j) f'(v_{n_j} - \tilde{v}_j) - |f(\tilde{v}_j)|^{22^*-2} f(\tilde{v}_j) f'(\tilde{v}_j) \Big] \psi \\
& + \lambda \int h_{n_j} \varphi + \lambda \int g_{n_j} \psi,
\end{aligned}
$$

where $h_{n_j}(x)$ and $g_{n_j}(x)$ are stated in Lemma 3.4. Noticing the boundedness of $(\mathbf{u}_{n_j})$ in $\mathbb{E}$, the equality

$$
\begin{aligned}
\frac{d |f(t)|^{22^*-2} f(t) f'(t)}{dt} &= C |f(t)|^{22^*-2} |f'(t)|^2 + |f(t)|^{22^*-2} f(t) f''(t) \\
&= C |f(t)|^{22^*-2} |f'(t)|^2 - 2 |f(t)|^{22^*} |f'(t)|^4,
\end{aligned}
$$

the mean value theorem, Lemma 2.1 (vii), (ix) and the Hölder inequality, we have for $R > 0$

$$
\begin{aligned}
\int_{B_R^c} & \Big| |f(u_{n_j})|^{22^*-2} f(u_{n_j}) f'(u_{n_j}) - |f(u_{n_j} - \tilde{u}_j)|^{22^*-2} f(u_{n_j} - \tilde{u}_j) f'(u_{n_j} - \tilde{u}_j) \Big| |\varphi| dx \\
& \leq C \int_{B_R^c} \Big[|f(u_{n_j} - \theta \tilde{u}_j)|^{22^*-2} |f'(u_{n_j} - \theta \tilde{u}_j)|^2 + |f(u_{n_j} - \theta \tilde{u}_j)|^{22^*} |f'(u_{n_j} - \theta \tilde{u}_j)|^4 \Big] |\tilde{u}_j| |\varphi| dx \\
& \leq C \int_{B_R^c} |u_{n_j} - \theta \tilde{u}_j|^{2^*-2} |\tilde{u}_j| |\varphi| dx \\
& \leq C \left(\int |u_{n_j} - \theta \tilde{u}_j|^{2^*} \right)^{\frac{2^*-2}{2^*}} \left(\int_{B_R^c} |\tilde{u}_j|^{2^*} dx \right)^{\frac{1}{2^*}} \left(\int |\varphi|^{2^*} \right)^{\frac{1}{2^*}} \\
& \leq C \left(\int_{B_R^c} |u|^{2^*} dx \right)^{\frac{1}{2^*}} \|\varphi\|_{1,\lambda}.
\end{aligned}
$$

We have also that

$$
\begin{aligned}
\int_{B_R^c} \Big| |f(\tilde{u}_j)|^{22^*-2} f(\tilde{u}_j) f'(\tilde{u}_j) \Big| |\varphi| dx & \leq C \int_{B_R^c} |\tilde{u}_j|^{2^*-1} |\varphi| dx \\
& \leq C \left(\int_{B_R^c} |\tilde{u}_j|^{2^*} dx \right)^{\frac{2^*-1}{2^*}} \left(\int |\varphi|^{2^*} \right)^{\frac{1}{2^*}} \\
& \leq C \left(\int_{B_R^c} |u|^{2^*} dx \right)^{\frac{2^*-1}{2^*}} \|\varphi\|_{1,\lambda}.
\end{aligned}
$$

Thus, for every $\epsilon > 0$, there exists $R = R_\epsilon > 0$ such that for any $\|\varphi\|_{1,\lambda} \le 1$

$$\left| \int_{B_R^c} \left[|f(u_{n_j} - \tilde{u}_j)|^{22^*-2} f(u_{n_j}) f'(u_{n_j}) \right. \right.$$
$$\left. \left. - |f(u_{n_j} - \tilde{u}_j)|^{22^*-2} f(u_{n_j} - \tilde{u}_j) f'(u_{n_j} - \tilde{u}_j) - |f(\tilde{u}_j)|^{22^*-2} f(\tilde{u}_j) f'(\tilde{u}_j) \right] |\varphi| dx \right| \le \epsilon.$$

On the other hand, applying the Rellich compact embedding theorem, we have

$$\lim_{n\to\infty} \int_{B_R} \left[|f(u_{n_j} - \tilde{u}_j)|^{22^*-2} f(u_{n_j}) f'(u_{n_j}) \right.$$
$$\left. - |f(u_{n_j} - \tilde{u}_j)|^{22^*-2} f(u_{n_j} - \tilde{u}_j) f'(u_{n_j} - \tilde{u}_j) - |f(\tilde{u}_j)|^{22^*-2} f(\tilde{u}_j) f'(\tilde{u}_j) \right] |\varphi| dx = 0$$

uniformly for $\|\varphi\|_{1,\lambda} \le 1$. Hence, by (K), we get that

$$\lim_{n\to\infty} \int K_1(x) \left[|f(u_{n_j})|^{22^*-2} f(u_{n_j}) f'(u_{n_j}) \right.$$
$$\left. - |f(u_{n_j} - \tilde{u}_j)|^{22^*-2} f(u_{n_j} - \tilde{u}_j) f'(u_{n_j} - \tilde{u}_j) - |f(\tilde{u}_j)|^{22^*-2} f(\tilde{u}_j) f'(\tilde{u}_j) \right] \varphi = 0$$

uniformly for $\|\varphi\|_{1,\lambda} \le 1$. Similarly, we know

$$\lim_{j\to\infty} \int V_1(x) \left[f(u_{n_j}) f'(u_{n_j}) - f(u_{n_j} - \tilde{u}_j) f'(u_{n_j} - \tilde{u}_j) - f(\tilde{u}_j) f'(\tilde{u}_j) \right] \varphi = 0,$$

$$\lim_{j\to\infty} \int V_2(x) \left[f(v_{n_j}) f'(v_{n_j}) - f(v_{n_j} - \tilde{v}_j) f'(v_{n_j} - \tilde{v}_j) - f(\tilde{v}_j) f'(\tilde{v}_j) \right] \psi = 0,$$

$$\lim_{j\to\infty} \int K_2(x) \left[|f(v_{n_j})|^{22^*-2} f(v_{n_j}) f'(v_{n_j}) \right.$$
$$\left. - |f(v_{n_j} - \tilde{v}_j)|^{22^*-2} f(v_{n_j} - \tilde{v}_j) f'(v_{n_j} - \tilde{v}_j) - |f(\tilde{v}_j)|^{22^*-2} f(\tilde{v}_j) f'(\tilde{v}_j) \right] \psi = 0$$

uniformly for $\|\mathbf{w}\|_\lambda \le 1$. Since $\Phi'_\lambda(\mathbf{u}_{n_j}) \to 0$ and $\Phi'_\lambda(\tilde{\mathbf{u}}_j) = 0$, we get the conclusion 2 by Lemma 3.4. $\qquad\square$

