Electronic Journal of Qualitative Theory of Differential Equations
2016, No. **64**, 1–38; doi: 10.14232/ejqtde.2016.1.64 http://www.math.u-szeged.hu/ejqtde/

Properties of the resolvent of a linear Abel integral equation: implications for a complementary fractional equation

Dedicated to Professor Tibor Krisztin on the occasion of his 60th birthday

Leigh C. Becker ✉

Christian Brothers University, 650 E. Parkway S., Memphis, TN 38104, USA

Received 18 June 2016, appeared 12 September 2016
Communicated by Jeff R. L. Webb

Abstract. New and known properties of the resolvent of the kernel of linear Abel integral equations of the form

$$x(t) = f(t) - \lambda \int_0^t (t-s)^{q-1} x(s)\, ds, \tag{A_λ}$$

where $\lambda > 0$ and $q \in (0,1)$, are assembled and derived here. First, *a priori* bounds on potential solutions of the *resolvent equation*

$$R(t) = \lambda t^{q-1} - \lambda \int_0^t (t-s)^{q-1} R(s)\, ds \tag{R_λ}$$

are obtained. Second, it is proven—using these bounds, Banach's contraction mapping principle, new continuation and translation results, and Schaefer's fixed point theorem—that (R_λ) has a unique continuous solution on $(0, \infty)$, which is called the *resolvent* in the literature and denoted here by $R(t)$. Third, both known and new properties of $R(t)$ are derived. Fourth, $R(t)$ is shown to be completely monotone and the unique continuous solution of the initial value problem of fractional order q:

$$D^q x(t) = -\lambda \Gamma(q) x(t), \qquad \lim_{t \to 0^+} t^{1-q} x(t) = \lambda, \tag{I_λ}$$

where D^q denotes the Riemann–Liouville fractional differential operator. Finally, the *resolvent integral function* $\int_0^t R(s)\, ds$ is shown to be the unique continuous solution of an integral equation closely related to (R_λ). Closed-form expressions for it and $R(t)$ are derived.

Keywords: Abel integral equations, fixed points, fractional differential equations, Mittag-Leffler functions, resolvents, Riemann–Liouville operators, singular kernels, Volterra integral equations.

2010 Mathematics Subject Classification: 34A08, 34A12, 45D05, 45E10.

✉ Email: lbecker@cbu.edu

1 Introduction

This is both an expository and research paper. It is expository in that many of the results for the resolvent of the kernel $\lambda(t-s)^{q-1}$ have been collected from various sources and are conveniently assembled here. It is also a research paper in that many of the derivations of the properties of the resolvent have not appeared before—additionally, there are new results, such as explicit formulas for the resolvent and the resolvent integral function mentioned in the abstract and showing that the resolvent equation is equivalent to a fractional initial value problem. Our approach is to first prove that the resolvent equation has a unique continuous solution on the entire interval $(0,\infty)$, which will then serve as a springboard for deriving important and useful properties of this solution, both known and new.

Resolvents are used to express the solutions of Volterra equations. Volterra integral and integro-differential equations aid in modeling a host of situations: ranging from the integro-differential equations of population [23] and pharmacokinetics [35] models; to the integral equations of renewal theory [12, 16]; to the partial differential equation models of heat conduction and diffusion problems that can be recast as integral equations, such as are found in [23, 27]; and the list goes on. More recently, Heese and Freyberger state that the coupled Heisenberg equations of motion that appear in their paper [21, p. 4] take on the form of a Volterra integro-differential equation—after explicitly solving the dynamics of certain particles—and that its general solution can formally be expressed by means of a resolvent [2, 20]. Gorenflo and Vessella [18, p. 142] model the temperature $u(x,t)$ along a one-dimensional, semi-infinite rod ($x \geq 0$) with the heat equation and assume that Newtonian heating takes place at the boundary $x = 0$. The model of the inside boundary temperature $u(0,t)$ simplifies to a Volterra integral equation which they then solve using the method of Laplace transforms. However, there is the alternative of expressing the solution in terms of a resolvent [4, p. 4840].

The material in this paper revolves around the *resolvent equation*

$$R(t) = \lambda t^{q-1} - \lambda \int_0^t (t-s)^{q-1} R(s)\, ds \tag{R_λ}$$

and its fractional differential equation counterpart, where λ and q are positive constants with $q \in (0,1)$. The singularities of the functions λt^{q-1} and $\lambda(t-s)^{q-1}$ present challenges, but they can be surmounted by choosing a suitable Banach space and employing a translation to circumvent the singularity at $t = 0$. The fact that (R_λ) plays a role in a multitude of diverse applications, especially when $q = 1/2$, warrants a thorough investigation of this equation.

This is one in a series of papers [6–10] constituting a study of the scalar Volterra integral equation

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds \tag{1.1}$$

together with the scalar fractional differential equation

$$D^q x(t) = f(t, x(t)) \tag{1.2a}$$

subject to the initial condition

$$\lim_{t\to 0^+} t^{1-q} x(t) = x^0, \tag{1.2b}$$

where $f\colon (0,T] \times I \to \mathbb{R}$ denotes a continuous function and $I \subseteq \mathbb{R}$ an unbounded interval while x^0, q denote constants with $x^0 \neq 0$ and $q \in (0,1)$. The symbol D^q denotes the *Riemann–*

Liouville fractional differential operator of order q, which for $0 < q < 1$ is defined by

$$D^q x(t) := \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} x(s)\, ds$$

where Γ is Euler's Gamma function.

One of the main results in [6] is an "equivalence theorem", which asserts that if a function $x(t)$ is continuous on an interval $(0, T]$ and if both $x(t)$ and $f(t, x(t))$ are absolutely integrable on $(0, T]$, then $x(t)$ satisfies the Volterra integral equation (1.1) on $(0, T]$ if and only if it satisfies the initial value problem (1.2) on this same interval. In [7] we proved a couple of theorems that establish the existence and uniqueness of a continuous, absolutely integrable solution of (1.1) and (1.2) on an interval $(0, T]$ when $f(t, x)$ satisfies either a Lipschitz condition or a generalized Lipschitz condition. We also discussed the continuation of such a solution beyond T by means of a transformation of the integral equation (1.1) followed by a translation. In [9] we obtain an existence theorem for (1.1) that is atypical of the standard existence theorems in the literature in that (i) a growth condition obviates the need for f to be bounded or even to satisfy a Lipschitz condition and (ii) the existence of solutions is dependent on the value of q. Equations with closed-form solutions were constructed to illustrate the main results in all of these papers.

2 Related equations

In this paper our interest in (1.1) and (1.2) is confined to

$$f(t, x) = -\lambda \Gamma(q) x \tag{2.1}$$

with $\lambda > 0$ and $q \in (0, 1)$. For the most part, attention will be directed to the integral equation (1.1) rather than to (1.2); note that any results obtained for it will also pertain to (1.2) because of the aforementioned equivalence theorem in [6]. Generally speaking, we use $x(t)$ to denote the unknown function; however, we reserve $z(t)$ for denoting the unknown function in (1.1) when $f(t, x)$ has the special form (2.1). In this case, (1.1) simplifies to

$$z(t) = x^0 t^{q-1} - \lambda \int_0^t (t-s)^{q-1} z(s)\, ds \tag{2.2}$$

where $x^0 \neq 0$, $\lambda > 0$, and $q \in (0, 1)$.

Equation (2.2) and the resolvent equation (R_λ) are intertwined: since one can be easily converted into the other, any solutions they may have are related by a simple formula. Specifically, multiplying (R_λ) by x^0/λ yields

$$\frac{x^0}{\lambda} R(t) = x^0 t^{q-1} - \lambda \int_0^t (t-s)^{q-1} \frac{x^0}{\lambda} R(s)\, ds.$$

Then replacing $(x^0/\lambda) R(t)$ with $z(t)$, we get

$$z(t) = x^0 t^{q-1} - \lambda \int_0^t (t-s)^{q-1} z(s)\, ds,$$

which is (2.2). So if a function $R(t)$ is a solution of (R_λ), then the function

$$z(t) := \frac{x^0}{\lambda} R(t). \tag{2.3}$$

is a solution of (2.2). The actual proof of the existence of a continuous solution $R(t)$ of (R_λ) is found in the next section.

Strictly speaking, the function

$$C(t-s) := \lambda(t-s)^{q-1} \tag{2.4}$$

is known as the *kernel* of equations (2.2) and (R_λ); nevertheless, for ease of communication, we will also refer to $C(t)$ as the kernel.

Changing variables, we obtain the following alternate form of the resolvent equation (R_λ):

$$R(t) = \lambda t^{q-1} - \lambda \int_0^t R(t-s)s^{q-1}\, ds. \tag{R_λ^a}$$

3 Existence and uniqueness of solutions

Some of the results in this paper were originally obtained by R. K. Miller in 1968 and 1971 and G. Gripenberg in 1978 (using results and techniques in a 1963 paper by A. Friedman [17]) for a Volterra integral equation, namely (R_A) below, that includes (R_λ) as a special case. However, by investigating (R_λ) directly, we can employ arguments less intricate in their details and which do not presuppose familiarity with the works of the aforementioned authors. Moreover, we obtain a number of new results.

Miller proves that

$$\widehat{R}(t) = A(t) - \int_0^t A(t-s)\widehat{R}(s)\, ds \tag{R_A}$$

has a unique continuous solution on the entire open interval $(0,\infty)$ if the kernel A is positive, continuous, and nonincreasing on $(0,\infty)$; $A \in L^1(0,1)$; and for each $T > 0$, the quotient $A(t)/A(t+T)$ is a nonincreasing function of t on $(0,\infty)$. For details, consult Miller [26, Thm. 2] who gives an existence and uniqueness proof based on a result in the 1963 paper by Friedman [17, pp. 384–387]—or see Miller's monograph [27, Thm. 6.2]. Since these conditions are satisfied when $A(t) = \lambda t^{q-1}$ and $\lambda > 0$, it follows that (R_λ) has a unique continuous solution on $(0,\infty)$. More recently, a constructive proof for the equation (R_λ) that exploits $\lambda(t-s)^{q-1}$ being weakly singular and employs iterated kernels to derive and express the unique continuous solution of (R_λ) in terms of a series can be found in [4, Thm. 4.2].

Here we present another proof of the existence and uniqueness of a continuous solution of (R_λ), which delineates a procedure that could be adapted to certain types of nonlinear Volterra integral equations to establish the local existence of a continuous solution on an interval $[0,\tau]$ and then to continue that solution beyond τ. (Recent local existence results for nonlinear equations can be found in [7, Thms. 2.7, 4.1]. Another result appears in [9, Thm. 3.1].) By focusing on the specific kernel $\lambda(t-s)^{q-1}$ instead of on the kernel of (R_A), we will be able to obtain results for (R_λ) that may not necessarily pertain to the more general (R_A).

The steps that we will employ for the resolvent equation (R_λ) are the following.

(S1) Determine *a priori* bounds for any continuous solutions of (R_λ) that may exist.

(S2) Prove the existence and uniqueness of a continuous solution of (R_λ) on an interval $(0,\tau]$ with an appropriate fixed point mapping.

(S3) Prove that if the local solution obtained in (S2) can be continued past τ, then that continuation is unique.

(S4) Insert a parameter $\nu \in (0, 1]$ into (R_λ) (cf. (3.12)) to set it up for the eventual use of Schaefer's fixed point theorem. Then translate this modified equation in order to bypass the singularity of the forcing function at $t = 0$. Determine *a priori* bounds for any possible solutions of the translated equation.

(S5) Apply Schaefer's theorem to prove that the translated equation with $\nu = 1$ has a continuous solution on every finite interval $[0, b]$. Prove each of these solutions is unique. Use this uniqueness to prove the existence of a unique continuous solution on $[0, \infty)$.

(S6) Finally, construct the unique continuous solution of (R_λ) on $(0, \infty)$ by splicing together the solutions obtained in (S2) and (S5).

The details of each of these steps are spelled out in the rest of this section. In the following subsection, we will establish the existence of *a priori* bounds for any continuous solutions of (R_λ) that may exist.

3.1 The resolvent equation and *a priori* bounds

Since the purpose of carrying out steps (S1)–(S6) is to prove the existence of a continuous solution of (R_λ), the first thing to consider is whether the improper Riemann integral

$$\Phi(t) := \int_0^t (t-s)^{q-1} \phi(s)\, ds \tag{3.1}$$

exists when $\phi(s)$ is continuous for $s > 0$, and if so, whether the integral itself is continuous. This can be answered with the help of the following lemma whose proof is found in [4, (2.6)–(2.8)]. (It is also very similar to the proof of Lemma 8.1 that appears later in this paper.)

Lemma 3.1. *If a function ϕ is continuous on an interval $[0, b]$, then the improper Riemann integral $\Phi(t)$ defined by (3.1) defines a function that is also continuous on $[0, b]$. Furthermore,*

$$|\Phi(t_1) - \Phi(t_2)| \le \frac{2m}{q} |t_1 - t_2|^q$$

for all $t_1, t_2 \in [0, b]$, where $m := \sup\{|\phi(t)| : 0 \le t \le b\}$.

But this is not quite the result that we need here since any solution of (R_λ) is necessarily undefined at $t = 0$; nonetheless, it can be used to prove the following lemma for functions that are both continuous and absolutely integrable on the left-open interval $(0, b]$. Details are provided in [6, pp. 7–8].

Lemma 3.2. *If a function ϕ is continuous and absolutely integrable on $(0, b]$, then the integral $\Phi(t)$ defined by (3.1) is continuous on $(0, b]$.*

With the help of the next lemma, we will prove that there are *a priori* bounds for continuous solutions of (R_λ) should any exist. Notice that if one does exist, call it $R(t)$, then equation (R_λ) suggests that

$$t^{1-q} R(t) \to \lambda \quad \text{as } t \to 0^+. \tag{3.2}$$

In fact, see (3.11) and the subsequent sentence.

Lemma 3.3. *If there exists a continuous solution $R(t)$ of (R_λ) on an interval $(0, T]$ that satisfies (3.2), then it is initially positive. That is, a $T_0 \in (0, T]$ exists such that $R(t) > 0$ on $(0, T_0]$.*

Proof. For a given $\epsilon \in (0, \lambda)$, there exists a $T_0 \in (0, T]$ such that the postulated solution $R(t)$ satisfies

$$|t^{1-q} R(t) - \lambda| < \epsilon$$

for $0 < t \le T_0$. Hence, on this interval,

$$R(t) > \frac{\lambda - \epsilon}{t^{1-q}} > 0. \qquad \square$$

Theorem 3.4. *If there exists a continuous solution $R(t)$ of* (R_λ) *on an interval $(0, T]$ that satisfies* (3.2)*, then it must lie in the strip bounded by the t-axis and the forcing function. That is,*

$$0 \le R(t) \le \lambda t^{q-1}$$

for $0 < t \le T$.

