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Multiple solutions for fourth order m-point boundary value problems with sign-changing nonlinearity *

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Abstract. Using a fixed point theorem in ordered Banach spaces with lattice structure founded by Liu and Sun, this paper investigates the multiplicity of nontrivial solutions for fourth order m-point boundary value problems with sign-changing nonlinearity. Our results are new and improve on those in the literature.
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1 Introduction

Consider the following fourth order differential equation

$$x^{(4)}(t) = f(t, x(t)), \quad t \in (0,1) \tag{1.1}$$

subject to one of the following two classes of m-point boundary value conditions

$$\begin{cases} x(0) = 0, \quad x(1) = \sum\limits_{i=1}^{m-2} \alpha_i x(\eta_i); \\ \\ x''(0) = 0, \quad x''(1) = \sum\limits_{i=1}^{m-2} \alpha_i x''(\eta_i), \end{cases} \tag{1.2}$$

and

$$\begin{cases} x'(0) = 0, \quad x(1) = \sum\limits_{i=1}^{m-2} \alpha_i x(\eta_i); \\ \\ x'''(0) = 0, \quad x''(1) = \sum\limits_{i=1}^{m-2} \alpha_i x''(\eta_i), \end{cases} \tag{1.3}$$

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where $f:\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ is a given sign-changing continuous function, $m\geq 3$, $\eta_i\in(0,1)$ and $\alpha_i>0$ for $i=1,\cdots,m-2$ with

$$\sum_{i=1}^{m-2}\alpha_i<1. \tag{1.4}$$

The existence of nontrivial or positive solutions of nonlinear multi-point boundary value problems (BVP, for short) for fourth order differential equations has been extensively studied and lots of excellent results have been established by using fixed point index for cone mappings, standard upper and lower solution arguments, fixed point theorems for cone mappings and so on (see [2, 8-10] and the references therein). For example, in [9], Wei and Pang studied the following fourth order differential equation

$$x^{(4)}(t)=f(x(t),-x''(t)),\quad t\in(0,1)$$

with the boundary condition (1.2).

By means of fixed point index theory in a cone and the Leray-Schauder degree, the existence and multiplicity of nontrivial solutions are obtained.

Recently Professor Jingxian Sun advanced a new approach to compute the topological degree when the concerned operators are not cone mappings in ordered Banach spaces with lattice structure. He established some interesting results for such nonlinear operators (for details, see [3, 6, 7]). To our best knowledge, there is no paper to use this new method to study fourth order m-point boundary value problems. We try to fill this gap in the present paper.

Suppose the following conditions are satisfied throughout.

(H0) the sequence of positive solutions of the equation

$$\sin\sqrt{s}=\sum_{i=1}^{m-2}\alpha_i\sin\eta_i\sqrt{s}$$

is $0<\lambda_1<\lambda_2<\cdots<\lambda_n<\lambda_{n+1}<\cdots$.

(H0′) the sequence of positive solutions of the equation

$$\cos\sqrt{s}=\sum_{i=1}^{m-2}\alpha_i\cos\eta_i\sqrt{s}$$

is $0<s_1<s_2<\cdots<s_n<s_{n+1}<\cdots$.

(H1) $\lim_{x\to+\infty}\frac{f(t,x)}{x}=\alpha$ uniformly on $t\in[0,1]$.

(H2) $\lim_{x\to-\infty}\frac{f(t,x)}{x}=\beta$ uniformly on $t\in[0,1]$.

(H3) $f(t,0)\equiv 0$, $\lim_{x\to 0}\frac{f(t,x)}{x}=\gamma$ uniformly on $t\in[0,1]$.

This paper is organized as follows. In Section 2, we present some basic definitions of the lattice and some lemmas that will be used to prove the main results. In Section 3, we shall give our main results and their proofs.

2 Preliminaries

We first recall some properties of a lattice and some operators (see [3, 7]).