Lemma 3.6. *If the conditions (V_1), (K) and (H_1)–(H_3) hold. There is a constant $\alpha_0 > 0$ being independent of λ such that, for any $(PS)_c$ sequence $(\mathbf{u}_n)$ for Φ_λ with $\mathbf{u}_n \rightharpoonup \mathbf{u}$, either $\mathbf{u}_n \to \mathbf{u}$ along a subsequence or $c - \Phi_\lambda(\mathbf{u}) \ge \alpha_0 \lambda^{1-\frac{N}{2}}$.*

Proof. Set $\mathbf{u}_j^1 = \mathbf{u}_{n_j} - \tilde{\mathbf{u}}_j = (u_j^1, v_j^1)$. Then $\mathbf{u}_{n_j} - \mathbf{u} = \mathbf{u}_j^1 + (\tilde{\mathbf{u}}_j - \mathbf{u})$ and by (3.3), $\mathbf{u}_{n_j} \to \mathbf{u}$ if and only if $\mathbf{u}_j^1 \to 0$. If $(\mathbf{u}_n)$ has no convergent subsequence, we have $\liminf_{n\to\infty} \|\mathbf{u}_n - \mathbf{u}\|_\lambda > 0$. From Lemma 3.5, $\Phi_\lambda(\mathbf{u}_j^1) \to c - \Phi_\lambda(\mathbf{u})$ and $\Phi'_\lambda(\mathbf{u}_j^1) \to 0$ along a subsequence. Let $V_{i,b}(x) = \max\{V_i(x), b\}$, $i \in \{1, 2\}$. Since (V_1) and the fact $\mathbf{u}_j^1 \to 0$ in $L^2_{\text{loc}} \times L^2_{\text{loc}}$, we have that

$$\int V_1(x) |f(u_j^1)|^2 + \int V_2(x) |f(v_j^1)|^2 = \int V_{1,b}(x) |f(u_j^1)|^2 + \int V_{2,b}(x) |f(v_j^1)|^2 + o(1). \quad (3.11)$$

It follows from (H_2) and (H_3) that

$$|h_1(x, f(u_j^1), f(v_j^1)) f^2(u_j^1)| + |h_2(x, f(u_j^1), f(v_j^1)) f^2(v_j^1)|$$
$$\le \epsilon |(f(u_j^1), f(v_j^1))|^2 + C_\epsilon |(f(u_j^1), f(v_j^1))|^{22^*}$$
$$\le \epsilon (|(f(u_j^1)|^2 + |f(v_j^1))|^2) + C_\epsilon (|(f(u_j^1)|^{22^*} + |f(v_j^1))|^{22^*}).$$

Obviously, there exists a constant $\gamma_{\frac{b}{2}} > 0$ such that

$$K_1(x)|f(u_j^1)|^{22^*} + K_2(x)|f(v_j^1)|^{22^*} + |h_1(x, f(u_j^1), f(v_j^1))f^2(u_j^1)| + |h_2(x, f(u_j^1), f(v_j^1))f^2(v_j^1)|$$
$$\leq \frac{b}{2}(|f(u_j^1)|^2 + |f(v_j^1)|^2) + \gamma_{\frac{b}{2}}(|f(u_j^1)|^{22^*} + |f(v_j^1)|^{22^*}). \tag{3.12}$$

Then from Lemma 2.1 (vi), (vii), (3.11) and (3.12), we obtain that

$$\frac{S}{2}2^{-\frac{2}{2^*}}\left[|f(u_j^1)|_{22^*}^{22^*} + |f(v_j^1)|_{22^*}^{22^*}\right]^{\frac{2}{2^*}}$$
$$\leq \frac{S}{2}(|f(u_j^1)|_{22^*}^4 + |f(v_j^1)|_{22^*}^4)$$
$$\leq S(|u_j^1|_{2^*}^2 + |v_j^1|_{2^*}^2)$$
$$\leq 2\int |\nabla u_j^1|^2 + \lambda V_1(x)f(u_j^1)f'(u_j^1)u_j^1 + |\nabla v_j^1|^2 + \lambda V_2(x)f(v_j^1)f'(v_j^1)v_j^1$$
$$\quad - \lambda \int (V_1(x)f^2(u_j^1) + V_2(x)f^2(v_j^1))$$
$$= 2\lambda \int \left[K_1(x)|f(u_j^1)|^{22^*-2}f(u_j^1)f'(u_j^1)u_j^1 + K_2(x)|f(v_j^1)|^{22^*-2}f(v_j^1)f'(v_j^1)v_j^1\right.$$
$$\quad\quad \left. + h_1(x, f(u_j^1), f(v_j^1))f(u_j^1)f'(u_j^1)u_j^1 + h_2(x, f(u_j^1), f(v_j^1))f(v_j^1)f'(v_j^1)v_j^1\right]$$
$$\quad - \lambda \int (V_{1,b}(x)f^2(u_j^1) + V_{2,b}(x)f^2(v_j^1)) + o(1)$$
$$\leq 2\lambda \int \left[K_1(x)|f(u_j^1)|^{22^*} + K_2(x)|f(v_j^1)|^{22^*} + h_1(x, f(u_j^1), f(v_j^1))f^2(u_j^1)\right.$$
$$\quad\quad \left. + h_2(x, f(u_j^1), f(v_j^1))f^2(v_j^1)\right] - \lambda b\int (f^2(u_j^1) + f^2(v_j^1)) + o(1)$$
$$\leq 2\lambda\gamma_{\frac{b}{2}}(|f(u_j^1)|_{22^*}^{22^*} + |f(v_j^1)|_{22^*}^{22^*}) + o(1).$$