Proof. According to Lemma 3.3, a $T_0 \in (0, T]$ exists such that $R(t) > 0$ for $0 < t \le T_0$. Since it is trivially true that $R(t) \ge 0$ for $0 < t \le T$ if $T_0 = T$, assume $T_0 < T$. Suppose to the contrary that $R(t)$ eventually becomes negative at a point $t_0 \in (T_0, T)$; namely, suppose $R(t) > 0$ for $t \in (0, t_0)$, $R(t_0) = 0$, and $R(t) < 0$ for $t \in (t_0, t_1)$ for some $t_1 \in (t_0, T]$. Thus, for an arbitrary $t \in (t_0, t_1)$, $R(t) < 0$ with

$$\begin{aligned} R(t) &= \lambda t^{q-1} - \lambda \left[\int_0^{t_0} (t-s)^{q-1} R(s)\, ds + \int_{t_0}^{t} (t-s)^{q-1} R(s)\, ds \right] \\ &\ge \lambda t^{q-1} - \lambda \int_0^{t_0} (t-s)^{q-1} R(s)\, ds \\ &= \frac{t^{q-1}}{t_0^{q-1}} \left[\lambda t_0^{q-1} - \lambda \int_0^{t_0} \frac{t_0^{q-1}}{t^{q-1}} (t-s)^{q-1} R(s)\, ds \right]. \end{aligned}$$

Since

$$\frac{t_0}{t} \ge \frac{t_0 - s}{t - s}$$

and t^{q-1} is decreasing on $(0, T]$, we have

$$\left(\frac{t_0}{t} \right)^{q-1} \le \left(\frac{t_0 - s}{t - s} \right)^{q-1}$$

for $0 \le s < t_0 < t$. And so

$$\frac{t_0^{q-1}}{t^{q-1}} (t-s)^{q-1} \le (t_0 - s)^{q-1}.$$

Consequently, as $R(s) \ge 0$ on $(0, t_0]$,

$$R(t) \ge \frac{t^{q-1}}{t_0^{q-1}} \left[\lambda t_0^{q-1} - \lambda \int_0^{t_0} (t_0 - s)^{q-1} R(s)\, ds \right] = \frac{t^{q-1}}{t_0^{q-1}} R(t_0) = 0.$$

But this contradicts $R(t) < 0$. Therefore, $R(t) \ge 0$ for all $t \in (0, T]$. This in turn implies

$$R(t) = \lambda t^{q-1} - \lambda \int_0^t (t-s)^{q-1} R(s)\, ds \le \lambda t^{q-1}. \qquad \square$$

3.2 Local solution of the resolvent equation

Now that we have found *a priori* bounds for continuous solutions of (R_λ) should any exist, we proceed to the next step (S2). We will prove that a continuous solution does in fact exist—at least on a short interval. Moreover, it is unique. In the rest of this section, $R(t)$ designates this unique continuous solution.

Theorem 3.5. *For each $q \in (0,1)$ and $\lambda > 0$, a unique continuous solution $R(t)$ of the resolvent equation (R_λ) exists on the interval $(0,\tau]$ where*

$$\tau := \left[\frac{\Gamma(2q)}{2\lambda\Gamma^2(q)}\right]^{1/q}. \tag{3.3}$$

$R(t)$ is absolutely integrable on $(0,\tau]$ and $t^{1-q}R(t)$ has the limit:

$$\lim_{t\to 0^+} t^{1-q}R(t) = \lambda. \tag{3.4}$$

Also,

$$0 \le R(t) \le \lambda t^{q-1} \tag{3.5}$$

for $0 < t \le \tau$.

Remark 3.6. The interval $(0,\tau]$ is really quite short unless $\lambda \ll 1$. For instance, with $\lambda = 1$ and $q = 1/2$,

$$\tau = \left[\frac{\Gamma(1)}{2(1)\Gamma^2(\frac{1}{2})}\right]^2 = \frac{1}{4\pi^2} \approx 0.025.$$

Proof. For τ given by (3.3), let $C(0,\tau]$ denote the vector space of all continuous functions $\phi\colon (0,\tau] \to \mathbb{R}$. Define the subset $X \subset C(0,\tau]$ by

$$X := \left\{\phi \in C(0,\tau] : \sup_{0<t\le\tau} \frac{|\phi(t)|}{t^{q-1}} < \infty\right\} \tag{3.6}$$

and the function $|\cdot|_g\colon X \to \mathbb{R}$ by

$$|\phi|_g := \sup_{0<t\le\tau} \frac{|\phi(t)|}{g(t)} \tag{3.7}$$

for $\phi \in X$, where $g(t) := t^{q-1}$. It is a straightforward exercise to prove X is a subspace of $C(0,\tau]$ and $|\cdot|_g$ a norm on X. Moreover, $(X, |\cdot|_g)$ is a Banach space (cf. [7, Thm. 2.3] for details).

Define the set

$$M := \{\phi \in C(0,\tau] : |\phi(t)| \le 2\lambda t^{q-1}\}. \tag{3.8}$$

Let ρ denote the metric provided by the norm $|\cdot|_g$; i.e., for $\phi, \psi \in X$,

$$\rho(\phi,\psi) := |\phi - \psi|_g.$$

Since one can show that M is a closed subset of the complete metric space (X,ρ), the metric space (M,ρ) is also complete. And so if we can establish that the mapping P defined by

$$(P\phi)(t) := \lambda t^{q-1} - \lambda \int_0^t (t-s)^{q-1}\phi(s)\, ds \tag{3.9}$$

for $\phi \in M$ maps M into itself and is a contraction on M, then it would follow from Banach's contraction mapping principle that P has a unique fixed point in M.

First we show $P\colon M \to M$. So let $\phi \in M$. By virtue of being in M, it is absolutely integrable on $(0, \tau]$. As a result, Lemma 3.2 implies that $P\phi \in C(0, \tau]$. Furthermore, we see from (3.8) that

$$\begin{aligned} |(P\phi)(t)| &\le \lambda t^{q-1} + \lambda \int_0^t (t-s)^{q-1} |\phi(s)|\, ds \\ &\le \lambda t^{q-1} + \lambda \int_0^t (t-s)^{q-1} \left(2\lambda s^{q-1}\right) ds \end{aligned}$$

for $0 < t \le \tau$. Let us evaluate the integral with the following formula that is derived from the Beta function (cf. [6, (4.4)]):

$$\int_0^t (t-s)^{p-1} s^{q-1}\, ds = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\, t^{p+q-1} \quad (t > 0); \tag{3.10}$$

it converges if and only if $p, q > 0$. Then, using τ from (3.3), we have

$$|(P\phi)(t)| \le \left[\lambda + 2\lambda^2 \frac{\Gamma^2(q)}{\Gamma(2q)} t^q\right] t^{q-1} \le \left[\lambda + \frac{2\lambda^2\Gamma^2(q)}{\Gamma(2q)} \tau^q\right] t^{q-1} = 2\lambda t^{q-1}$$

for $0 < t \le \tau$. Thus, $P\colon M \to M$.

In order to show that P is a contraction on M, select any $\phi, \psi \in M$. Then $\phi, \psi \in X$ as $M \subset X$. Consequently,

$$|\phi(t) - \psi(t)| \le t^{q-1} |\phi - \psi|_g$$

and

$$\begin{aligned} \frac{|(P\phi)(t) - (P\psi)(t)|}{t^{q-1}} &\le \lambda t^{1-q} \int_0^t (t-s)^{q-1} |\phi(s) - \psi(s)|\, ds \\ &\le \lambda t^{1-q} \int_0^t (t-s)^{q-1} s^{q-1} |\phi - \psi|_g\, ds \end{aligned}$$

for $0 < t \le \tau$. Hence, because of (3.10),

$$\begin{aligned} \frac{|(P\phi)(t) - (P\psi)(t)|}{t^{q-1}} &\le \lambda t^{1-q} |\phi - \psi|_g \int_0^t (t-s)^{q-1} s^{q-1}\, ds \\ &\le \lambda t^{1-q} |\phi - \psi|_g \frac{\Gamma^2(q)}{\Gamma(2q)} t^{2q-1} \le \left[\frac{\lambda\Gamma^2(q)}{\Gamma(2q)} \tau^q\right] |\phi - \psi|_g \end{aligned}$$

for $0 < t \le \tau$. Let

$$\alpha := \frac{\lambda\Gamma^2(q)}{\Gamma(2q)} \tau^q.$$

In fact, because of the value of τ given by (3.3), $\alpha = 1/2$. Thus we have

$$\sup_{0<t\le\tau} \frac{|(P\phi)(t) - (P\psi)(t)|}{t^{q-1}} \le \alpha |\phi - \psi|_g.$$

In other words,

$$|P\phi - P\psi|_g \le \alpha |\phi - \psi|_g$$

where $\alpha \in (0, 1)$. And so P is a contraction on M.

Therefore, as $P\colon M \to M$ is a contraction mapping, there is a unique point $R \in M$ such that $PR = R$. In other words, R is the only continuous function residing in the set M that satisfies the resolvent equation (R_λ) on the interval $(0, \tau]$.

Since $R \in M$, $|R(t)| \le 2\lambda t^{q-1}$ for $0 < t \le \tau$. Thus it is absolutely integrable on $(0, \tau]$ since

$$\int_0^\tau |R(s)|\, ds \le \int_0^\tau 2\lambda s^{q-1}\, ds = \frac{2\lambda}{q}\, \tau^q < \infty.$$

Also observe from (3.10) that

$$\begin{aligned}\left| t^{1-q} \int_0^t (t-s)^{q-1} R(s)\, ds \right| &\le t^{1-q} \int_0^t (t-s)^{q-1} |R(s)|\, ds \\ &\le 2\lambda\, t^{1-q} \frac{\Gamma^2(q)}{\Gamma(2q)}\, t^{2q-1} = \frac{2\lambda\Gamma^2(q)}{\Gamma(2q)}\, t^q.\end{aligned}$$

Hence,

$$\lim_{t\to 0^+} t^{1-q} \int_0^t (t-s)^{q-1} R(s)\, ds = 0. \tag{3.11}$$

And so from

$$t^{1-q} R(t) = \lambda - \lambda t^{1-q} \int_0^t (t-s)^{q-1} R(s)\, ds,$$

we see that $t^{1-q}R(t) \to \lambda$ as $t \to 0^+$, which proves (3.4).

Now that we have established that the continuous solution R satisfies (3.4), it follows from Theorem 3.4 that R actually resides in a subset of M, namely $\{\phi \in C(0, \tau] : 0 \le \phi(t) \le \lambda t^{q-1}\}$. Moreover, this theorem rules out any continuous solutions existing outside this set. Thus $R(t)$ is the only continuous solution of (R_λ) on the interval $(0, \tau]$. □

3.3 Continuation of the solution

Let us move on to (S3) and show that there is only one possible way to continue the local solution of Theorem 3.5.

Theorem 3.7. *If, in Theorem 3.5, the local continuous solution $R(t)$ of the resolvent equation (R_λ) can be extended beyond τ so that it remains a continuous solution of (R_λ), then that continued solution is unique.*

Proof. Suppose to the contrary that R_1 and R_2 are continuous solutions of (R_λ) that separate at a point $t_1 \ge \tau$ but which are identical on the interval $(0, t_1]$. Then for $\delta > 0$ sufficiently small, the difference $R_2(t) - R_1(t)$ is nonzero and does not change sign for $t \in (t_1, t_1 + \delta]$. However, this results in a contradiction since the left- and right-hand sides of

$$R_2(t_1+\delta) - R_1(t_1+\delta) = -\lambda \int_{t_1}^{t_1+\delta} (t_1+\delta-s)^{q-1} \big[R_2(s) - R_1(s)\big]\, ds$$

have opposite signs. □

Now we will move on to steps (S4)–(S6) to prove that $R(t)$ can indeed be extended beyond $(0, \tau]$ to the entire interval $(0, \infty)$ so that it remains a continuous solution of the resolvent equation (R_λ). Before we do so, let us summarize what has been proven so far:

(a) The resolvent equation (R_λ) has a unique, continuous solution $R(t)$ on an interval $(0, \tau]$. Moreover, it is nonnegative and bounded above by the forcing function λt^{q-1}. And $t^{q-1}R(t)$ has the limit given by (3.4). (Cf. Theorem 3.5.)

(b) If a continuous solution of (R_λ) exists on an interval $(0, L]$ that is longer than $(0, \tau]$, then it is the only such continuous solution. Also, it is nonnegative and bounded above by the forcing function λt^{q-1} for $0 < t \leq L$. (Cf. Theorems 3.4 and 3.7.)

3.4 The translated equation and *a priori* bounds

In order to prove that the solution $R(t)$ can be continued beyond the point τ (cf. (3.3)), we first insert a parameter $\nu \in (0, 1]$ into the resolvent equation (R_λ), thereby obtaining the equation

$$r(t) = \nu \left[\lambda t^{q-1} - \lambda \int_0^t (t-s)^{q-1} r(s)\, ds \right]. \tag{3.12}$$

This is to get ready for the eventual application of Schaefer's fixed point theorem (cf. Theorem 3.10).

Actually we will work with a translation of (3.12) (cf. (S4)) so as to bypass the singularity of the forcing function $\nu\lambda t^{q-1}$ at $t = 0$ (more on this later). Note that (R_λ) becomes (3.12) when λ is replaced with $\nu\lambda$. Consequently, for a given $\nu \in (0, 1]$, items (a) and (b) in the summary also pertain to (3.12) if in those statements λ and $R(t)$ are replaced with $\nu\lambda$ and $r(t)$, respectively, where the corresponding value of τ is

$$\tau = \left[\frac{\Gamma(2q)}{2\nu\lambda\Gamma^2(q)} \right]^{1/q}. \tag{3.13}$$

Of course, with $\nu = 1$, then $r(t)$ is precisely $R(t)$ and (3.12) is the resolvent equation

$$R(t) = \lambda t^{q-1} - \lambda \int_0^t (t-s)^{q-1} R(s)\, ds. \tag{R_λ}$$

To obtain the translation of (3.12) to which we just alluded, first let

$$x(t) := r(t+T) \quad \text{where} \quad T := \tfrac{1}{2}\tau. \tag{3.14}$$

Then replace t in (3.12) with $t + T$:

$$\begin{aligned} r(t+T) &= \nu \left[\lambda (t+T)^{q-1} - \lambda \int_0^{t+T} (t+T-s)^{q-1} r(s)\, ds \right] \\ &= \nu\lambda (t+T)^{q-1} - \nu\lambda \int_0^T (t+T-s)^{q-1} r(s)\, ds \\ &\quad - \nu\lambda \int_T^{t+T} (t+T-s)^{q-1} r(s)\, ds. \end{aligned}$$

With the change of variable $u = s - T$, we get

$$\begin{aligned} r(t+T) = \nu\lambda (t+T)^{q-1} &- \nu\lambda \int_0^T (t+T-s)^{q-1} r(s)\, ds \\ &- \nu\lambda \int_0^t (t-u)^{q-1} r(u+T)\, du. \end{aligned}$$

Now use (3.14) to express this in terms of x:

$$x(t) = \nu\lambda \left[(t+T)^{q-1} - \int_0^T (t+T-s)^{q-1} \, r(s) \, ds \right] \\ - \nu\lambda \int_0^t (t-u)^{q-1} x(u) \, du.$$

And so we have

$$x(t) = \nu \left[\lambda F(t) - \lambda \int_0^t (t-s)^{q-1} \, x(s) \, ds \right] \tag{3.15}$$

where

$$F(t) := (t+T)^{q-1} - \int_0^T (t+T-s)^{q-1} \, r(s) \, ds. \tag{3.16}$$

Recall from the discussion before (3.13) with regard to items (a) and (b) that for each value of $\nu \in (0,1]$ there is a unique continuous solution of (3.12) on the interval $(0,\tau] \equiv (0,2T]$, where τ is given by (3.13). Letting $r(t)$ denote this solution, it trivially follows from the preceding work that (3.15) always has a solution on the interval $[0,T]$, namely (3.14), which is identical to the piece of $r(t)$ on the interval $[T,2T]$. We show next that if (3.12) has a solution $x(t)$ extending beyond T, then it and $r(t)$ can be spliced together to obtain a continuation of the solution $r(t)$.