Let E be an ordered Banach space in which the partial ordering $\leq$ is induced by a cone $P \subseteq E$. P is called normal if there exists a constant $N > 0$ such that $\theta \leq x \leq y$ implies $\| x \| \leq N \| y \|$.

Definition 2.1.[7] We call E a lattice under the partial ordering $\leq$, if $\sup\{x, y\}$ and $\inf\{x, y\}$ exist for arbitrary $x, y \in E$.

Definition 2.2.[3] Let E be a Banach space with a cone P and $A : E \to E$ be a nonlinear operator. We say that A is a unilaterally asymptotically linear operator along $P_w = \{x \in E : x \geq w,\ w \in E\}$, if there exists a bounded linear operator L such that

$$\lim_{x \in P_w,\ \|x\| \to \infty} \frac{\| Ax - Lx \|}{\| x \|} = 0.$$

L is said to be the derived operator of A along P_w and will be denoted by A'_{P_w}. Similarly, we can also define a unilaterally asymptotically linear operator along $P^w = \{x \in E : x \leq w,\ w \in E\}$.

Remark 2.1. If $w = 0$ in Definition 2.2, A is a unilaterally asymptotically linear operator along P and $(-P)$. It is remarkable that A is not assumed to be a cone mapping.

Definition 2.3.[7] Let $D \subseteq E$ and $A : D \to E$ be a nonlinear operator. A is said to be quasi-additive on a lattice, if there exists $v^* \in E$ such that

$$Ax = Ax_+ + Ax_- + v^*,\ \forall x \in D,$$

where $x_+ = x^+ = \sup\{x, \theta\},\ x_- = -x^- = -\sup\{-x, \theta\}$.

The following lemma is important for us to obtain the main results.

Lemma 2.1.[3] *Suppose E is an ordered Banach space with a lattice structure, P is a normal cone of E, and the nonlinear operator A is quasi-additive on the lattice. Assume that*

(i) A is strongly increasing on P and $(-P)$;

(ii) both A'_P and $A'_{(-P)}$ exist with $r(A'_P) > 1$ and $r(A'_{-P}) > 1$, and 1 is not an eigenvalue of A'_P and $A'_{(-P)}$ corresponding a positive eigenvector;

(iii) $A\theta = \theta$; the Frechet derivative A'_θ of A at θ is strongly positive and $r(A'_\theta) < 1$;

(iv) the Frechet derivative A'_∞ of A at ∞ exists; 1 is not an eigenvalue of A'_∞; the sum β of the algebraic multiplicities for all eigenvalues of A'_∞ lying in the interval $(1,\ \infty)$ is an even number.

Then A has at least three nontrivial fixed points containing one sign-changing fixed point.

Let $E = C[0,\ 1]$ with the norm $\| x \| = \max_{t \in [0,\ 1]} | x(t) |$ and $P = \{x \in E : x(t) \geq 0, t \in [0,\ 1]\}$. Then E is a Banach space and P is a normal cone of E. It is easy to see that E is a lattice under the partial ordering $\leq$ that is deduced by P.

Using the same method as in [9], we can easily convert BVP (1.1) and (1.2) into the following operator equation

$$x(t) = (L_1^2 F x)(t), \tag{2.1}$$

where the operators F and L_1 are defined by

$$(Fx)(t) = f(t, x(t)),\ \forall t \in [0,\ 1],\ x \in E; \tag{2.2}$$

$$(L_1x)(t) = \int_0^1 H_1(t,s)x(s)ds,\ \forall t \in [0,\ 1],\ x \in E; \tag{2.3}$$

where

$$H_1(t,s) = G_1(t,s) + \frac{\sum\limits_{i=1}^{m-2} \alpha_i G(\eta_i, s)}{1 - \sum\limits_{i=1}^{m-2} \alpha_i \eta_i} t,$$

$$G_1(t,s) = \begin{cases} (1-t)s,\ 0 \le s \le t \le 1; \\ (1-s)t,\ 0 \le t \le s \le 1. \end{cases}$$