Additionally, (K) and (H_1) imply that

$$\Phi_\lambda(\mathbf{u}_j^1) - \frac{1}{2}\Phi_\lambda'(\mathbf{u}_j^1)\mathbf{u}_j^1 \geq \left(\frac{1}{4} - \frac{1}{22^*}\right)\lambda \int \left[K_1(x)|f(u_j^1)|^{22^*} + K_2(x)|f(v_j^1)|^{22^*}\right]$$
$$\quad\quad + \lambda \int \frac{1}{4}\left[h_1(x, f(u_j^1), f(v_j^1))f^2(u_j^1) + h_2(x, f(u_j^1), f(v_j^1))f^2(v_j^1)\right]$$
$$\quad\quad - H(x, f(u_j^1), f(v_j^1))$$
$$\quad \geq \lambda C\left[|f(u_j^1)|_{22^*}^{22^*} + |f(v_j^1)|_{22^*}^{22^*}\right].$$

So, we obtain that

$$|f(u_j^1)|_{22^*}^{22^*} + |f(v_j^1)|_{22^*}^{22^*} \leq \frac{c - \Phi_\lambda(\mathbf{u})}{\lambda C} + o(1).$$

Hence, we get

$$\frac{S}{2}2^{-\frac{2}{2^*}} \leq 2\lambda\gamma_{\frac{b}{2}}\left[|f(u_j^1)|_{22^*}^{22^*} + |f(v_j^1)|_{22^*}^{22^*}\right]^{\frac{2^*-2}{2^*}} + o(1)$$
$$\leq C\lambda^{\frac{2}{2^*}}\gamma_{\frac{b}{2}}(c - \Phi_\lambda(\mathbf{u}))^{\frac{2^*-2}{2^*}} + o(1),$$

that is, there is $\alpha_0 > 0$ being independent of the parameter λ such that $\alpha_0\lambda^{1-\frac{N}{2}} \leq c - \Phi_\lambda(\mathbf{u})$. $\quad\square$

From Lemma 3.6, we have the following conclusions.

Corollary 3.1. *There exists $\alpha_0 > 0$ being independent of λ such that Φ_λ satisfies the $(PS)_c$ condition for all $c < \alpha_0 \lambda^{1-\frac{N}{2}}$.*

Corollary 3.2. *There exists $\alpha_0 > 0$ being independent of λ such that Φ_λ^+ satisfies the $(PS)_c$ condition for all $c < \alpha_0 \lambda^{1-\frac{N}{2}}$.*

4 The mountain pass geometry

The following lemmas imply that Φ_λ possesses the mountain pass geometry.

Lemma 4.1. *There exist $\rho, \alpha > 0$ such that $\int |\nabla u|^2 + \lambda V_1(x) f^2(u) + |\nabla v|^2 + \lambda V_2(x) f^2(v) \geq \alpha \|\mathbf{u}\|_\lambda^2$, whenever $\|\mathbf{u}\|_\lambda = \rho$.*

Proof. As in [15], suppose that there is $\mathbf{u}_n \to \mathbf{0}$ in $\mathbb{E}$ such that

$$\int |\nabla w_n^1|^2 + \lambda V_1(x) \frac{f^2(u_n)}{u_n^2} (w_n^1)^2 + \int |\nabla w_n^2|^2 + \lambda V_2(x) \frac{f^2(v_n)}{v_n^2} (w_n^2)^2 \to 0, \qquad (4.1)$$

where $\mathbf{w}_n := (w_n^1, w_n^2) = (\frac{u_n}{\|\mathbf{u}_n\|_\lambda}, \frac{v_n}{\|\mathbf{u}_n\|_\lambda})$. (4.1) is equivalent to the both limits

$$\int |\nabla w_n^1|^2 + \lambda V_1(x) \frac{f^2(u_n)}{u_n^2} (w_n^1)^2 \to 0 \qquad (4.2)$$

and

$$\int |\nabla w_n^2|^2 + \lambda V_2(x) \frac{f^2(v_n)}{v_n^2} (w_n^2)^2 \to 0. \qquad (4.3)$$

We get that $(u_n, v_n) \to (0,0)$ in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, $(u_n, v_n) \to (0,0)$ a.e., $(w_n^1, w_n^2) \rightharpoonup (w^1, w^2)$ in $E_1 \times E_2$, $(w_n^1, w_n^2) \to (w^1, w^2)$ in $L_{\text{loc}}^2 \times L_{\text{loc}}^2$, $(w_n^1, w_n^2) \to (w^1, w^2)$ a.e. up to a subsequence. We consider two cases:

If $w^1 \neq 0$, Fatou's lemma and Lemma 2.1 (iv) imply that

$$\liminf_{n \to \infty} \int |\nabla w_n^1|^2 + \lambda V_1(x) \frac{f^2(u_n)}{u_n^2} (w_n^1)^2 \geq \int |\nabla w^1|^2 + \lambda V_1(x)(w^1)^2 > 0,$$

which contradicts to (4.2).

The other case is $w^1 = 0$. (4.2) ensures that

$$\int |\nabla w_n^1|^2 + \lambda V_1(x)(w_n^1)^2 + \lambda V_1(x) \left(\frac{f^2(u_n)}{u_n^2} - 1 \right) (w_n^1)^2 \to 0.$$

Since $u_n \to 0$ in $L^2(\mathbb{R}^N)$, for every $\epsilon > 0$, $\mathrm{m}\{x \in \mathbb{R}^N : |u_n(x)| > \epsilon\} \to 0$ as $n \to \infty$. By (V_2), Lemma 2.1 (iii) and the Hölder inequality, we have

$$\left| \int_{|u_n|>\epsilon} \lambda V_1(x) \left(\frac{f^2(u_n)}{u_n^2} - 1 \right) (w_n^1)^2 dx \right| \leq C \int_{|u_n|>\epsilon} (w_n^1)^2$$

$$\leq (\mathrm{m}\{x \in \mathbb{R}^N : |u_n(x)| > \epsilon\})^{\frac{2^*-2}{2^*}} |w_n^1|_{2^*}^2 \to 0.$$

It follows that $\int_{|u_n|<\epsilon} \lambda V_1(x) \big(\frac{f^2(u_n)}{u_n^2} - 1 \big)(w_n^1)^2 dx$ is small as ϵ is small. So $w_n^1 \to 0$ in E_1. Similarly, we can get $w_n^2 \to 0$ in E_2, which contradicts to $\|\mathbf{w}_n\|_\lambda = 1$. $\qquad \square$