Theorem 3.8. *For a given $\nu \in (0,1]$, let $r(t)$ be the unique continuous solution of (3.12) on the interval $(0,\tau]$ where τ is given by (3.13). Let $T = \tau/2$. If for some $b > T$ there exists a continuous solution $x(t)$ of the translated equation (3.15) on the interval $[0,b]$, then the function*

$$r_c(t) := \begin{cases} r(t), & \text{if } \; 0 < t < T \\ x(t-T), & \text{if } \; T \le t \le b+T \end{cases} \tag{3.17}$$

is a continuation of $r(t)$ to the interval $(0, b+T]$. Furthermore, it is the only continuous solution of (3.12) on this interval and

$$0 \le r_c(t) \le \nu\lambda t^{q-1} \tag{3.18}$$

for all $0 < t \le b+T$.

Proof. Since $r_c \equiv r$ on $(0,T)$, it follows from (3.12) that

$$r_c(t) = \nu\lambda t^{q-1} - \nu\lambda \int_0^t (t-s)^{q-1} r_c(s) \, ds \quad (0 < t < T). \tag{3.19}$$

Thus, $r_c(t)$ is a continuous solution of (3.12) on $(0,T)$.

Now consider r_c to the right of T. By hypothesis, $x(t)$ is a continuous solution of (3.15) on $[0,b]$; so

$$\begin{aligned} x(t) &= \nu\lambda F(t) - \nu\lambda \int_0^t (t-s)^{q-1} \, x(s) \, ds \\ &= \nu\lambda (t+T)^{q-1} - \nu\lambda \int_0^T (t+T-s)^{q-1} r(s) \, ds - \nu\lambda \int_0^t (t-s)^{q-1} x(s) \, ds. \end{aligned}$$

With the change of variable $u = s+T$, this becomes

$$x(t) = \nu\lambda (t+T)^{q-1} - \nu\lambda \int_0^T (t+T-s)^{q-1} r(s) \, ds - \nu\lambda \int_T^{t+T} (t+T-u)^{q-1} x(u-T) \, du.$$

Expressed in terms of the function r_c, this simplifies to

$$x(t) = \nu\lambda(t+T)^{q-1} - \nu\lambda \int_0^{t+T} (t+T-s)^{q-1} r_c(s)\, ds$$

for $0 \le t \le b$. Or, equivalently,

$$x(t-T) = \nu\lambda t^{q-1} - \nu\lambda \int_0^{t} (t-s)^{q-1} r_c(s)\, ds$$

for $T \le t \le b+T$. So,

$$r_c(t) = \nu\lambda t^{q-1} - \nu\lambda \int_0^{t} (t-s)^{q-1} r_c(s)\, ds \quad (T \le t \le b+T). \tag{3.20}$$

Thus $r_c(t)$ is a continuous solution of (3.12) on $[T, b+T]$.

Since $r(t)$ is continuous at T and $x(t-T)$ is continuous from the right at this point, we see from (3.12), (3.15), (3.16), and (3.17) that

$$\begin{aligned}\lim_{t\to T^-} r_c(t) = \lim_{t\to T^-} r(t) = r(T) &= \nu\left[\lambda T^{q-1} - \lambda \int_0^{T} (T-s)^{q-1} r(s)\, ds\right] \\ &= \nu\lambda F(0) = x(0) = \lim_{t\to T^+} x(t-T) = \lim_{t\to T^+} r_c(t).\end{aligned}$$

Hence,

$$\lim_{t\to T} r_c(t) = x(0) = r_c(T).$$

This and the continuity of $r(t)$ and $x(t-T)$ on their respective intervals in (3.17) imply $r_c(t)$ is continuous on the entire interval $(0, b+T]$. Therefore, it follows from (3.19) and (3.20) that $r_c(t)$ is a continuous solution of (3.12) on $(0, b+T]$.

As we discussed earlier, item (b) applies not only to (R_λ) but to (3.12) as well. Hence, $r_c(t)$ is the only continuous solution of this equation on $(0, b+T]$. Moreover, from (b) we also have (3.18). □

In the context of Theorem 3.8, the uniqueness of the solutions $r(t)$ and $r_c(t)$ on their respective domains implies that $r_c(t)$ is identical to $r(t)$ on the interval $(0, 2T]$. A corollary of this theorem is that there are *a priori* bounds for solutions of (3.15)—for those existing on $[0, T]$ and for any possibly existing on a longer interval—and that the bounds are independent of the interval.

Corollary 3.9. *Suppose, for a given $\nu \in (0, 1]$, a solution x of the translated equation (3.15) exists on an interval $[0, b]$, where the function r in the integrand of (3.16) is the unique continuous solution of (3.12) on the interval $(0, \tau]$ with τ given by (3.13). Then, irrespective of the values of b and ν,*

$$0 \le x(t) \le \lambda T^{q-1} \tag{3.21}$$

for $0 \le t \le b$, where $T = \tau/2$.

Proof. Recall that there is always a solution of (3.15) on $[0, T]$, namely $x(t) = r(t+T)$, and that

$$0 \le r(t+T) \le \nu\lambda(t+T)^{q-1}$$

for $0 \le t \le T$. Thus, for a given positive $b < T$,

$$0 \le x(t) \le \nu\lambda(t+T)^{q-1}$$

for $0 \le t \le b$. This implies (3.21) since t^{q-1} is a decreasing function and $0 < \nu \le 1$.

Now suppose $b > T$ and that there is a solution $x(t)$ of (3.15) on $[0, b]$. It then follows from (3.17) that $x(t) = r_c(t+T)$ for $0 \le t \le b$. By (3.18),

$$0 \le r_c(t+T) \le \nu\lambda(t+T)^{q-1}.$$

Thus (3.21) also holds for $0 \le t \le b$ when $b > T$. □

3.5 The translated equation and solutions

At this juncture, we move on to (S5). Now our objective is to prove that there actually is a continuous solution of (3.15) when $\nu = 1$ on every finite interval $[0, b]$. This will be achieved with Corollary 3.9 and the version of Schaefer's fixed point theorem [33] for normed spaces [34, p. 29] that is stated below in Theorem 3.10. Then we show that this equation has a unique continuous solution on $[0, \infty)$.

So we will address existence as a fixed point problem and work with the following Banach space and mapping. For a given $b > 0$, let $\mathcal{B}$ denote the Banach space of continuous functions $\phi\colon [0, b] \to \mathbb{R}$ with the supremum norm

$$\|\phi\| = \sup\{|\phi(t)| : 0 \le t \le b\}. \tag{3.22}$$

Define the mapping $\mathcal{P}$ on $\mathcal{B}$ by

$$(\mathcal{P}\phi)(t) := \lambda F(t) - \lambda \int_0^t (t-s)^{q-1}\phi(s)\,ds. \tag{3.23}$$

Expressed in terms of this operator, (3.15) is

$$x(t) = \nu(\mathcal{P}x)(t). \tag{3.24}$$

Referring back to Corollary 3.9, we see that the set of all continuous solutions of (3.24) is bounded, irrespective of the length of the interval $[0, b]$ or the value of $\nu \in (0, 1]$. Consequently, for the mapping $\mathcal{P}$ defined by (3.23), this rules out the alternative labeled (ii) in the following statement of Schaefer's fixed point theorem.

Theorem 3.10 (Schaefer)**.** *Let $(\mathcal{B}, \|\cdot\|)$ be a normed space, $\mathcal{P}$ a continuous mapping of $\mathcal{B}$ into $\mathcal{B}$ which is compact on each bounded subset Ω of $\mathcal{B}$. Then either*

(i) *the equation $x = \nu\mathcal{P}x$ has a solution for $\nu = 1$, or*

(ii) *the set of all solutions of $x = \nu\mathcal{P}x$ for $0 < \nu < 1$ is unbounded.*

This leaves (i), namely, the existence of a continuous solution of the equation $x = \mathcal{P}x$ on the interval $[0, b]$, provided $\mathcal{P}$ satisfies the hypotheses of Schaefer's theorem. Showing this then becomes our present task. Let us begin with the next two lemmas to prove that the function F in (3.23) is uniformly continuous on $[0, \infty)$. Recall that the function r appearing in its definition (cf. (3.16)) is the unique continuous solution of (3.12) on $(0, \tau] \equiv (0, 2T]$ and that it is nonnegative and bounded above by $\nu\lambda t^{q-1}$.

Lemma 3.11. *The function*

$$\mathsf{G}(t) := \int_0^T (t+T-s)^{q-1} r(s)\,ds$$

is uniformly continuous on $[a, \infty)$ for each $a > 0$.

Proof. Let $a > 0$ be given. Choose $\epsilon > 0$. Since the function t^{q-1} is continuous on $[a, \infty)$ and $t^{q-1} \to 0$ as $t \to \infty$, it is uniformly continuous on $[a, \infty)$. Hence, for a given $\eta > 0$, there is a $\gamma > 0$ such that distinct $t_1, t_2 \in [a, \infty)$ and

$$|(t_1 + T - s) - (t_2 + T - s)| = |t_1 - t_2| < \gamma$$

and $T - s \geq 0$ imply that

$$\left|(t_1 + T - s)^{q-1} - (t_2 + T - s)^{q-1}\right| < \eta.$$

Hence for these t_i,

$$\begin{aligned} |\mathsf{G}(t_1) - \mathsf{G}(t_2)| &\leq \int_0^T \left|(t_1 + T - s)^{q-1} - (t_2 + T - s)^{q-1}\right| |r(s)| \, ds \\ &\leq \eta \int_0^T |r(s)| \, ds. \end{aligned}$$

Since $r(t)$ is nonnegative and bounded above by $\nu\lambda t^{q-1}$,

$$|\mathsf{G}(t_1) - \mathsf{G}(t_2)| \leq \eta \int_0^T \nu\lambda s^{q-1} \, ds = \frac{\eta\nu\lambda}{q} T^q.$$

Choose any $\eta < q\epsilon/(\nu\lambda T^q)$. Therefore, for the given $\epsilon > 0$, there is a $\gamma > 0$ such that $|t_1 - t_2| < \gamma$ implies $|G(t_1) - G(t_2)| < \epsilon$. □

Lemma 3.12. *The function F defined by* (3.16), *namely*

$$F(t) := (t + T)^{q-1} - \mathsf{G}(t),$$

is uniformly continuous on $[0, \infty)$.

Proof. First observe that the term $(t + T)^{q-1}$ is not only continuous on $[0, \infty)$ but it is also uniformly continuous on this interval because $(t + T)^{q-1} \to 0$ as $t \to \infty$.

Solving for $F(t)$ in (3.15), we get

$$F(t) = \frac{1}{\nu\lambda} x(t) + \mathcal{X}(t)$$

where

$$\mathcal{X}(t) := \int_0^t (t - s)^{q-1} x(s) \, ds.$$

Since $r(t)$ denotes the unique continuous solution of (3.12) on $(0, 2T]$, the function $x(t)$, defined by (3.14) as the translation of $r(t)$ to the left by T units, is continuous on $(-T, T]$. It then follows from Lemma 3.1 that $\mathcal{X}(t)$ is continuous on the subinterval $[0, T]$. Hence $F(t)$ is continuous on $[0, T]$. This and the continuity of the first term in (3.16) imply that $\mathsf{G}(t)$ is also continuous on $[0, T]$.

Thus $\mathsf{G}(t)$ is uniformly continuous on $[0, T]$. By Lemma 3.11 it is also uniformly continuous on $[T/2, \infty)$. Consequently, $\mathsf{G}(t)$ is uniformly continuous on $[0, \infty)$. Therefore, because both $(t + T)^{q-1}$ and $\mathsf{G}(t)$ are uniformly continuous on $[0, \infty)$, so is $F(t)$. □

We will also use the following lemma to help establish the uniqueness of the solution in Theorem 3.14 below.

Lemma 3.13. *The only continuous solution of*

$$x(t) = -\lambda \int_0^t (t-s)^{q-1} x(s)\, ds \quad (\lambda > 0) \tag{3.25}$$

on $[0, \infty)$ *is the trivial solution* $x(t) \equiv 0$.

Proof. First we prove with the contraction mapping principle that $x(t) \equiv 0$ is the only continuous solution on an interval $[0, b]$ if b is sufficiently small. To this end, take $b = (q/(2\lambda))^{1/q}$. Then let $\mathcal{B}$ be the Banach space of continuous functions on $[0, b]$ with the sup norm (3.22). For $\phi \in \mathcal{B}$, define the mapping $\mathcal{Z}$ by

$$(\mathcal{Z}\phi)(t) := -\lambda \int_0^t (t-s)^{q-1} \phi(s)\, ds.$$

It follows from Lemma 3.1 that $\mathcal{Z}\colon \mathcal{B} \to \mathcal{B}$. Moreover, $\mathcal{Z}$ is a contraction on $\mathcal{B}$ since for $\phi, \psi \in \mathcal{B}$ we have

$$\begin{aligned} |(\mathcal{Z}\phi)(t) - (\mathcal{Z}\psi)(t)| &\le \lambda \int_0^t (t-s)^{q-1} |\phi(s) - \psi(s)|\, ds \\ &\le \lambda \|\phi - \psi\| \int_0^t (t-s)^{q-1}\, ds \le \frac{\lambda}{q} b^q \|\phi - \psi\|. \end{aligned}$$

And so

$$\|\mathcal{Z}\phi - \mathcal{Z}\psi\| \le \frac{1}{2} \|\phi - \psi\|.$$

Thus $\mathcal{Z}$ has a unique fixed point in $\mathcal{B}$. So it must be the trivial fixed point.