Lemma 4.2. *For the above ρ, there exists a constant $\beta > 0$ such that $\inf_{\|\mathbf{u}\|_\lambda = \rho} \Phi_\lambda(\mathbf{u}) \geq \beta$.*

Proof. Due to (K), Lemma 2.1 (7) and the Sobolev embedding inequality, it is easy to obtain that

$$\int K_1(x)|f(u)|^{2 \cdot 2^*} + K_2(x)|f(v)|^{2 \cdot 2^*} \leq C \int (|u|^{2^*} + |v|^{2^*})$$

$$\leq CS^{-\frac{2^*}{2}} \left[\left(\int |\nabla u|^2 \right)^{\frac{2^*}{2}} + \left(\int |\nabla v|^2 \right)^{\frac{2^*}{2}} \right]$$

$$\leq C\|\mathbf{u}\|_\lambda^{2^*}.$$

Based on Lemma 2.1 (iii), (vii), (H_2) and (H_3), it is obvious that for all $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\int H(x, f(u), f(v)) \leq \epsilon \int |(f(u), f(v))|^2 + C_\epsilon \int |(f(u), f(v))|^{2 \cdot 2^*}$$

$$\leq \epsilon \int (|f(u)|^2 + |f(v)|^2) + C_\epsilon \int (|f(u)|^{2 \cdot 2^*} + |f(v)|^{2 \cdot 2^*})$$

$$\leq \epsilon \int (|u|^2 + |v|^2) + C_\epsilon \int (|u|^{2^*} + |v|^{2^*})$$

$$\leq \epsilon \nu_2^2 \|\mathbf{u}\|_\lambda^2 + C_\epsilon \|\mathbf{u}\|_\lambda^{2^*}.$$

Therefore, combining the above inequalities and Lemma 4.1, we obtain that

$$\Phi_\lambda(\mathbf{u}) \geq \frac{\alpha}{2}\|\mathbf{u}\|_\lambda^2 - \lambda C\|\mathbf{u}\|_\lambda^{2^*} - \epsilon \lambda \nu_2^2 \|\mathbf{u}\|_\lambda^2 - C_\epsilon \lambda \|\mathbf{u}\|_\lambda^{2^*}$$

$$= \left(\frac{\alpha}{2} - \lambda \nu_2^2 \epsilon \right) \rho^2 - C\lambda \rho^{2^*} - C_\epsilon \lambda \rho^{2^*}$$

for every $\|\mathbf{u}\|_\lambda = \rho$. Choosing for all $\epsilon \in \left(0, \frac{\alpha}{2\lambda \nu_2^2}\right)$ and ρ sufficiently small, we derive that there exists a constant $\beta > 0$ with $\inf_{\|\mathbf{u}\|_\lambda = \rho} \Phi(\mathbf{u}) \geq \beta$. $\qquad\square$

Lemma 4.3. *For any $\sigma > 0$, there exists $\Lambda_\sigma > 0$ such that for each $\lambda \geq \Lambda_\sigma$, there is $\mathbf{e}_\lambda \in \mathbb{E}$ with $\|\mathbf{e}_\lambda\|_\lambda > \rho$, $\Phi_\lambda(\mathbf{e}_\lambda) < 0$ and*

$$\max_{t \in [0,1]} \Phi_\lambda(t\mathbf{e}_\lambda) < \sigma \lambda^{1-\frac{N}{2}}.$$

Proof. By a standard argument, (H_1) implies that given $C_1 > 0$ there exists $C_2 > 0$ such that

$$H(x, u, v) \geq C_1|(u, v)|^\mu - C_2$$

for all $(x, u, v) \in \bar{B}_1 \times \mathbb{R} \times \mathbb{R}$, where $\bar{B}_1$ is the unit ball in $\mathbb{R}^N$. Choosing any $\varphi \in C_0^\infty(\mathbb{R}^N, [0,1])$ and writing $\mathbf{u} = (\varphi, \varphi)$ such that $\operatorname{supp} \varphi = \bar{B}_1$ and $0 < \varphi(x) \leq 1$ for all $x \in \bar{B}_1$, we have

$$\Phi_\lambda(t\mathbf{u}) \leq \frac{t^2}{2} \int_{B_1} 2|\nabla\varphi|^2 + [V_1(x) + V_2(x)]\varphi^2 dx - \lambda C_1 \int_{B_1} |(f(t\varphi), f(t\varphi))|^\mu + C_2|B_1|.$$

By Lemma 2.1 (6), we know that $\frac{f(t)}{t}$ is decreasing for $t > 0$. Since $0 < t\varphi(x) \leq t$ for $x \in B_1$ and $t > 0$, we obtain $f(t\varphi(x)) \geq f(t)\varphi(x)$, which implies that

$$\Phi_\lambda(t\mathbf{u}) \leq \frac{t^2}{2} \left[\int_{B_1} 2|\nabla\varphi|^2 + [V_1(x) + V_2(x)]\varphi^2 dx - \lambda C_1 \frac{f^\mu(t)}{t^2} \int_{B_1} |\varphi|^\mu + \frac{C_2}{t^2}|B_1| \right].$$

It follows from Lemma 2.1 (5) that $\Phi_\lambda(t\mathbf{u}) \to -\infty$ as $t \to +\infty$. Notice that

$$\inf\left\{\int |\nabla\varphi|^2 : \varphi \in C_0^\infty(\mathbb{R}^N),\ |\varphi|_q = 1\right\} = 0.$$

For any $\delta > 0$, one can choose $\varphi_\delta \in C_0^\infty(\mathbb{R}^N)$ such that $|\varphi_\delta|_q = 1$, supp $\varphi_\delta \subset B_{r_\delta}(0)$ and $|\nabla\varphi_\delta|_2^2 < \frac{\delta}{2}$. Set

$$\mathbf{e}_\lambda'(x) := (e_\lambda(x), e_\lambda(x)) = (\varphi_\delta(\lambda^{\frac{1}{2}}x), \varphi_\delta(\lambda^{\frac{1}{2}}x)).$$

For $t \geq 0$, Lemma 2.1 (iii), (viii) and (H_4) imply that

$$\Phi_\lambda(t\mathbf{e}_\lambda') \leq \frac{t^2}{2}\int |\nabla e_\lambda|^2 + \frac{\lambda}{2}\int V_1(x)f^2(te_\lambda) + \frac{t^2}{2}\int |\nabla e_\lambda|^2 + \frac{\lambda}{2}\int V_2(x)f^2(te_\lambda)$$
$$- \lambda\int H(x, f(te_\lambda), f(te_\lambda))$$
$$\leq t^2\int |\nabla e_\lambda|^2 + \frac{\lambda t^2}{2}\int V_1(x)|e_\lambda|^2 + V_2(x)|e_\lambda|^2 - C\lambda\int [f^2(te_\lambda) + f(te_\lambda)]^q$$
$$\leq \lambda^{1-\frac{N}{2}}\left\{t^2\int |\nabla\varphi_\delta|^2 + \frac{t^2}{2}\int (V_1(\lambda^{-\frac{1}{2}}x) + V_2(\lambda^{-\frac{1}{2}}x))|\varphi_\delta|^2 - Ct^q\int |\varphi_\delta|^q\right\}.$$

Since $V_i(0) = 0$ and supp $\varphi_\delta \subset B_{r_\delta}(0)$, there is $\Lambda_\delta' > 0$ such that

$$V_i(\lambda^{-\frac{1}{2}}x) < \frac{\delta}{2|\varphi_\delta|_2^2}, \qquad \forall |x| \leq r_\delta,\ \lambda > \Lambda_\delta',\ i = 1, 2.$$

Consequently, there holds

$$\max_{t\geq 0}\Phi_\lambda(t\mathbf{e}_\lambda') \leq C\delta^{\frac{q}{q-2}}\lambda^{1-\frac{N}{2}}, \qquad \forall \lambda > \Lambda_\delta'.$$

Choose $\delta > 0$ small enough such that $C\delta^{\frac{q}{q-2}} \leq \sigma$ and take $\Lambda_\sigma = \Lambda_\delta'$. Then $\Phi_\lambda(\mathbf{e}_\lambda) < 0$ and $\max_{t\in[0,1]}\Phi_\lambda(t\mathbf{e}_\lambda) \leq \sigma\lambda^{1-\frac{N}{2}}$, where $\mathbf{e}_\lambda = t_1\mathbf{e}_\lambda'$ and t_1 is large enough. $\qquad\square$

5 Proof of Theorem 2.1

In this section we will prove Theorem 2.1.

Proof of Theorem 2.1. Lemmas 4.2–4.3 imply that for any $\alpha_0 > \sigma > 0$ there exists $\Lambda_\sigma > 0$ such that for each $\lambda \geq \Lambda_\sigma$, there is $\beta > 0$ and a $(PS)_c$ sequence $(\mathbf{u}_n)$ satisfying $0 < \beta < c \leq \sigma\lambda^{1-\frac{N}{2}}$. In virtue of Corollary 3.1, we get that $(PS)_c$ condition holds for Φ_λ at c. Thus there is $\mathbf{u}_\lambda = (u_\lambda, v_\lambda) \in \mathbb{E}$ such that $\Phi_\lambda'(\mathbf{u}_\lambda) = 0$ and $\Phi_\lambda(\mathbf{u}_\lambda) = c$. So $(f(u_\lambda), f(v_\lambda))$ must solve system (2.1).

In order to get the multiplicity of critical points, we will use the index theory defined by the Krasnoselski genus. Define the set of all symmetric (in the sense that $-A = A$) and closed subsets of E as Σ. For all $A \in \Sigma$, denote gen(A) by the Krasnoselski genus and

$$i(A) := \min_{h\in\Gamma}\text{gen}(h(A) \cap S_\lambda),$$

where Γ is the set of all odd homeomorphisms $h \in C(\mathbb{E}, \mathbb{E})$ and S_λ is the closed symmetric set

$$S_\lambda := \{\mathbf{u} \in \mathbb{E} : \|\mathbf{u}\|_\lambda = \rho\}$$

satisfying $\Phi_\lambda|_{S_\lambda} \geq \beta > 0$. Then i is a version of Benci's pseudoindex (see [2]). (H_5) implies that Φ_λ is even. Let

$$c_{\lambda_j} := \inf_{i(A) \geq j} \sup_{\mathbf{u} \in A} \Phi_\lambda(\mathbf{u}), \qquad 1 \leq j \leq m.$$

Then if c_{λ_j} is finite and the (PS) condition holds for Φ_λ at c_{λ_j}, we know that c_{λ_j} is a critical value for Φ_λ. However, the (PS) condition does not hold in general. In order to show that Φ_λ satisfies the (PS) condition for λ large enough and c_{λ_j} sufficiently small, as in [31] we will construct here small minimax levels for Φ_λ when λ large enough. Similar to the proof in Lemma 4.3, for any $m \in \mathbb{N}$, $\delta > 0$ and $j = 1, 2, \ldots, m$, one can choose m functions $\varphi_\delta^j \in C_0^\infty(\mathbb{R}^N)$ with $\operatorname{supp} \varphi_\delta^i \cap \operatorname{supp} \varphi_\delta^k = \varnothing$ if $i \neq k$, $|\varphi_\delta^j|_q = 1$ and $|\nabla \varphi_\delta^j|_2^2 < \frac{\delta}{2}$. Let $r_{\delta m} > 0$ be such that $\operatorname{supp} \varphi_\delta^j \subset B_{r_{\delta m}}(0)$. Set

$$\mathbf{e}_\lambda^j(x) := (e_\lambda^j(x), e_\lambda^j(x)) = (\varphi_\delta^j(\lambda^{\frac{1}{2}}x), \varphi_\delta^j(\lambda^{\frac{1}{2}}x))$$

and define

$$H_\lambda^m = \operatorname{Span}\{\mathbf{e}_\lambda^1, \mathbf{e}_\lambda^2, \ldots, \mathbf{e}_\lambda^m\}.$$