Now suppose there is another continuous solution $y(t)$ on the entire interval $[0, \infty)$ besides $x(t) \equiv 0$. Then, in view of the uniqueness of the latter on $[0, b]$, there are $t_1 \ge b$ and $\delta > 0$ such that $y(t) \equiv 0$ on $[0, t_1]$ but which is strictly positive or strictly negative on $(t_1, t_1 + \delta]$. As a result,

$$y(t_1 + \delta) = -\lambda \int_{t_1}^{t_1+\delta} (t_1 + \delta - s)^{q-1} y(s)\, ds,$$

a contradiction since the sides are opposite in sign. □

Finally we are poised to prove that the translated equation (3.15) with $\nu = 1$ has a unique continuous solution on the entire interval $[0, \infty)$ and that it is bounded and nonnegative.

Theorem 3.14. *The translated equation*

$$x(t) = \lambda F(t) - \lambda \int_0^t (t-s)^{q-1} x(s)\, ds, \tag{3.26}$$

where F is defined by (3.16), *has a unique continuous solution $x(t)$ on $[0, \infty)$. Moreover,*

$$0 \le x(t) \le \lambda T^{q-1} \tag{3.27}$$

for all $t \ge 0$, where $T = \tau/2$ and τ is defined by (3.3).

Proof. Choose any $b > 0$. As before, let $(\mathcal{B}, \|\cdot\|)$ denote the Banach space of continuous functions on $[0, b]$ with the sup norm $\|\cdot\|$. And let $\mathcal{P}$ be the mapping on $\mathcal{B}$ defined by (3.23). It follows from Lemmas 3.1 and 3.12 that the function $\mathcal{P}\phi$ is continuous on $[0, b]$ for each $\phi \in \mathcal{B}$. Thus, $\mathcal{P}\colon \mathcal{B} \to \mathcal{B}$.

To prove $\mathcal{P}$ is continuous on $\mathcal{B}$, choose any $\phi_1 \in \mathcal{B}$. Choose $\epsilon > 0$. Let $\delta = (q\epsilon)/(\lambda b^q)$. Then for $\phi_2 \in \mathcal{B}$ with $\|\phi_1 - \phi_2\| < \delta$, we have

$$|(\mathcal{P}\phi_1)(t) - (\mathcal{P}\phi_2)(t)| \le \lambda \int_0^t (t-s)^{q-1} |\phi_1(s) - \phi_2(s)|\, ds$$
$$\le \lambda \|\phi_1 - \phi_2\| \int_0^t (t-s)^{q-1}\, ds \le \lambda \|\phi_1 - \phi_2\| \frac{t^q}{q}.$$

Thus,

$$|(\mathcal{P}\phi_1)(t) - (\mathcal{P}\phi_2)(t)| \le \frac{\lambda b^q}{q} \|\phi_1 - \phi_2\|$$

for all $t \in [0, b]$. So

$$\|\mathcal{P}\phi_1 - \mathcal{P}\phi_2\| \le \frac{\lambda b^q}{q} \|\phi_1 - \phi_2\| < \frac{\lambda b^q}{q} \delta = \epsilon.$$

Thus, $\mathcal{P}\colon \mathcal{B} \to \mathcal{B}$ is continuous.

Next we will prove that $\mathcal{P}$ is compact on each bounded subset of $\mathcal{B}$, to wit: the image $\mathcal{P}\Omega$ of each bounded set $\Omega \subset \mathcal{B}$ is contained in a compact subset of $\mathcal{B}$. We will show that for each bounded set Ω it follows that $\mathcal{P}\Omega$ is bounded, equicontinuous, and contained in a closed set. Application of Arzelà–Ascoli's theorem will then complete the proof of compactness of the mapping $\mathcal{P}$. First let us establish that the set $\mathcal{P}\Omega$ of functions is uniformly bounded and equicontinuous on $[0, b]$.

Choose $a > 0$ large enough so that the closed ball

$$B_a := \{\phi \in \mathcal{B} : \|\phi\| \le a\}$$

contains Ω. For any $\phi \in \Omega$,

$$|(\mathcal{P}\phi)(t)| \le \lambda \left[|F(t)| + \int_0^t (t-s)^{q-1} |\phi(s)|\, ds \right]$$
$$\le \lambda \left[\|F\| + \|\phi\| \int_0^t (t-s)^{q-1}\, ds \right] = \lambda \left[\|F\| + \|\phi\| \frac{t^q}{q} \right].$$

Hence, as $\Omega \subset B_a$,

$$|(\mathcal{P}\phi)(t)| \le \lambda \left[\|F\| + a \frac{b^q}{q} \right] \tag{3.28}$$

for all $\phi \in \Omega$ and all $t \in [0, b]$. Thus the set $\mathcal{P}\Omega$ of functions is uniformly bounded on $[0, b]$.

To prove $\mathcal{P}\Omega$ is equicontinuous on $[0, b]$, choose any $\phi \in \Omega$. Then we see from Lemma 3.1 that

$$|(\mathcal{P}\phi)(t_1) - (\mathcal{P}\phi)(t_2)|$$
$$\le \lambda |F(t_1) - F(t_2)| + \lambda \left| \int_0^{t_1} (t_1 - s)^{q-1} \phi(s)\, ds - \int_0^{t_2} (t_2 - s)^{q-1} \phi(s)\, ds \right|$$
$$\le \lambda |F(t_1) - F(t_2)| + \lambda \frac{2\|\phi\|}{q} |t_1 - t_2|^q$$

for all $t_1, t_2 \in [0, b]$. And so as $\|\phi\| \le a$,

$$|(\mathcal{P}\phi)(t_1) - (\mathcal{P}\phi)(t_2)| \le \lambda |F(t_1) - F(t_2)| + \frac{2a\lambda}{q} |t_1 - t_2|^q$$

for all $\phi \in \Omega$ and $t_1, t_2 \in [0, b]$. For a given $\epsilon > 0$, let

$$\eta = \left[\frac{q\epsilon}{4a\lambda}\right]^{1/q}.$$

Since F is uniformly continuous on $[0, b]$, there is a $\delta \in (0, \eta)$ such that

$$|F(t_1) - F(t_2)| < \frac{\epsilon}{2\lambda}$$

whenever $|t_1 - t_2| < \delta$. Consequently, $t_1, t_2 \in [0, b]$ and $|t_1 - t_2| < \delta$ imply

$$|(\mathcal{P}\phi)(t_1) - (\mathcal{P}\phi)(t_2)| < \lambda\left(\frac{\epsilon}{2\lambda}\right) + \frac{2a\lambda}{q}\eta^q = \frac{\epsilon}{2} + \frac{2a\lambda}{q}\left(\frac{q\epsilon}{4a\lambda}\right) = \epsilon$$

for all $\phi \in \Omega$, which concludes the proof of equicontinuity.

Since the set $\mathcal{P}\Omega$ of functions is equicontinuous on $[0, b]$, so is its closure $\overline{\mathcal{P}\Omega}$. This is easily seen with the following $\epsilon/3$ argument. Let $\epsilon > 0$. Suppose ψ is a limit point of $\mathcal{P}\Omega$. Then there is a sequence $\{\mathcal{P}\phi_n\}_{n=1}^{\infty} \subset \mathcal{P}\Omega$ with $\|\mathcal{P}\phi_n - \psi\| \to 0$ as $n \to \infty$. Thus the sequence converges uniformly on $[0, b]$ to ψ. Choose n_1 sufficiently large so that

$$|(\mathcal{P}\phi_{n_1})(t) - \psi(t)| < \frac{\epsilon}{3}$$

for all $t \in [0, b]$. And as $\mathcal{P}\Omega$ is equicontinuous on $[0, b]$, a $\delta > 0$ exists such that $t_1, t_2 \in [0, b]$ and $|t_1 - t_2| < \delta$ imply

$$|(\mathcal{P}\phi_{n_1})(t_1) - (\mathcal{P}\phi_{n_1})(t_2)| < \frac{\epsilon}{3}.$$

As a result,

$$\begin{aligned}|\psi(t_1) - \psi(t_2)| \le |\psi(t_1) - (\mathcal{P}\phi_{n_1})(t_1)| + |(\mathcal{P}\phi_{n_1})(t_1) - (\mathcal{P}\phi_{n_1})(t_2)|& \\ + |(\mathcal{P}\phi_{n_1})(t_2) - \psi(t_2)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon&\end{aligned}$$

whenever $t_1, t_2 \in [0, b]$ and $|t_1 - t_2| < \delta$. Therefore every limit point of $\mathcal{P}\Omega$ satisfies the same equicontinuity condition as do all the functions constituting $\mathcal{P}\Omega$.

Likewise, $\overline{\mathcal{P}\Omega}$ is uniformly bounded on $[0, b]$. As before, suppose ψ is a limit point of $\mathcal{P}\Omega$ with a sequence $\{\mathcal{P}\phi_n\}_{n=1}^{\infty}$ converging uniformly to it. Let $\mu > 0$. Then for a sufficiently large n_2, we see from (3.28) that

$$|\psi(t)| \le |\psi(t) - (\mathcal{P}\phi_{n_2})(t)| + |(\mathcal{P}\phi_{n_2})(t)| < \mu + \lambda\left[\|F\| + a\frac{b^q}{q}\right]$$

for all $t \in [0, b]$. It follows that $\lambda\left[\|F\| + ab^q/q\right]$ is not only an upper bound for all of the functions in $\mathcal{P}\Omega$ but also for all of the limit points of $\mathcal{P}\Omega$.

It follows from the Arzelà–Ascoli theorem that the set $\overline{\mathcal{P}\Omega}$ is compact since it is uniformly bounded and equicontinuous on $[0, b]$ and of course closed. Thus we have shown that $\mathcal{P}$ maps every bounded set Ω in $\mathcal{B}$ into a compact subset of $\mathcal{B}$, namely into $\overline{\mathcal{P}\Omega}$.

Therefore, we have established that the mapping $\mathcal{P}$ fulfills all of the conditions of Schaefer's fixed point theorem. Consequently we have alternative (i) in Theorem 3.10 since (ii) has already been ruled out. In other words, we conclude that (3.26) has a continuous solution on the interval $[0, b]$, no matter the value of $b > 0$. Moreover, Corollary 3.9 shows that this solution is nonnegative and bounded above by λT^{q-1}, which is the assertion of (3.27).

As for uniqueness, suppose that $x(t)$ and $y(t)$ are continuous solutions of (3.26) on $[0, b]$. Then

$$x(t) - y(t) = -\lambda \int_0^t (t-s)^{q-1}[x(s) - y(s)]\, ds.$$

By Lemma 3.13, $x(t) \equiv y(t)$ on $[0, b]$.

We have just proved equation (3.26) has a unique continuous solution on every finite interval $[0, b]$. In fact, because of this uniqueness, it has a unique continuous solution on the entire interval $[0, \infty)$. We justify this assertion with the following argument. Consider the sequence $\{x_n\}_{n=1}^{\infty}$, where x_n denotes the unique continuous solution of (3.26) on the interval $[0, n]$. Define the function $x\colon [0, \infty) \to \mathbb{R}$ as follows: For a given $t \geq 0$, choose $n \in \mathbb{N}$ (set of natural numbers) so that $n > t$; then let $x(t) := x_n(t)$. This function is well-defined because $x_{n+p} \equiv x_n$ on the interval $[0, n]$ for each $p \in \mathbb{N}$.

Also, for every $n \in \mathbb{N}$, define the function $\widetilde{x}_n\colon [0, \infty) \to \mathbb{R}$ by

$$\widetilde{x}_n(t) := \begin{cases} x_n(t), & \text{if } \; 0 \leq t \leq n \\ x_n(n), & \text{if } \; t > n. \end{cases} \tag{3.29}$$

Clearly the sequence $\{\widetilde{x}_n\}_{n=1}^{\infty}$ of functions converges uniformly to x on compact subsets of $[0, \infty)$. Thus x is continuous on $[0, n]$ for each $n \in \mathbb{N}$; and consequently on $[0, \infty)$.

For a given $t \in [0, \infty)$, choose $n > t$. Then, as x_n is a solution of (3.26) on $[0, n]$,

$$x_n(t) = \lambda F(t) - \lambda \int_0^t (t-s)^{q-1}\, x_n(s)\, ds.$$

So, as $x \equiv x_n$ on $[0, n]$,

$$x(t) = \lambda F(t) - \lambda \int_0^t (t-s)^{q-1}\, x(s)\, ds.$$

Moreover, $|x(t)| = |x_n(t)| \leq \lambda T^{q-1}$. □

3.6 Global solution of the resolvent equation

Now that we have established the existence of a unique continuous solution of the translated equation (3.26) on $[0, \infty)$, we splice it to the unique continuous solution of (R_λ) on the short interval $(0, \tau]$. As a result, we obtain the following global existence and uniqueness result.

Theorem 3.15. *Let $R(t)$ be the unique continuous solution of the resolvent equation (R_λ) on $(0, \tau]$, where τ is defined by (3.3). Let $x(t)$ be the unique continuous solution of the translated equation (3.26) on $[0, \infty)$. Let $T = \tau/2$. Then the function*

$$\widetilde{R}(t) := \begin{cases} R(t), & \textit{if } \; 0 < t < T \\ x(t-T), & \textit{if } \; t \geq T \end{cases} \tag{3.30}$$

is the unique continuous solution of the resolvent equation (R_λ) on $(0, \infty)$. Furthermore,

$$0 \leq \widetilde{R}(t) \leq \lambda t^{q-1} \tag{3.31}$$

for $t > 0$.

Proof. This follows from Theorem 3.14, Theorem 3.8 with $\nu = 1$, and a "sequence of functions" argument like the one at the end of the proof of Theorem 3.14. □

Note. Henceforth, the function $\widetilde{R}(t)$ defined by (3.30) will be renamed $R(t)$. That is, for the remainder of this paper, $R(t)$ will denote the unique continuous solution of the equation (R_λ), equivalently of (R_λ^q), on the entire interval $(0,\infty)$. In the literature of integral equations, $R(t)$ is known either as the *resolvent* [27, p. 21] or the *resolvent kernel* [23, p. 38] of $C(t) = \lambda t^{q-1}$. We prefer the term *resolvent* and will usually omit the mention of $C(t)$ since it is the only kernel considered in this paper.

The next result follows from the discussion about the function (2.3) and Theorem 3.15. It is also valid when $x_0 = 0$ (cf. Lemma 3.13).