Then $i(H_\lambda^m) = \dim H_\lambda^m = m$. Observe that for each $\mathbf{v} = \Sigma_{j=1}^m t_j \mathbf{e}_\lambda^j \in H_\lambda^m$,

$$\Phi_\lambda(\mathbf{v}) = \Sigma_{j=1}^m \Phi_\lambda(t_j \mathbf{e}_\lambda^j)$$

and for $t_j \geq 0$

$$\begin{aligned}
\Phi_\lambda(t_j \mathbf{e}_\lambda^j) \leq{}& \frac{t_j^2}{2} \int |\nabla e_\lambda^j|^2 + \frac{\lambda}{2} \int V_1(x) f^2(t_j e_\lambda^j) + \frac{t_j^2}{2} \int |\nabla e_\lambda^j|^2 + \frac{\lambda}{2} \int V_2(x) f^2(t_j e_\lambda^j) \\
& - \lambda \int H(x, f(t_j e_\lambda^j), f(t_j e_\lambda^j)) \\
\leq{}& \lambda^{1-\frac{N}{2}} \left\{ t_j^2 \int |\nabla \varphi_\delta^j|^2 + t_j^2 \int (V_1(\lambda^{-\frac{1}{2}}x) + V_2(\lambda^{-\frac{1}{2}}x))|\varphi_\delta^j|^2 - C t_j^q \int |\varphi_\delta^j|^q \right\}.
\end{aligned}$$

Set

$$\beta_\delta := \max\left\{ |\varphi_\delta^j|_2^2 : j = 1, 2, \ldots, m \right\}.$$

$V_i(0) = 0$ and $\operatorname{supp} \varphi_\delta^j \subset B_{r_{\delta m}}(0)$ imply that there is $\Lambda'_{\delta m} > 0$ such that

$$V_i(\lambda^{-\frac{1}{2}}x) < \frac{\delta}{2\beta_\delta}, \qquad \forall |x| \leq r_{\delta m}, \ \lambda > \Lambda'_{\delta m}, \ i = 1, 2.$$

Consequently, there holds

$$\sup_{\mathbf{v} \in H_\lambda^m} \Phi_\lambda(\mathbf{v}) \leq m C \delta^{\frac{q}{q-2}} \lambda^{1-\frac{N}{2}}, \qquad \forall \lambda > \Lambda'_{\delta m}.$$

Choose $\delta > 0$ so small that $m C \delta^{\frac{q}{q-2}} \lambda^{1-\frac{N}{2}} \leq \sigma$. Thus for any $m \in \mathbb{N}$ and $\sigma \in (0, \alpha_0)$, there exists $\Lambda_{\sigma m} = \Lambda'_{\delta m}$ such that for each $\lambda > \Lambda_{\sigma m}$, we can choose a m-dimensional subspace H_λ^m with $\max \Phi_\lambda(H_\lambda^m) \leq \sigma \lambda^{1-\frac{N}{2}}$.

Now we can define the minimax values c_{λ_j} by

$$c_{\lambda_j} := \inf_{i(A) \geq j} \sup_{\mathbf{u} \in A} \Phi_\lambda(\mathbf{u}).$$

Since $\Phi_\lambda|_{S_\lambda} \geq \beta > 0$ and $\max \Phi_\lambda(H_\lambda^m) \leq \sigma\lambda^{1-\frac{N}{2}}$, we know

$$\beta \leq c_{\lambda_1} \leq \cdots \leq c_{\lambda_m} \leq \sup_{\mathbf{u} \in H_\lambda^m} \Phi_\lambda(\mathbf{u}) \leq \sigma\lambda^{1-\frac{N}{2}}.$$

It follows from Corollary 3.1 that Φ_λ satisfies the (PS) condition at all levels c_{λ_j}, since $c_{\lambda_j} < \alpha_0\lambda^{1-\frac{N}{2}}$. Then all c_{λ_j} are critical values. Hence Φ_λ has at least m pairs of nontrival critical points satisfying

$$\beta \leq \Phi_\lambda(\mathbf{u}_\lambda) \leq \sigma\lambda^{1-\frac{N}{2}}.$$

Therefore, Φ_λ has at least m pairs of solutions $\mathbf{u}_\lambda = (u_\lambda, v_\lambda)$. And $(f(u_\lambda), f(v_\lambda))$ solves problem (2.1). Since $\mathbf{u}_\lambda$ is a critical point of Φ_λ, there holds for $\nu \in [4, 22^*]$

$$\sigma\lambda^{1-\frac{N}{2}} \geq \Phi_\lambda(\mathbf{u}_\lambda)$$
$$= \Phi_\lambda(\mathbf{u}_\lambda) - \frac{2}{\nu}(\Phi_\lambda'(\mathbf{u}_\lambda), \mathbf{u}_\lambda)$$
$$\geq \left(\frac{1}{2} - \frac{2}{\nu}\right) \int |\nabla u_\lambda|^2 + \lambda V_1(x)f^2(u_\lambda) + |\nabla v_\lambda|^2 + \lambda V_2(x)f^2(v_\lambda)$$
$$+ \left(\frac{1}{\nu} - \frac{1}{22^*}\right) \lambda \int K_1(x)|f(u_\lambda)|^{22^*} + K_2(x)|f(v_\lambda)|^{22^*}$$
$$+ \lambda\left(\frac{\mu}{\nu} - 1\right) \int H(x, f(u_\lambda), f(v_\lambda)).$$

Taking $\nu = 4$ gives

$$\frac{1}{2N} \int K_1(x)|f(u_\lambda)|^{22^*} + K_2(x)|f(v_\lambda)|^{22^*} + \frac{\mu-4}{4} \int H(x, f(u_\lambda), f(v_\lambda)) \leq \sigma\lambda^{-\frac{N}{2}}$$

and taking $\nu = \mu$ gives

$$\frac{\mu-4}{2\mu} \int |\nabla u_\lambda|^2 + \lambda V_1(x)f^2(u_\lambda) + |\nabla v_\lambda|^2 + \lambda V_2(x)f^2(v_\lambda) \leq \sigma\lambda^{1-\frac{N}{2}}.$$

From the above two inequalities and (H_4) it follows that

$$\begin{cases} \int |\nabla u_\lambda|^2 + \lambda V_1(x)f^2(u_\lambda) + |\nabla v_\lambda|^2 + \lambda V_2(x)f^2(v_\lambda) \leq C\sigma\lambda^{1-\frac{N}{2}}, \\ \int K_1(x)|f(u_\lambda)|^{22^*} + K_2(x)|f(v_\lambda)|^{22^*} \leq C\sigma\lambda^{-\frac{N}{2}}. \end{cases} \tag{5.1}$$

We claim that

$$\|\mathbf{u}_\lambda\|_\lambda^2 \leq C\sigma\lambda^{1-\frac{N}{2}}. \tag{5.2}$$