Corollary 3.16. *Let $x_0 \in \mathbb{R}$ and $\lambda > 0$. The function*

$$z(t) = \frac{x^0}{\lambda} R(t)$$

is the unique continuous solution of (2.2) *on* $(0,\infty)$. *In particular,*

$$z(t) = -R(t)$$

is the unique continuous solution of

$$z(t) = -\lambda t^{q-1} - \lambda \int_0^t (t-s)^{q-1} z(s)\, ds.$$

4 Properties of the resolvent

We now prove, based on our work in Section 3, some known properties of the resolvent R that have been used in some of the references at the end of this paper. We also obtain some new results. For example, we improve property (i) below significantly with Corollary 4.6; and with Theorem 4.5 we improve upon the bounds of $\int_0^t R(s)\, ds$ in (viii) by sandwiching the integral between functions that approach 1 as $t \to \infty$. In Theorem 4.10 we shall see that the kernel C of (R_λ) is the unique continuous solution of a complementary integral equation. Additional new results for R will be derived in the remaining sections of this paper.

Theorem 4.1. *The resolvent R has the following properties:*

(i) *For all $t > 0$, $0 \le R(t) \le \lambda t^{q-1}$.*

(ii) *For a given $b > 0$, $0 \le R(t) \le \lambda b^{q-1}$ for all $t \ge b$.*

(iii) *For all $t > 0$, $0 \le \int_0^t (t-s)^{q-1} R(s)\, ds \le t^{q-1}$.*

(iv) *As $t \to \infty$, $R(t) \to 0$.*

(v) *As $t \to 0^+$, $R(t) \to \infty$.*

(vi) *As $t \to 0^+$, $t^{1-q} \int_0^t (t-s)^{q-1} R(s)\, ds \to 0$.*

(vii) *As $t \to 0^+$, $t^{q-1} R(t) \to \lambda$.*

(viii) *For every $t > 0$, R is (improperly) integrable on $(0,t]$ and*

$$0 \le \int_0^t R(s)\, ds \le 1. \tag{4.1}$$

Proof. Property (i) is a restatement of (3.31). And as t^{q-1} is a decreasing function and $\lambda > 0$, $\lambda t^{q-1} \leq \lambda b^{q-1}$ for $t \geq b$; hence (ii). From (i) and (R_λ) we see that

$$0 \leq \lambda \left[t^{q-1} - \int_0^t (t-s)^{q-1} R(s)\, ds \right] \leq \lambda t^{q-1}$$

for all $t > 0$. This implies (iii) since $\lambda > 0$. Clearly (i) implies (iv). As for (v), use (i) and integration formula (3.10), namely

$$\int_0^t (t-s)^{p-1} s^{q-1}\, ds = t^{p+q-1} \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \qquad (p, q > 0),$$

to get

$$\int_0^t (t-s)^{q-1} R(s)\, ds \leq \lambda \int_0^t (t-s)^{q-1} s^{q-1}\, ds = \lambda t^{2q-1} \frac{\Gamma^2(q)}{\Gamma(2q)}.$$

Thus,

$$\begin{aligned} R(t) &= \lambda \left[t^{q-1} - \int_0^t (t-s)^{q-1} R(s)\, ds \right] \\ &\geq \lambda \left[t^{q-1} - \lambda t^{2q-1} \frac{\Gamma^2(q)}{\Gamma(2q)} \right] = \lambda t^{q-1} \left[1 - \frac{\lambda \Gamma^2(q)}{\Gamma(2q)} t^q \right]. \end{aligned}$$

Since the very last quantity increases without bound as $t \to 0^+$, (v) obtains. And (vi) follows from

$$0 \leq t^{1-q} \int_0^t (t-s)^{q-1} R(s)\, ds \leq \frac{\lambda \Gamma^2(q)}{\Gamma(2q)} t^q.$$

Property (vi) together with (R_λ) implies (vii).

Finally, we prove (viii). As t^{q-1} is decreasing on $(0, \infty)$,

$$\int_0^{t-\delta} R(t-s) s^{q-1}\, ds \geq t^{q-1} \int_0^{t-\delta} R(t-s)\, ds$$

for all $\delta \in (0, t)$. Consequently, from the alternate form (R_λ^a) of the resolvent equation, we have

$$\begin{aligned} R(t) &= \lambda \left[t^{q-1} - \int_0^t R(t-s) s^{q-1}\, ds \right] \leq \lambda \left[t^{q-1} - \int_0^{t-\delta} R(t-s) s^{q-1}\, ds \right] \\ &\leq \lambda \left[t^{q-1} - t^{q-1} \int_0^{t-\delta} R(t-s)\, ds \right] = \lambda t^{q-1} \left[1 - \int_\delta^t R(u)\, du \right]. \end{aligned}$$

Therefore, as $R(t) \geq 0$, we have

$$0 \leq \int_\delta^t R(u)\, du \leq 1$$

for all $\delta \in (0, t)$. Since, for a fixed $t > 0$, $R(u) \geq 0$ on $(0, t)$, the integral is a monotone function of δ for $0 < \delta < t$. And as it is also bounded, its one-sided limit as $\delta \to 0^+$ exists. Therefore,

$$\int_0^t R(u)\, du = \lim_{\delta \to 0^+} \int_\delta^t R(u)\, du \leq 1. \qquad \square$$

We show next that the integral in (viii) is dominated by a function whose value at every $t \geq 0$ is strictly less than 1.

Lemma 4.2. *For all* $t \geq 0$, $\int_0^t R(s)\,ds \leq 1 - e^{-\frac{\lambda}{q}t^q}$.

Proof. For a given $t > 0$, $(t-s)^{q-1}$ increases in s for $s \in [0,t)$. Thus, as $R(s) \geq 0$ for $s > 0$,

$$\int_0^t (t-s)^{q-1} R(s)\,ds \geq t^{q-1} \int_0^t R(s)\,ds.$$

Then, as $\lambda > 0$, it follows from the resolvent equation (R_λ) that

$$R(t) \leq \lambda t^{q-1} \left[1 - \int_0^t R(s)\,ds\right] \tag{4.2}$$

for all $t > 0$.

Define the function u by

$$u(t) := 1 - \int_0^t R(s)\,ds.$$

Since

$$\frac{du}{dt} = -R(t),$$

the inequality (4.2) expressed in terms of u is

$$\frac{du}{dt} + \lambda t^{q-1} u \geq 0.$$

Multiplying this by the integrating factor

$$\exp \int \lambda t^{q-1}\,dt = e^{\frac{\lambda}{q}t^q},$$

we obtain

$$e^{\frac{\lambda}{q}t^q} \frac{du}{dt} + \lambda t^{q-1} e^{\frac{\lambda}{q}t^q} u \geq 0$$

or

$$\frac{d}{dt}\left[e^{\frac{\lambda}{q}t^q} u(t)\right] \geq 0$$

for all $t > 0$. Hence, for a given $t > 0$ and an arbitrary $\delta \in (0,t)$,

$$e^{\frac{\lambda}{q}t^q} u(t) \geq e^{\frac{\lambda}{q}\delta^q} u(\delta) \geq u(\delta).$$

Consequently,

$$e^{\frac{\lambda}{q}t^q} u(t) \geq 1 - \int_0^\delta R(s)\,ds.$$

Since this is true for every $\delta \in (0,t)$,

$$e^{\frac{\lambda}{q}t^q} u(t) \geq 1.$$

Therefore,

$$1 - \int_0^t R(s)\,ds \geq e^{-\frac{\lambda}{q}t^q}$$

for every $t > 0$. Note this holds for $t = 0$ as well. □

Later on we appeal to either the following theorem or to its corollary to justify interchanging the order of integration in some of the proofs.

Theorem 4.3. *If a function φ is continuous and nonnegative on $(0,\infty)$ and integrable on $(0,1)$, then*

$$\int_0^t \int_0^u (u-s)^{q-1}\varphi(s)\,ds\,du = \int_0^t \int_s^t (u-s)^{q-1}\varphi(s)\,du\,ds$$

for all $t > 0$.

Proof. For a given $t > 0$, let Ω_t denote the interior of the triangular region

$$\Delta_t := \{(u,s) : 0 \le s \le u \le t\}.$$

Define the function f_t on the rectangle $[0,t] \times [0,t]$ by

$$f_t(u,s) := \begin{cases} (u-s)^{q-1}\varphi(s) & \text{if} \quad (u,s) \in \Omega_t \\ 0 & \text{if} \quad (u,s) \in \Delta_t \backslash \Omega_t. \end{cases}$$

Observe that f_t is discontinuous on the diagonal $\{(u,u) : 0 \le u \le t\}$. It is also discontinuous on the right side of the rectangle if $\varphi(t) \neq 0$ and on the bottom side unless φ is defined at $t = 0$ with $\varphi(0) = 0$. At all other points it is continuous. So f_t is continuous almost everywhere since these line segments have measure 0 in $\mathbb{R}^2$. Consequently, f_t is measurable on $[0,t] \times [0,t]$.

Next note that

$$\int_0^t \int_0^t |f_t(u,s)|\,du\,ds < \infty$$

since

$$\begin{aligned} \int_0^t \int_0^t |f_t(u,s)|\,du\,ds &= \int_0^t \left[\int_s^t (u-s)^{q-1}\,du \right] \varphi(s)\,ds \\ &= \frac{1}{q} \int_0^t (t-s)^q \varphi(s)\,ds \le \frac{t^q}{q} \int_0^t \varphi(s)\,ds < \infty, \end{aligned} \tag{4.3}$$

where the inequalities are consequences of t^q being an increasing function and the integrability of φ on every interval $(0,t)$. Then the finiteness of this iterated integral and the measurability of f_t imply both

$$\int_0^t \int_0^t f_t(u,s)\,du\,ds \quad \text{and} \quad \int_0^t \int_0^t f_t(u,s)\,ds\,du$$

exist and are equal by the Tonelli–Hobson test (cf. [28, p. 93]), [1, p. 415]). The first iterated integral can be written as (4.3) and the second one as

$$\int_0^t \int_0^t f_t(u,s)\,ds\,du = \int_0^t \int_0^u (u-s)^{q-1}\varphi(s)\,ds\,du. \qquad \square$$

According to Theorems 3.15 and 4.1, R is continuous and nonnegative on $(0,\infty)$ and integrable on $(0,1)$. So it satisfies the conditions of Theorem 4.3.

Corollary 4.4.

$$\int_0^t \int_0^u (u-s)^{q-1}R(s)\,ds\,du = \int_0^t \int_s^t (u-s)^{q-1}R(s)\,du\,ds$$

for all $t > 0$.

Theorem 4.5. *For given $q \in (0,1)$ and $\lambda > 0$,*

$$\frac{1}{1+\frac{q}{\lambda t^q}} \le \int_0^t R(s)\,ds \le 1 - e^{-\frac{\lambda t^q}{q}} \tag{4.4}$$

for all $t > 0$.

Proof. We only need to prove the left-hand inequality since the right-hand inequality has already been established with Lemma 4.2. Integrating (R_λ) and using Corollary 4.4 to interchange the order of integration, we obtain

$$\begin{aligned}\int_0^t R(u)\,du &= \lambda\left[\int_0^t u^{q-1}\,du - \int_0^t\int_0^u (u-s)^{q-1}R(s)\,ds\,du\right]\\ &= \lambda\left[\frac{t^q}{q} - \int_0^t\left(\int_s^t (u-s)^{q-1}\,du\right)R(s)\,ds\right]\\ &= \lambda\left[\frac{t^q}{q} - \int_0^t \frac{(t-s)^q}{q}R(s)\,ds\right].\end{aligned}$$

Thus, as t^q is increasing, we have

$$\int_0^t R(u)\,du \geq \lambda\left[\frac{t^q}{q} - \frac{t^q}{q}\int_0^t R(s)\,ds\right] = \frac{\lambda t^q}{q}\left[1 - \int_0^t R(s)\,ds\right].$$

Consequently,

$$\int_0^t R(s)\,ds \geq \frac{1}{1+\frac{q}{\lambda t^q}}$$

for all $t > 0$. □

Theorem 4.5 enables us to greatly improve upon the function bounding $R(t)$ from above in Theorem 4.1 (i). Later on, with Theorem 7.3, we show that $R(t)$ is strictly positive. Thus, the integrals in (iii) and (viii) of Theorem 4.1 never assume the value zero on the interval $(0,\infty)$.

Corollary 4.6. *For given $q \in (0,1)$ and $\lambda > 0$,*

$$0 < R(t) \leq \left(\frac{q}{q+\lambda t^q}\right)\lambda t^{q-1} \tag{4.5}$$

for all $t > 0$.

Proof. From (4.2) and (4.4), it follows that

$$\begin{aligned}R(t) &\leq \lambda t^{q-1}\left[1 - \int_0^t R(s)\,ds\right] \leq \lambda t^{q-1}\left[1 - \frac{1}{1+\frac{q}{\lambda t^q}}\right]\\ &\leq \lambda t^{q-1}\left[1 - \frac{\lambda t^q}{\lambda t^q + q}\right] = \frac{q\lambda t^{q-1}}{q+\lambda t^q}.\end{aligned}$$

□

Remark 4.7. Expressed in terms of $C(t) = \lambda t^{q-1}$, (4.5) is

$$0 < R(t) \leq \frac{C(t)}{1+\int_0^t C(s)\,ds}.$$

This is a special case of a result by Gripenberg [19, (1.9)].

Since the functions bounding the integral in Theorem 4.5 approach 1 as $t \to \infty$, we have the following important result.

Corollary 4.8. $\int_0^\infty R(t)\,dt = 1$ *for all $q \in (0,1)$ and $\lambda > 0$.*

Remark 4.9. Closed-form expressions for $\int_0^t R(s)\,ds$, which we call the *resolvent integral function*, are derived later on in Theorem 9.5.

Theorem 4.10. *Let $R(t)$ be the resolvent of $C(t) = \lambda t^{q-1}$. Then $C(t)$ is the unique continuous solution of*

$$y(t) = R(t) + \int_0^t R(t-s)y(s)\,ds \tag{4.6}$$

on the interval $(0,\infty)$.

Proof. With Theorem 3.15 we established that the resolvent equation (R_λ) has a unique continuous solution on $(0,\infty)$, which we now denote by $R(t)$. Thus, from (R_λ) we have

$$\lambda t^{q-1} = R(t) + \int_0^t \lambda(t-s)^{q-1}R(s)\,ds$$

for all $t > 0$. With the change of variable $u = t - s$, this becomes

$$\lambda t^{q-1} = R(t) + \int_0^t \lambda u^{q-1}R(t-u)\,du,$$

verifying that $C(t)$ is a solution of (4.6) on $(0,\infty)$.

As for uniqueness, suppose $D(t)$ is a continuous solution of (4.6) on an interval $(0,b]$. Then

$$C(t) - D(t) = \int_0^t R(t-s)[C(s) - D(s)]\,ds$$

for $t \in (0,b]$. Consequently,

$$|C(t) - D(t)| \le \|C - D)\| \int_0^t R(t-s)\,ds \le \|C - D\| \int_0^b R(s)\,ds,$$

where $\|\cdot\|$ denotes the sup norm (3.22). Thus,

$$\|C - D\| \le \|C - D\| \int_0^b R(s)\,ds.$$

This implies $D \equiv C$ on $(0,b]$ since $\int_0^b R(s)\,ds < 1$ (cf. Theorem 4.5). In other words, the only continuous solution of (4.6) on an interval $(0,b]$, regardless of the value of $b > 0$, is $C(t) = \lambda t^{q-1}$. It then follows from a "sequence of solutions" argument such as the one concluding the proof of Theorem 3.14 that $C(t)$ is the unique continuous solution of (4.6) on $(0,\infty)$. □

5 A complementary fractional differential equation

In this section, we show that for a given $\lambda > 0$ and $q \in (0,1)$ that the unique continuous solution of the resolvent equation (R_λ) is also the unique continuous solution of a related linear fractional differential equation. This result is a consequence of the following equivalence theorem, which is taken from [6, Thm. 6.2].

Theorem 5.1. *Let $q \in (0,1)$ and $x^0 \neq 0$. Let $f(t,x)$ be a function that is continuous on the set*

$$\mathcal{B} := \{(t,x) \in \mathbb{R}^2 : 0 < t \le T, x \in I\}$$

where $I \subseteq \mathbb{R}$ is an unbounded interval. Suppose a function $x\colon (0,T] \to I$ is continuous and both $x(t)$ and $f(t,x(t))$ are absolutely integrable on $(0,T]$. Then $x(t)$ satisfies the Volterra integral equation

$$x(t) = x^0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x(s))\,ds \tag{1.1}$$

on $(0, T]$ *if and only if it satisfies the initial value problem*

$$D^q x(t) = f(t, x(t)), \quad \lim_{t \to 0^+} t^{1-q} x(t) = x^0 \tag{1.2}$$

on $(0, T]$.

As we stated earlier in Section 1, the focus of this paper is on (1.1) and (1.2) when $f(t, x)$ is the linear function (2.1), where $q \in (0, 1)$ and $\lambda > 0$ throughout this paper. In this case, Theorem 5.1 implies the following.

Theorem 5.2. *The resolvent* $R(t)$ *is the unique continuous solution of the initial value problem*

$$D^q x(t) = -\lambda \Gamma(q) x(t), \quad \lim_{t \to 0^+} t^{1-q} x(t) = \lambda. \tag{I_λ}$$

on $(0, \infty)$.

Proof. We see from Corollary 4.8 that the resolvent $R(t)$ is absolutely integrable on $(0, \infty)$. Since it is the unique continuous solution of (1.1) on $(0, \infty)$ when $f(t, x(t)) = -\lambda\Gamma(q)x(t)$ and $x^0 = \lambda$, Theorem 5.1 implies that it is also the unique continuous solution of (I_λ) on $(0, \infty)$. □

6 The resolvent formula

There is a simple closed-form formula for the resolvent of a Volterra integral equation with a separable kernel (cf. [5, 23]). Is there a formula for the resolvent of the kernel of equation (R_λ)? The answer is yes, but it is not simple. We will now derive that formula using Laplace transforms, where $\mathcal{L}\{f(t)\}(s)$ denotes the Laplace transform of a function f.

Theorem 6.1. *The Laplace transform of the resolvent* $R(t)$ *exists and is*

$$\mathcal{L}\{R(t)\}(s) = \frac{\lambda\Gamma(q)}{s^q + \lambda\Gamma(q)} \tag{6.1}$$

for $s \geq 0$.

Proof. We begin by showing $\int_0^\infty e^{-st} R(t)\, dt$ converges for all $s \geq 0$. First let $s > 0$. It follows from Theorem 4.1 (i) that

$$0 \leq e^{-st} R(t) \leq \lambda e^{-st} t^{q-1} \tag{6.2}$$

for all $t > 0$. Fix $T > 0$. Then for any $\eta \in (0, T)$,

$$\int_\eta^T e^{-st} R(t)\, dt \leq \lambda \int_\eta^T e^{-st} t^{q-1}\, dt.$$

The integral on the right-hand side converges because

$$\int_0^\infty e^{-st} t^{q-1}\, dt = \mathcal{L}\{t^{q-1}\}(s) = \frac{\Gamma(q)}{s^q} \tag{6.3}$$

for $s > 0$. Thus $\int_0^T e^{-st} R(t)\, dt$ is convergent by the comparison test.

Similarly, as

$$\int_T^b e^{-st} R(t)\, dt \leq \lambda \int_T^b e^{-st} t^{q-1}\, dt$$

for $b > T$, the integral $\int_T^\infty e^{-st}R(t)\,dt$ is also convergent.

Therefore, for each $s > 0$, the integral $\int_0^\infty e^{-st}R(t)\,dt$ is convergent. As for $s = 0$, it follows from Corollary 4.8 that

$$\int_0^\infty e^{-st}R(t)\,dt = \int_0^\infty R(t)\,dt = 1.$$

Since we have established the existence of the Laplace transform

$$\mathcal{L}\{R(t)\}(s) := \int_0^\infty e^{-st}R(t)\,dt$$

for $s \geq 0$, we can now take the Laplace transform of equation (R_λ):

$$\begin{aligned}\mathcal{L}\{R(t)\} &= \lambda\mathcal{L}\{t^{q-1}\} - \lambda\mathcal{L}\left\{\int_0^t (t-s)^{q-1}R(s)\,ds\right\}\\ &= \lambda\mathcal{L}\{t^{q-1}\} - \lambda\mathcal{L}\{t^{q-1}\}\mathcal{L}\{R(t)\}\end{aligned}$$

noting that the integral is the convolution of t^q and $R(t)$. Thus,

$$\mathcal{L}\{R(t)\} = \frac{\lambda\mathcal{L}\{t^{q-1}\}}{1+\lambda\mathcal{L}\{t^{q-1}\}} = \frac{\lambda\frac{\Gamma(q)}{s^q}}{1+\lambda\frac{\Gamma(q)}{s^q}} = \frac{\lambda\Gamma(q)}{s^q+\lambda\Gamma(q)}$$

for $s \geq 0$. □

Let us now derive the resolvent formula for values of $q \in (0,1)$.

Theorem 6.2. *For a given $q \in (0,1)$, the resolvent is*

$$R(t) = \lambda\Gamma(q)t^{q-1}E_{q,q}(-\lambda\Gamma(q)t^q) \tag{6.4}$$

where $E_{m,n}(t)$ denotes the two-parameter Mittag-Leffler function:

$$E_{m,n}(t) := \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(km+n)} \tag{6.5}$$

where m, n are positive constants. That is, (6.4) is the unique continuous solution of

$$R(t) = \lambda t^{q-1} - \lambda\int_0^t (t-s)^{q-1}R(s)\,ds \tag{R_λ}$$

on $(0,\infty)$. It is also the unique continuous solution of the initial value problem

$$D^q x(t) = -\lambda\Gamma(q)x(t), \qquad \lim_{t\to 0^+} t^{1-q}x(t) = \lambda. \tag{I_λ}$$

on $(0,\infty)$.

Proof. In terms of the constant $a := \lambda\Gamma(q)$, the right-hand side of (6.1) is

$$\frac{a}{s^q+a} = \frac{a}{s^q(1+\frac{a}{s^q})} = as^{-q}\sum_{k=0}^{\infty}(-1)^k\left(\frac{a}{s^q}\right)^k = a\sum_{k=0}^{\infty}(-a)^k s^{-(kq+q)}$$

if $s^q > a$ ($a > 0$ as $\lambda > 0$). Since

$$s^{-r} = \mathcal{L}\left\{\frac{t^{r-1}}{\Gamma(r)}\right\} \qquad (r > 0, s > 0), \tag{6.6}$$

we have

$$\frac{a}{s^q+a} = a\sum_{k=0}^{\infty}(-a)^k \mathcal{L}\left\{\frac{t^{kq+q-1}}{\Gamma(kq+q)}\right\} = a\sum_{k=0}^{\infty}(-a)^k \int_0^\infty e^{-st}\frac{t^{kq+q-1}}{\Gamma(kq+q)}\,dt.$$

With the intent of justifying interchanging the order of summation and integration, consider the series of absolute values:

$$\sum_{k=0}^{\infty}\int_0^\infty \left|(-a)^k e^{-st}\frac{t^{kq+q-1}}{\Gamma(kq+q)}\right| dt.$$

Rewriting this as

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{a^k}{\Gamma(kq+q)}\int_0^\infty e^{-st}t^{kq+q-1}\,dt &= \sum_{k=0}^{\infty}\frac{a^k}{\Gamma(kq+q)}\mathcal{L}\{t^{kq+q-1}\}\\ &= \sum_{k=0}^{\infty}\frac{a^k}{\Gamma(kq+q)}\cdot\frac{\Gamma(kq+q)}{s^{kq+q}} = \frac{1}{s^q}\sum_{k=0}^{\infty}\left(\frac{a}{s^q}\right)^k,\end{aligned}$$

we see that it is a geometric series that converges if $s^q > a$. Thus, by Levi's theorem ([1, p. 269], [32, p. 36]) for series, the interchange is allowed. As a result, by (6.5) we have

$$\frac{a}{s^q+a} = \int_0^\infty e^{-st}\left(at^{q-1}\sum_{k=0}^{\infty}\frac{(-at^q)^k}{\Gamma(kq+q)}\right)dt = \mathcal{L}\left\{at^{q-1}E_{q,q}(-at^q)\right\}.$$

And so

$$\mathcal{L}\left\{\lambda\Gamma(q)t^{q-1}E_{q,q}(-\lambda\Gamma(q)t^q)\right\} = \frac{\lambda\Gamma(q)}{s^q+\lambda\Gamma(q)} \tag{6.7}$$

for $s^q > \lambda\Gamma(q)$.

We see from (6.1) and (6.7) that

$$\mathcal{L}\{R(t)\}(s) = \mathcal{L}\left\{\lambda\Gamma(q)t^{q-1}E_{q,q}(-\lambda\Gamma(q)t^q)\right\}(s)$$

for $s^q > \lambda\Gamma(q)$. Since the Mittag-Leffler function $E_{q,q}(z)$ is known to be entire in the complex plane (cf. [15, p. 68]), it follows that the argument of the operator $\mathcal{L}$ on the right-hand side is continuous on $(0,\infty)$. Of course, the resolvent $R(t)$ is also continuous on $(0,\infty)$. Therefore, by Lerch's theorem (cf. [1, p. 342] or [36, Thm. 6.3, p. 63]), $R(t)$ is given by (6.4).

Finally, by Theorem 5.2, (6.4) is also the unique continuous solution of (I_λ) for $t > 0$. □

Remark 6.3. In 1970 Prabhakar [31] introduced the complex-valued function

$$E^{\rho}_{\alpha,\beta}(z) := \sum_{k=0}^{\infty}\frac{(\rho)_k z^k}{\Gamma(k\alpha+\beta)k!}$$

where α, β and ρ are complex constants with $\operatorname{Re}\alpha > 0$ and $(\rho)_k$ denotes Pochhammer's symbol:

$$(\rho)_0 := 1 \ \text{ and } \ (\rho)_k := \rho(\rho+1)\cdots(\rho+k-1) \qquad (k\in\mathbb{N}).$$

In particular, $E^{1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z)$ since $(1)_k = k!$.

Results from this paper include the Laplace transform formula

$$\mathcal{L}\left\{t^{\beta-1}E^{\rho}_{\alpha,\beta}(\omega t^\alpha)\right\}(s) = \frac{s^{\rho\alpha-\beta}}{(s^\alpha-\omega)^\rho} \tag{6.8}$$

for $\operatorname{Re}\beta > 0$, $\operatorname{Re}s > 0$, and $|s| > |\omega|^{\frac{1}{\operatorname{Re}\alpha}}$. With $\rho = 1$, $\alpha = \beta = q$, $\omega = -\lambda\Gamma(q)$, and $s \in \mathbb{R}$ positive, (6.8) becomes

$$\mathcal{L}\left\{t^{q-1}E_{q,q}(-\lambda\Gamma(q)t^q)\right\}(s) = \frac{1}{s^q + \lambda\Gamma(q)}$$

for $s^q > \lambda\Gamma(q)$. Multiplying both sides by $\lambda\Gamma(q)$, we obtain (6.7).

Remark 6.4. There is yet another way to derive (6.4) and that is to use the series representation for the resolvent that is found in [4, Thm. 4.2], which was derived by means of fixed point theorems. It is shown there that

$$\widehat{R}(t) := \sum_{i=1}^{\infty} \frac{[\beta\lambda(q)]^i}{\lambda(iq)} t^{iq-1}, \tag{6.9}$$

where $\beta \neq 0$ and $q \in (0,1)$, defines a unique continuous function that satisfies the equation

$$\widehat{R}(t) = \beta t^{q-1} + \int_0^t \beta(t-s)^{q-1}\widehat{R}(s)\,ds \tag{6.10}$$

on $(0,\infty)$. Let us convert this equation to the form of (R_λ) by first rewriting it as

$$-\widehat{R}(t) = -\beta t^{q-1} - (-\beta)\int_0^t (t-s)^{q-1}[-\widehat{R}(s)]\,ds.$$

Then let $R(t) := -\widehat{R}(t)$ and $\lambda := -\beta$ obtaining

$$R(t) = \lambda t^{q-1} - \lambda\int_0^t (t-s)^{q-1}R(s)\,ds$$

for any $\lambda \neq 0$. Expressed in terms of λ and R, the series (6.9) is

$$R(t) = -\sum_{i=1}^{\infty} \frac{[-\lambda\Gamma(q)]^i}{\Gamma(iq)} t^{iq-1}. \tag{6.11}$$

Changing the index of summation to $k = i-1$, (6.11) becomes

$$\begin{aligned} R(t) &= -\sum_{k=0}^{\infty} \frac{[-\lambda\Gamma(q)]^{k+1}}{\Gamma((k+1)q)} t^{(k+1)q-1} = \lambda\Gamma(q)t^{q-1}\sum_{k=0}^{\infty}\frac{[-\lambda\Gamma(q)]^k}{\Gamma(kq+q)}t^{kq} \\ &= \lambda\Gamma(q)t^{q-1}\sum_{k=0}^{\infty}\frac{[-\lambda\Gamma(q)t^q]^k}{\Gamma(kq+q)} = \lambda\Gamma(q)t^{q-1}E_{q,q}\left(-\lambda\Gamma(q)t^q\right), \end{aligned}$$

which is (6.4).

Note that even though we have assumed throughout this paper that $\lambda > 0$, we see here that (6.4) is the unique continuous solution of (R_λ) for any $\lambda \neq 0$.