In fact, from (5.1), we only need to prove that $\lambda \int V_1(x)u_\lambda^2 + V_2(x)v_\lambda^2 \leq C\sigma\lambda^{1-\frac{N}{2}}$. We write that

$$\lambda \int V_1(x)u_\lambda^2 = \lambda \int_{|u_\lambda|\geq 1} V_1(x)u_\lambda^2 dx + \lambda \int_{|u_\lambda|\leq 1} V_1(x)u_\lambda^2 dx.$$

Combining (V_2), (K), (5.1) and Lemma 2.1 (viii), we have

$$\lambda \int_{|u_\lambda|\geq 1} V_1(x)u_\lambda^2 dx \leq C\lambda \max V_1 \int_{|u_\lambda|\geq 1} |f(u_\lambda)|^{22^*} dx$$
$$\leq C\lambda \int K_1(x)|f(u_\lambda)|^{22^*} dx$$
$$\leq C\sigma\lambda^{1-\frac{N}{2}}$$

and

$$\lambda \int_{|u_\lambda| \leq 1} V_1(x) u_\lambda^2 dx \leq \frac{\lambda}{C^2} \int_{|u_\lambda| \leq 1} V_1(x) f^2(u_\lambda) dx \leq C\sigma \lambda^{1-\frac{N}{2}}.$$

Thus $\lambda \int V_1(x) u_\lambda^2 \leq C\sigma \lambda^{1-\frac{N}{2}}$. Similarly, we can get $\lambda \int V_2(x) v_\lambda^2 \leq C\sigma \lambda^{1-\frac{N}{2}}$. Then we conclude that (5.2) holds, which shows $(u_\lambda, v_\lambda) \to (0,0)$ in $\mathbb{E}$ as $\lambda \to \infty$. Meanwhile, we also have by Lemma 2.1 (ii)

$$\frac{\mu - 4}{2\mu} \int |\nabla f(u_\lambda)|^2 + \lambda V_1(x) f^2(u_\lambda) + |\nabla f(v_\lambda)|^2 + \lambda V_2(x) f^2(v_\lambda)$$
$$\leq \frac{\mu - 4}{2\mu} \int |\nabla u_\lambda|^2 + \lambda V_1(x) f^2(u_\lambda) + |\nabla v_\lambda|^2 + \lambda V_2(x) f^2(v_\lambda)$$
$$\leq \sigma \lambda^{1-\frac{N}{2}},$$

which shows that $(f(u_\lambda), f(v_\lambda)) \to (0,0)$ in $\mathbb{E}$ as $\lambda \to \infty$. It follows from $\lambda = \varepsilon^{-2}$ that Theorem 1.1 is completed. $\qquad\square$

Remark 5.1. The same arguments applied to Φ_λ^+ can give the existence of multiple positive solutions for system (2.2).

6 Proof of Theorem 1.2

In this section, we shall give some crucial lemmas and prove Theorem 1.2 under the conditions (V_3), (K'), (H_1)–(H_4) and (H_6).

Let $\varepsilon = 1$. We redefine the functional

$$\Phi(\mathbf{u}) := \frac{1}{2} \int |\nabla u|^2 + V_1(x) f^2(u) + |\nabla v|^2 + V_2(x) f^2(v)$$
$$- \frac{1}{22^*} \int K_1(x) |f(u)|^{22^*} + K_2(x) |f(v)|^{22^*} - \int H(x, f(u), f(v)).$$

Similar to Lemmas 4.1–4.3, Φ satisfies the mountain pass geometry in $\mathbb{E}$. And the $(PS)_c$ sequence $(\mathbf{u}_n)$ for Φ is bounded in $\mathbb{E}$ by Lemma 3.1.

Some propositions and lemmas are needed and their proofs are similar as in [26]. We just state them briefly and omit their proofs.

Proposition 6.1. *Let* $(\mathbf{u}_n) = ((u_n, u_n)) \subset \mathbb{E}$ *be a* $(PS)_c$ *sequence with* $0 < c < \frac{1}{N} K_*^{\frac{2-N}{2}} S^{\frac{N}{2}}$, *where* $K_* = \max\{\|K_1\|_\infty, \|K_2\|_\infty\}$, *and* $\mathbf{u}_n \rightharpoonup \mathbf{0}$ *in* $\mathbb{E}$. *Then there exists a sequence* $(y_n) \subset \mathbb{R}^N$ *and* $r, \eta > 0$ *such that* $|y_n| \to +\infty$ *and*

$$\limsup_{n\to\infty} \int_{B_r(y_n)} u_n^2 \geq \eta > 0.$$

Given $\epsilon > 0$, we study the function $w_\epsilon : \mathbb{R}^N \to \mathbb{R}$ defined by

$$w_\epsilon(x) = C(N) \frac{\epsilon^{\frac{N-2}{2}}}{(\epsilon + |x|^2)^{\frac{N-2}{2}}},$$

where $C(N) = [N(N-2)]^{\frac{N-2}{4}}$. Recall that by [1,29], $\{w_\epsilon\}_{\epsilon>0}$ is a family of functions at which the infimum, that defines the best constant S, for the Sobolev imbedding $D^{1,2}(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N)$, is attained. Moreover, one has

$$w_\epsilon \in L^{2^*}(\mathbb{R}^N), \quad \nabla w_\epsilon \in L^2(\mathbb{R}^N), \quad \int |\nabla w_\epsilon|^2 = \int |w_\epsilon|^{2^*} = S^{\frac{N}{2}}.$$

We also consider $\phi \in C_0^\infty(\mathbb{R}^N, [0,1])$, $\phi \equiv 1$ for $|x - x_0| \leq r$, $\phi \equiv 0$ for $|x - x_0| \geq 2r$, where $r > 0$ is a small enough constant. Define $u_\epsilon(x) = \phi(x)w_\epsilon(x - x_0)$. We get the following estimations (e.g. [4,5]).