Next we use Theorem 6.2 with $q = 1/2$ to express the resolvent in terms of the error function (cf. (6.12) below). An equivalent formula was derived in 1965 in a paper by Brakhage et al. [11, p. 297] using methods different from those in this paper. Instead of "resolvent" the authors used the term "lösender Kern" (German for "solvent kernel"). Other derivations can be found in [3, Ex. 3.2] and [4, Ex. 5.1].

Corollary 6.5. *For $q = 1/2$, the resolvent is*

$$R(t) = \frac{\lambda}{\sqrt{t}} - \pi\lambda^2 e^{\pi\lambda^2 t}\left(1 - \mathrm{erf}(\lambda\sqrt{\pi t})\right) \tag{6.12}$$

where erf(u) *is the* error function*:*

$$\mathrm{erf}(u) := \frac{2}{\sqrt{\pi}} \int_0^u e^{-v^2}\, dv.$$

That is, (6.12) *is the unique continuous solution of*

$$R(t) = \frac{\lambda}{\sqrt{t}} - \lambda \int_0^t \frac{R(s)}{\sqrt{t-s}}\, ds \tag{6.13}$$

and of

$$D^{1/2}x(t) = -\lambda\sqrt{\pi}x(t), \qquad \lim_{t\to 0^+} \sqrt{t}x(t) = \lambda \tag{6.14}$$

on the interval $(0,\infty)$*.*

Proof. Take $q = 1/2$ in Theorem 6.2. Then (6.4) becomes

$$R(t) = \lambda\Gamma(\tfrac{1}{2})t^{-1/2}E_{\frac{1}{2},\frac{1}{2}}(-\lambda\Gamma(\tfrac{1}{2})t^{1/2}) = \lambda\sqrt{\pi}t^{-1/2}E_{\frac{1}{2},\frac{1}{2}}(-\lambda\sqrt{\pi t}).$$

Employing the formula (cf. [25, (4.4), p. 12])

$$E_{\frac{1}{2},\frac{1}{2}}(-x) = \frac{1}{\sqrt{\pi}} - xe^{x^2}(1 - \mathrm{erf}(x)),$$

we obtain

$$R(t) = \lambda\frac{\sqrt{\pi}}{\sqrt{t}}\left[\frac{1}{\sqrt{\pi}} - \lambda\sqrt{\pi t}\, e^{\lambda^2\pi t}\left(1 - \mathrm{erf}(\lambda\sqrt{\pi t})\right)\right],$$

which simplifies to (6.12). Moreover, with $q = 1/2$, (R_λ) and (I_λ) become (6.13) and (6.14), respectively. □

7 Complete monotonicity of the resolvent

Definition 7.1. A function f is said to be *completely monotone* on an interval $I \subset \mathbb{R}$ if it has derivatives of every order at each point $t \in I$ and if

$$(-1)^n f^{(n)}(t) \geq 0$$

for all $t \in I$ and for $n = 0, 1, 2, \ldots$

An example is the kernel $C(t) = \lambda t^{q-1}$ studied in this paper. Clearly, as $\lambda > 0$, it is completely monotone on the interval $(0,\infty)$. Another is the function (cf. (7.1))

$$E_1(-t) := \sum_{k=0}^{\infty} \frac{(-t)^k}{\Gamma(k+1)}.$$

It is completely monotone on $\mathbb{R}$ since it is equal to e^{-t}.

Theorem 7.2. *The resolvent* $R(t)$ *is completely monotone on* $(0,\infty)$*.*

Proof. First consider the factor $E_{q,q}(-\lambda\Gamma(q)t^q)$ in (6.4). Let E_n, where $n > 0$, denote the Mittag-Leffler function with parameter n, which is defined by

$$E_n(t) := \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(kn+1)}. \tag{7.1}$$

Pollard [30] gives a direct proof that $E_n(-t)$ is completely monotone on $[0,\infty)$ for $0 < n \le 1$, a fact originally discovered by W. Feller using methods of probability theory and communicated to Pollard. Miller and Samko [24] use Pollard's result to prove that the two-parameter Mittag-Leffler function $E_{m,n}(-t)$ is completely monotone on $[0,\infty)$ for $0 < m \le 1$ and $n \ge m$ (cf. also [25, p. 11]).

As $\lambda > 0$, $\lambda\Gamma(q)t^q \in [0,\infty)$ for all $t \ge 0$. Thus, as $0 < q < 1$, $E_{q,q}(-\lambda\Gamma(q)t^q)$ is completely monotone on $[0,\infty)$. Certainly the function $\lambda\Gamma(q)t^{q-1}$ is completely monotone on the open interval $(0,\infty)$. Consequently, the resolvent

$$R(t) = \lambda\Gamma(q)t^{q-1}E_{q,q}(-\lambda\Gamma(q)t^q)$$

is completely monotone on $(0,\infty)$ since the product of two completely monotone functions is itself completely monotone (cf. [25, Thm. 1]). □

By Theorem 4.1 (i), $R(t)$ is nonnegative. In fact, it is strictly positive.

Theorem 7.3. *For all $t > 0$, $R(t) > 0$.*

Proof. Suppose that $R(t_1) = 0$ at some point $t_1 > 0$. Since R is completely monotone on $(0,\infty)$, $R'(t) \le 0$ for all $t > 0$. This and $R(t) \ge 0$ imply $R(t) \equiv 0$ on $[t_1,\infty)$. It then follows from Theorem 4.5 that

$$\int_0^\infty R(s)\,ds = \int_0^{t_1} R(s)\,ds \le 1 - e^{-(\lambda t_1^q)/q} < 1,$$

a contradiction of Corollary 4.8. □

8 Variation of parameters for Abel's integral equation

With the establishment of the existence and uniqueness of the resolvent in Section 3, a list of some of its most important properties in Section 4, and the derivation of the resolvent formula (6.4) in Section 6, both classical and new results are obtainable without resorting to series methods. Consider, for example, the Abel integral equation of the second kind

$$x(t) = f(t) - \int_0^t C(t-s)x(s)\,ds \tag{A$_\lambda$}$$

where $C(t) = \lambda t^{q-1}$. In 1930 Hille and Tamarkin [22, pp. 520–525] derived a solution of (A$_\lambda$) using a Liouville–Neumann series for the resolvent. But since we already have formula (6.4), we take a different tack. We will use the resolvent equation (R$_\lambda$) to formally obtain a solution of (A$_\lambda$) by following the steps presented in Miller's monograph [27, pp. 189–191]. First multiply (A$_\lambda$) by $R(t-s)$ and then integrate with respect to s. Then formally interchange the order of integration of the iterated integral. With a suitable change of variable and the use of (R$_\lambda$), the inner integral can be replaced with $C(t-s) - R(t-s)$. Simplification of the equation yields

$$\int_0^t C(t-s)x(s)\,ds = \int_0^t R(t-s)f(s)\,ds.$$

This suggests that

$$x(t) = f(t) - \int_0^t R(t-s) f(s)\, ds$$

is a solution of (A_λ). In fact, assuming that f is continuous on $[0, \infty)$, we will prove that it is the only continuous solution of (A_λ) on $[0, \infty)$. Our proof relies on the following lemma.

Lemma 8.1. *If a function ψ is continuous on an interval $[0, b]$, then the integral function Ψ defined by*

$$\Psi(t) := \int_0^t R(t-s)\psi(s)\, ds \qquad (0 < q < 1) \tag{8.1}$$

is also continuous on $[0, b]$. Moreover,

$$|\Psi(t_1) - \Psi(t_2)| \leq 2m \left(1 - e^{-\frac{\lambda}{q}|t_1 - t_2|^q}\right) \leq \frac{2m\lambda}{q} |t_1 - t_2|^q \tag{8.2}$$

for all $t_1, t_2 \in [0, b]$, where $m := \sup\{|\psi(t)| : 0 \leq t \leq b\}$.

Proof. For a given $t \in (0, b]$,

$$|R(t-s)\psi(s)| = |R(t-s)||\psi(s)| \leq mR(t-s)$$

for $0 \leq s < t$. Now

$$\begin{aligned} \int_0^t mR(t-s)\, ds &= m \lim_{\epsilon \to 0^+} \int_0^{t-\epsilon} R(t-s)\, ds \\ &= m \lim_{\epsilon \to 0^+} \int_\epsilon^t R(u)\, du = m \int_0^t R(u)\, du. \end{aligned}$$

Since this integral converges, then so does the integral $\Psi(t)$. Consequently, Ψ has a finite value at each $t \in [0, b]$, where $\Psi(0) = 0$. In fact, it follows from Theorem 4.1 (viii) that

$$|\Psi(t)| \leq m \int_0^t R(u)\, du \leq m$$

for $0 \leq t \leq b$.

As to continuity, first consider $|\Psi(t_1) - \Psi(t_2)|$ for $0 < t_1 < t_2 \leq b$:

$$\begin{aligned} |\Psi(t_2) - \Psi(t_1)| &= \left| \int_0^{t_2} R(t_2 - s)\psi(s)\, ds - \int_0^{t_1} R(t_1 - s)\psi(s)\, ds \right| \\ &\leq \int_0^{t_1} |R(t_2 - s) - R(t_1 - s)||\psi(s)|\, ds + \int_{t_1}^{t_2} |R(t_2 - s)||\psi(s)|\, ds. \end{aligned}$$

Since $R(t)$ is positive and decreasing for $t > 0$,

$$\begin{aligned} |\Psi(t_2) - \Psi(t_1)| &\leq m \int_0^{t_1} \big[R(t_1 - s) - R(t_2 - s)\big]\, ds + m \int_{t_1}^{t_2} R(t_2 - s)\, ds \\ &= m \int_0^{t_1} R(u)\, du - m \int_{t_2 - t_1}^{t_2} R(u)\, du + m \int_0^{t_2 - t_1} R(u)\, du \\ &= -m \left[\int_0^{t_2} R(u)\, du - \int_0^{t_1} R(u)\, du \right] + 2m \int_0^{t_2 - t_1} R(u)\, du \\ &\leq 2m \int_0^{t_2 - t_1} R(u)\, du. \end{aligned}$$

Then, because of Theorem 4.5,

$$|\Psi(t_2) - \Psi(t_1)| \le 2m\left(1 - e^{-\frac{\lambda}{q}(t_2-t_1)^q}\right) \tag{8.3}$$

for $0 < t_1 < t_2 \le b$. It also follows from this theorem that for $t_1 = 0$,

$$|\Psi(t_2) - \Psi(t_1)| = |\Psi(t_2)| \le m\int_0^{t_2} R(u)\,du \le m\left(1 - e^{-\frac{\lambda}{q}t_2^q}\right).$$

When $t_1 = t_2$, (8.3) trivially holds. Thus (8.3) holds for $0 \le t_1 \le t_2 \le b$.

Reversing the roles of t_1 and t_2, we conclude that the first inequality in (8.2) holds for all $t_1, t_2 \in [0, b]$. The second inequality is true because $e^x \ge 1 + x$ for all $x \in \mathbb{R}$. □

Remark 8.2. We see from (8.2) that the function Ψ is *Lipschitz continuous of order q* on the interval $[0, b]$; that is,

$$|\Psi(t_1) - \Psi(t_2)| \le \frac{2\lambda\|\psi\|}{q}\,|t_1 - t_2|^q$$

where $\|\psi\| = \sup\{|\psi(t)| : 0 \le t \le b\}$. Note from Lemma 3.1 that this inequality still holds if $R(t-s)$ in (8.1) is replaced with $\lambda(t-s)^{q-1}$. Part of the proof of Lemma 8.1 was adapted from [14, Thm. 5.1].

Equation (8.5) in the next theorem is known as the *linear variation of parameters formula* for (8.4). In this regard, see Miller's monograph [27]; in particular, note that equations (1.3) and (1.4) in Chapter IV of this work simplify to (1.1) and (1.2), respectively, when the nonlinear term $G(t, x) \equiv 0$. Also, see Burton [13, pp. 17, 24].

Theorem 8.3 (Linear variation of parameters). *Let $f\colon [0,\infty) \to \mathbb{R}$ be continuous. Let $C(t) = \lambda t^{q-1}$ where $\lambda > 0$ and $q \in (0, 1)$. The Abel integral equation*

$$x(t) = f(t) - \int_0^t C(t-s)x(s)\,ds \tag{8.4}$$

has a unique continuous solution on $[0,\infty)$, namely,

$$x(t) = f(t) - \int_0^t R(t-s)f(s)\,ds. \tag{8.5}$$

This solution expressed in terms of the Mittag-Leffler function $E_{q,q}$ is

$$x(t) = f(t) - \lambda\Gamma(q)\int_0^t f(t-s)s^{q-1}E_{q,q}(-\lambda\Gamma(q)s^q)\,ds. \tag{8.6}$$

Proof. Since the function f is continuous on $[0,\infty)$, it follows from Lemma 8.1 that

$$y(t) := f(t) - \int_0^t R(t-s)f(s)\,ds$$

is also continuous on $[0,\infty)$. Now let us verify that it is a solution.

Evaluating the integral in (8.4) with $y(s)$ substituted for $x(s)$, we obtain

$$\begin{aligned}\int_0^t C(t-s)y(s)\,ds &= \int_0^t C(t-s)\left[f(s) - \int_0^s R(s-u)f(u)\,du\right]ds\\ &= \int_0^t C(t-s)f(s)\,ds - \int_0^t\int_0^s C(t-s)R(s-u)f(u)\,du\,ds.\end{aligned}$$

A slight alteration of the proof of Theorem 4.3 justifies the following change in the order of integration:

$$\int_0^t \int_0^s C(t-s)R(s-u)f(u)\,du\,ds = \int_0^t \left(\int_u^t C(t-s)R(s-u)\,ds\right) f(u)\,du$$
$$= \int_0^t \left(\int_0^{t-u} C(t-u-v)R(v)\,dv\right) f(u)\,du.$$

From the resolvent equation (R_λ), we have

$$\int_0^{t-u} C(t-u-v)R(v)\,dv = C(t-u) - R(t-u).$$

Therefore,

$$\int_0^t C(t-s)y(s)\,ds = \int_0^t C(t-s)f(s)\,ds - \int_0^t \left[C(t-u) - R(t-u)\right] f(u)\,du$$
$$= \int_0^t R(t-u)f(u)\,du = f(t) - y(t),$$

which shows that $y(t)$ is truly a solution of (8.4). That it is the only continuous solution on $[0,\infty)$ follows from Lemma 3.13. Finally, with (6.4) we can write the resolvent in terms of a Mittag-Leffler function to obtain (8.6). □

Remark 8.4. Solution (8.6) agrees with the formula found by Hille and Tamarkin [22, cf. (25.1), (25.4), (25.5)]. This can be seen from (9.11), which is derived in the next section.

9 The resolvent integral function

In this concluding section, we will use the resolvent and variation of parameters formulas to derive some useful results. As has been the case throughout the paper, $R(t)$ denotes the resolvent of the kernel $C(t) = \lambda t^{q-1}$ where $q \in (0,1)$ and $\lambda > 0$.

Theorem 9.1. *Let $a, x^0 \in \mathbb{R}$ with $a > 0$ and $x^0 \neq 0$. The initial value problem*

$$D^q x(t) = -ax(t), \quad \lim_{t\to 0^+} t^{1-q}x(t) = x^0 \tag{9.1}$$

has the unique continuous solution

$$x(t) = x^0\Gamma(q)t^{q-1}E_{q,q}(-at^q) \tag{9.2}$$

on $(0,\infty)$. It is positive and completely monotone on $(0,\infty)$ if $x^0 > 0$. If $x^0 < 0$, then it is negative and $-x(t)$ is completely monotone on $(0,\infty)$.

Proof. We see from (1.1) and (1.2) that the complementary integral equation when $f(t,x) = -ax$ is

$$x(t) = x^0 t^{q-1} - \frac{a}{\Gamma(q)} \int_0^t (t-s)^{q-1} x(s)\,ds. \tag{9.3}$$

Let $\lambda := a/\Gamma(q)$. By (6.4) the resolvent is $R(t) = at^{q-1}E_{q,q}(-at^q)$. And so by Corollary 3.16, the unique continuous solution of (9.3) is

$$x(t) = \frac{x^0}{\lambda}R(t) = x^0\Gamma(q)\frac{R(t)}{a} = x^0\Gamma(q)t^{q-1}E_{q,q}(-at^q).$$

Since $x(t)$ and $f(t, x(t))$ are constant multiples of $R(t)$, they are continuous and absolutely integrable on $(0, \infty)$. Hence, Theorem 5.1 implies that $x(t)$ is also the unique continuous solution of (9.1) on $(0, \infty)$.

If $x^0 > 0$, then $x(t) = (x^0\Gamma(q)/a)R(t) > 0$ for $t > 0$ since $R(t) > 0$. Moreover, $x(t)$ is completely monotone on $(0, \infty)$ by Theorem 7.2. Similarly if $x^0 < 0$, then $x(t) < 0$ and $-x(t)$ is completely monotone. □

Remark 9.2. If $a \leq 0$, (9.2) is also the unique continuous solution of (9.1). This follows from Remark 6.4 and the statement containing (2.3) if $a < 0$. If $a = 0$, (9.2) reduces to $x(t) = x^0 t^{q-1}$, the only continuous solution of (9.1).

Remark 9.3. In the literature (e.g., [29, pp. 131–132]), it is shown that (9.2) is the solution of $D^q x(t) = -ax(t)$ satisfying the initial condition

$$\lim_{t\to 0^+} \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} x(s)\, ds = b$$

where $b = x^0\Gamma(q)$. In [6, pp. 17–19], it is proven that this initial condition is equivalent to the initial condition in (9.1). It appears that monotonicity results for fractional differential equations, such as in Theorem 9.1, have gone largely unnoticed.

Theorem 9.4. *The resolvent integral function*

$$x(t) = \int_0^t R(s)\, ds \tag{9.4}$$

is the unique continuous solution of the integral equation

$$x(t) = \frac{\lambda}{q} t^q - \lambda \int_0^t (t-s)^{q-1} x(s)\, ds \tag{9.5}$$

on the interval $[0, \infty)$.

Proof. Using the variation of parameters formula (8.5) with $f(t) = \lambda t^q/q$, we find that the unique continuous solution of (9.5) is

$$x(t) = \frac{\lambda}{q} t^q - \frac{\lambda}{q} \int_0^t (t-s)^q R(s)\, ds. \tag{9.6}$$

Integrating (R_λ) and applying Corollary 4.4, we get

$$\begin{aligned} \int_0^t R(s)\, ds &= \lambda \int_0^t s^{q-1}\, ds - \lambda \int_0^t \int_0^s (s-u)^{q-1} R(u)\, du\, ds \\ &= \frac{\lambda}{q} t^q - \lambda \int_0^t \left(\int_u^t (s-u)^{q-1}\, ds \right) R(u)\, du \\ &= \frac{\lambda}{q} t^q - \frac{\lambda}{q} \int_0^t (t-u)^q R(u)\, du. \end{aligned}$$

This and (9.6) imply that (9.4) is the unique continuous solution. □

Next we derive a pair of equivalent formulas for (9.4), which was alluded to earlier in Remark 4.9.

Theorem 9.5. *At each $t \geq 0$, the value of the resolvent integral function is*

$$\int_0^t R(s)\,ds = \lambda\Gamma(q)t^q E_{q,q+1}(-\lambda\Gamma(q)t^q). \tag{9.7}$$

It can also be expressed in terms of the Mittag-Leffler function of order q as follows:

$$\int_0^t R(s)\,ds = 1 - E_q(-\lambda\Gamma(q)t^q) \tag{9.8}$$

where

$$1 - E_q(-\lambda\Gamma(q)t^q) \leq 1 - e^{-\frac{\lambda t^q}{q}}. \tag{9.9}$$

Proof. Letting $a := \lambda\Gamma(q)$ and using (6.4), we have

$$\int_0^t R(s)\,ds = \int_0^t as^{q-1}E_{q,q}(-as^q)\,ds = -\frac{1}{q}\int_0^{-at^q} E_{q,q}(u)\,du.$$

Since the power series defining $E_{m,n}$ in (6.5) converges for all complex numbers z (cf. [15, p. 68]), the integral can be evaluated by integrating the series for $E_{q,q}(u)$ term by term. In so doing and then employing the recurrence formula $z\Gamma(z) = \Gamma(z+1)$, we find that

$$\begin{aligned}
\int_0^t R(s)\,ds &= -\frac{1}{q}\int_0^{-at^q}\sum_{k=0}^{\infty}\frac{u^k}{\Gamma(kq+q)}\,du = -\frac{1}{q}\sum_{k=0}^{\infty}\int_0^{-at^q}\frac{u^k}{\Gamma(kq+q)}\,du \\
&= -\sum_{k=0}^{\infty}\frac{(-at^q)^{k+1}}{(k+1)q\Gamma((k+1)q)} = -\sum_{k=0}^{\infty}\frac{(-at^q)^{k+1}}{\Gamma((k+1)q+1)} \\
&= at^q\sum_{k=0}^{\infty}\frac{(-at^q)^k}{\Gamma(kq+(q+1))} = at^q E_{q,q+1}(-at^q).
\end{aligned}$$

This concludes the proof of (9.7).

Changing the index in the next-to-last sum, we have

$$\begin{aligned}
at^q E_{q,q+1}(-at^q) &= -\sum_{i=1}^{\infty}\frac{(-at^q)^i}{\Gamma(iq+1)} = 1 - \left[1 + \sum_{i=1}^{\infty}\frac{(-at^q)^i}{\Gamma(iq+1)}\right] \\
&= 1 - E_q(-at^q) = 1 - E_q(-\lambda\Gamma(q)t^q),
\end{aligned}$$

which proves (9.8). Inequality (9.9) follows from (4.4). □

The following fractional differential equation (9.10) has infinitely many continuous solutions on $(0, \infty)$ but only one on the half-closed interval $[0, \infty)$.

Theorem 9.6. *There is one and only one continuous solution of*

$$D^q x(t) = -ax(t) + a \qquad (a > 0) \tag{9.10}$$

on the interval $[0, \infty)$, namely

$$x(t) = \int_0^t R(s)\,ds = 1 - E_q(-at^q), \tag{9.11}$$

where $R(t)$ is the resolvent of $C(t) = (a/\Gamma(q))t^{q-1}$.

Proof. Expressed in terms of $\lambda := a/\Gamma(q)$, (9.10) is

$$D^q x(t) = -\lambda\Gamma(q)x(t) + \lambda\Gamma(q).$$

It follows from Lemma 3.1 that if a continuous solution $x(t)$ of this equation exists on $[0,\infty)$, then

$$\lim_{t\to 0^+} \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} x(s)\, ds = 0.$$

With this we can adapt the line of reasoning in [6, p. 12] to convert (9.10) to the integral equation (9.5):

$$x(t) = \frac{\lambda}{q} t^q - \lambda \int_0^t (t-s)^{q-1} x(s)\, ds.$$

Then one concludes that if (9.10) has a continuous solution on $[0,\infty)$, then it is also a continuous solution of (9.5) on $[0,\infty)$. Conversely, we can argue as in [6, p. 15] that any continuous solution of (9.5) on this same interval must also be a continuous solution of (9.10). Theorems 9.4 and 9.5 imply this continuous solution is unique and given by (9.11). □

References

[1] T. M. Apostol, *Mathematical analysis*, second ed., Addison-Wesley, Reading MA, 1974. MR0344384

[2] L. C. Becker, Principal matrix solutions and variation of parameters for a Volterra integro-differential equation and its adjoint, *Electron. J. Qual. Theory Differ. Equ.* **2006**, No. 14, 1–22. MR2240716; url

[3] L. C. Becker, Resolvents and solutions of weakly singular linear Volterra integral equations, *Nonlinear Anal.* **74**(2011), 1892–1912. MR2764388; url

[4] L. C. Becker, Resolvents for weakly singular kernels and fractional differential equations, *Nonlinear Anal.* **75**(2012), 4839–4861. MR2927548; url

[5] L. C. Becker, Resolvents and solutions of singular Volterra integral equations with separable kernels, *Appl. Math. Comput.* **219**(2013), No. 24, 11265–11277. MR3073278; url

[6] L. C. Becker, T. A. Burton, I. K. Purnaras, Complementary equations: A fractional differential equation and a Volterra integral equation, *Electron. J. Qual. Theory Differ. Equ.* **2015**, No. 12, 1–24. MR3325915; url

[7] L. C. Becker, T. A. Burton, I. K. Purnaras, An inversion of a fractional differential equation and fixed points, *Nonlinear Dyn. Syst. Theory* **15**(2015), No. 3, 242–271. MR3410724

[8] L. C. Becker, T. A. Burton, I. K. Purnaras, Fractional differential equations, transformations, and fixed points, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **22**(2015), 333–361. MR3423277

[9] L. C. Becker, T. A. Burton, I. K. Purnaras, Existence of solutions of nonlinear fractional differential equations of Riemann–Liouville type, *J. Fract. Calc. Appl.* **7**(2016), No. 2, 20–39. MR3457215

[10] L. C. Becker, T. A. Burton, I. K. Purnaras, Integral and fractional equations, positive solutions, and Schaefer's fixed point theorem, *Opuscula Math.* **36**(2016), No. 4, 431–458. MR3488500

[11] H. Brakhage, K. Nickel, P. Rieder, Auflösung der Abelschen Integralgleichung 2. Art, *Z. Angew. Math. Phys.* **16**(1965), No. 2, 295–298. url

[12] T. A. Burton, *Volterra integral and differential equations*, second ed., Mathematics in Science and Engineering, Vol. 202, Elsevier B. V., Amsterdam, 2005. MR2155102

[13] T. A. Burton, *Liapunov theory for integral equations with singular kernels and fractional differential equations*, Theodore Allen Burton, Port Angeles, WA, 2012. MR3088486

[14] T. A. Burton, B. Zhang, *Fixed points and fractional differential equations: examples*, Fixed Point Theory **14**(2013), No. 2, 313–325. MR3137175

[15] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Heidelberg, 2010. MR2680847

[16] W. Feller, On the integral equation of renewal theory, *Ann. Math. Statist.* **12**(1941), No. 3, 243–267. MR0005419

[17] A. Friedman, On integral equations of Volterra type, *J. Analyse Math.* **11**(1963), 381–413. MR0158232

[18] R. Gorenflo, S. Vessella, *Abel integral equations. Analysis and applications*, Lecture Notes in Mathematics, Vol. 1461, Springer-Verlag, Berlin, 1991. MR1095269

[19] G. Gripenberg, On Positive, Nonincreasing resolvents of Volterra equations, *J. Differential Equations* **30**(1978), 380–390. MR0521860

[20] S. I. Grossman, R. K. Miller, Perturbation theory for Volterra integrodifferential systems, *J. Differential Equations* **8**(1970), 457–474. MR0270095

[21] R. Heese, M. Freyberger, Entropic uncertainty bound for open pointer-based simultaneous measurements of conjugate observables, *J. Phys. A* **48**(2015), No. 13, 135304, 19 pp. MR3323370

[22] E. Hille, J. D. Tamarkin, On the Theory of linear integral equations, *Ann. Math.* **31**(1930), No. 3, 479–528. MR1502959

[23] P. Linz, *Analytical and numerical methods for Volterra equations*, SIAM Studies in Applied Mathematics, Vol. 7, Philadelphia, 1985. MR0796318

[24] K. S. Miller, S. G. Samko, A note on the complete monotonicity of the generalized Mittag-Leffler function, *Real Anal. Exchange* **23**(1997/98), No. 2, 753–755. MR1639957

[25] K. S. Miller, S. G. Samko, Completely monotonic functions, *Integral Transform. Spec. Funct.* **12**(2001), No. 4, 389–402. MR1872377

[26] R. K. Miller, On Volterra integral equations with nonnegative integrable resolvents, *J. Math. Anal. Appl.* **22**(1968), 319–340. MR0227707

[27] R. K. Miller, *Nonlinear Volterra integral equations*, W. A. Benjamin, Inc., Menlo Park, CA, 1971. MR0511193

[28] I. P. Natanson, *Theory of functions of a real variable, Vol. II*, Translated from the Russian by Leo F. Boron, Frederick Ungar Publishing Co., New York, 1961. MR0148805

[29] I. Podlubny, *Fractional differential equations*, Mathematics in Science and Engineering, Vol. 198, Academic Press, San Diego, 1999. MR1658022

[30] H. Pollard, The completely monotonic character of the Mittag-Leffler function $E_\alpha(-x)$, *Bull. Amer. Math. Soc.* **54**(1948), 1115–1116. MR0027375

[31] T. R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel, *Yokohama Math. J.* **19**(1970), 7–15. MR0293349

[32] F. Riesz, B. Sz.-Nagy, *Functional analysis*, translated from the second French edition by L. F. Boron, Dover, New York, 1990. MR1068530

[33] H. Schaefer, Über die Methode der a priori-Schranken (in German), *Math. Ann.* **129**(1955), 415–416. MR0071723

[34] D. R. Smart, *Fixed point theorems*, Cambridge Tracts in Mathematics, No. 66, Cambridge Univ. Press, 1980. MR0467717

[35] D. Verotta, Fractional dynamics pharmacokinetics–pharmacodynamic models, *J. Pharmacokinet. Pharmacodyn.* **37**(2010), 257–276. url

[36] D. V. Widder, *The Laplace transform*, second printing, Princeton University Press, Princeton, N. J., 1946. MR0005923