Lemma 6.1. $u_\epsilon(x)$ satisfies the following estimations: as $\epsilon \to 0$,

$$\int |\nabla u_\epsilon|^2 = S^{N/2} + O\left(\epsilon^{\frac{N-2}{2}}\right), \qquad \int |u_\epsilon|^{2^*} = S^{N/2} + O\left(\epsilon^{\frac{N}{2}}\right),$$

$$\int |u_\epsilon| \leq C\epsilon^{\frac{N-2}{4}}, \qquad \int |u_\epsilon|^{2^*-1} \leq C\epsilon^{\frac{N-2}{4}}, \qquad \int |\nabla u_\epsilon| \leq C\epsilon^{\frac{N-2}{4}}$$

and

$$\int |u_\epsilon|^2 = \begin{cases} C\epsilon + O\left(\epsilon^{\frac{N-2}{2}}\right), & N \geq 5, \\ C\epsilon \ln|\epsilon| + O\left(\epsilon^{\frac{N-2}{2}}\right), & N = 4, \\ C\epsilon^{1/2}, & N = 3. \end{cases}$$

Lemma 6.2. *There exists* $\mathbf{u}_0 \in \mathbb{E} \backslash \{0\}$ *such that*

$$\sup_{t \geq 0} \Phi(t\mathbf{u}_0) < \frac{1}{N} K_*^{\frac{2-N}{2}} S^{\frac{N}{2}}. \tag{6.1}$$

Proof. Write $\mathbf{u}_\epsilon = (u_\epsilon, u_\epsilon)$. For $\epsilon > 0$ small enough, since $\Phi(0) = 0$ and $\lim_{t \to +\infty} \Phi(t\mathbf{u}_\epsilon) = -\infty$, there exists a constant $t_\epsilon > 0$ such that

$$\Phi(t_\epsilon \mathbf{u}_\epsilon) = \max_{t \geq 0} \Phi(t\mathbf{u}_\epsilon)$$

and there exist positive constants A_1 and A_2 being independent of ϵ such that

$$0 < A_1 \leq t_\epsilon \leq A_2 < +\infty. \tag{6.2}$$

Notice that $\text{supp}\, u_\epsilon \subset B_{2r}(x_0)$, by (V_3), Lemma 2.1 (vii), (x) and (H_4), we can see that

$$\Phi(t_\epsilon \mathbf{u}_\epsilon) \leq t_\epsilon^2 \int |\nabla u_\epsilon|^2 + Ct_\epsilon \int u_\epsilon - \frac{2^{\frac{N}{N-2}}}{22^*} \int_{t_\epsilon u_\epsilon \geq T} (K_1(x) + K_2(x))(t_\epsilon u_\epsilon)^{2^*}$$

$$+ C_2 \int_{t_\epsilon u_\epsilon \geq T} (t_\epsilon u_\epsilon)^{2^*-1} \ln(t_\epsilon u_\epsilon) + C_3 \int_{t_\epsilon u_\epsilon \geq T} (t_\epsilon u_\epsilon)^{2^*-1} - Ct_\epsilon^q \int u_\epsilon^q, \tag{6.3}$$

where $T > 1$ is large enough. From Lemma 6.1 and (6.2), as $\epsilon \to 0$, it follows that

$$0 < \int_{t_\epsilon u_\epsilon \geq T} (t_\epsilon u_\epsilon)^{2^*-1} \ln(t_\epsilon u_\epsilon)$$

$$\leq \int (t_\epsilon u_\epsilon)^{2^*-1} (C_4 + C_5|\ln\epsilon|)$$

$$= O\left(\epsilon^{\frac{N-2}{4}} |\ln\epsilon|\right). \tag{6.4}$$

Choosing $\epsilon_0 \in (0, r^2)$ such that for $\epsilon \leq \epsilon_0$, $u_\epsilon = w_\epsilon \geq C\epsilon^{-\frac{N-2}{4}}$ and $t_\epsilon u_\epsilon \geq T$ when $|x - x_0| \leq \sqrt{\epsilon}$, by Lemma 6.1 one can verify that

$$\int u_\epsilon^q \geq \int_{B_{\sqrt{\epsilon}}(x_0)} u_\epsilon^q = O\left(\epsilon^{\frac{N}{2} - \frac{q(N-2)}{4}}\right) \tag{6.5}$$

and for $i = 1, 2$

$$\int_{t_\epsilon u_\epsilon \geq T} K_i(x)(u_\epsilon)^{2^*} \geq \int_{B_{\sqrt{\epsilon}}(x_0)} K_i(x)(w_\epsilon)^{2^*} = K_i(x_0)S^{N/2} + O(\epsilon). \tag{6.6}$$

Combining (6.2)–(6.6) and Lemma 6.1, we have

$$
\begin{aligned}
\Phi(t_\epsilon \mathbf{u}_\epsilon) &\leq t_\epsilon^2 S^{N/2} + O\left(\epsilon^{\frac{N-2}{2}}\right) + O\left(\epsilon^{\frac{N-2}{4}}\right) - \frac{2^{\frac{2}{N-2}}}{2^*}(K_1(x_0) + K_2(x_0))S^{N/2}t_\epsilon^{2^*} \\
&\quad + O(\epsilon) + O\left(\epsilon^{\frac{N-2}{4}}|\ln \epsilon|\right) - O\left(\epsilon^{\frac{N}{2} - \frac{q(N-2)}{4}}\right) \\
&\leq \frac{1}{N}K_*^{\frac{2-N}{2}}S^{N/2} + O(\epsilon) + O\left(\epsilon^{\frac{N-2}{4}}|\ln \epsilon|\right) - O\left(\epsilon^{\frac{N}{2} - \frac{q(N-2)}{4}}\right) \\
&=: \frac{1}{N}K_*^{\frac{2-N}{2}}S^{N/2} + J. \tag{6.7}
\end{aligned}
$$

In the case of $3 \leq N < 6$, $\frac{N+2}{N-2} < q < 2^*$, we can see that $\frac{N}{2} - \frac{q(N-2)}{4} < \frac{N-2}{4} < 1$, which gives that $J < 0$ as $\epsilon > 0$ sufficiently small. In the other case of $N \geq 6$, $2 < q < 2^*$, we can see that $\frac{N}{2} - \frac{q(N-2)}{4} < 1 \leq \frac{N-2}{4}$, which gives that $J < 0$ as $\epsilon > 0$ sufficiently small. Taking $\mathbf{u}_0 = \mathbf{u}_\epsilon$ for ϵ small enough, we see that (6.1) holds. □

Proof of Theorem 1.2. Similar to the proof of Theorem 2 in [26], we can complete the proof of Theorem 1.2 and omit it here. □

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