Similarly, we can convert BVP (1.1) and (1.3) into the following operator equation

$$x(t) = (L_2^2 Fx)(t), \tag{2.4}$$

where

$$(L_2x)(t) = \int_0^1 H_2(t,s)x(s)ds,\ \forall t \in [0,\ 1],\ x \in E, \tag{2.5}$$

$$H_2(t,s) = G_2(t,s) + \frac{\sum\limits_{i=1}^{m-2} \alpha_i G(\eta_i, s)}{1 - \sum\limits_{i=1}^{m-2} \alpha_i \eta_i},$$

$$G_2(t,s) = \begin{cases} 1-t,\ 0 \le s \le t \le 1; \\ 1-s,\ 0 \le t \le s \le 1. \end{cases}$$

Define

$$A_1 = L_1^2 F,\ A_2 = L_2^2 F, \tag{2.6}$$

then the following lemma is obvious.

Lemma 2.2. *$x(t)$ is a solution of the BVP (1.1) and (1.2) (BVP (1.1) and (1.3)) if and only if $x(t)$ is a solution of the operator equation*

$$x(t) = (A_1x)(t)\ (x(t) = (A_2x)(t)).$$

Lemma 2.3. *(i) $L_i^2 : E \to E (i = 1, 2)$ is a completely continuous linear operator;*

(ii) $F : E \to E$ is quasi-additive on the lattice;

(iii) $A_i = L_i^2 F (i = 1, 2)$ is quasi-additive on the lattice;

(iv) the sequences of all eigenvalues of the operators L_1^2 and L_2^2 are $\{\frac{1}{\lambda_n^2},\ n = 1, 2, \cdots\}$ and $\{\frac{1}{s_n^2},\ n = 1, 2, \cdots\}$, respectively, where λ_n and s_n are respectively defined by (H0) and (H0′), and the algebraic multiplicities of $\frac{1}{\lambda_n^2}$ and $\frac{1}{s_n^2}$ are 1.

(v) $r(L_1^2) = \frac{1}{\lambda_1^2},\ r(L_2^2) = \frac{1}{s_1^2}$, where $r(L_i)$ is the spectral radius of the operator L_i $(i = 1, 2)$.

Proof. The proof of (i)-(iii) is obvious. Now we start to prove the conclusions (iv) and (v).

Let μ be a positive eigenvalue of the linear operator L_1^2, and $y \in E \setminus \{\theta\}$ be an eigenfunction corresponding to the eigenvalue μ. Then we have

$$\begin{cases} \mu y^{(4)} = y, \quad t \in [0,1]; \\ y(0) = 0, \quad y(1) = \sum\limits_{i=1}^{m-2} \alpha_i y(\eta_i); \\ y''(0) = 0, \quad y''(1) = \sum\limits_{i=1}^{m-2} \alpha_i y''(\eta_i). \end{cases} \tag{2.7}$$

Define $D = \frac{d}{dt}$, $L = \mu D^4 - 1$; then there exist two constants $r_1,\ r_2$ such that

$$Ly = \mu(D^2 + r_1)(D^2 + r_2)y.$$

By properties of differential operators, if (2.7) has a nonzero solution, then there exists $r_s,\ s \in \{1, 2\}$ such that $r_s = \lambda_k,\ k \in N$. In this case, $\sin t\sqrt{\lambda_k}$ is a nonzero solution of (2.7). Substituting this solution into (2.7), we have

$$\mu\lambda_k^2 - 1 = 0.$$

Hence, $\{\frac{1}{\lambda_k^2},\ k = 1, 2, \cdots\}$ is the sequence of all eigenvalues of the operator L_1^2. Then μ is one of the values

$$\frac{1}{\lambda_1^2} > \frac{1}{\lambda_2^2} > \cdots > \frac{1}{\lambda_n^2} \cdots$$

and the eigenfunction corresponding to the eigenvalue $\frac{1}{\lambda_n^2}$ is

$$y_n(t) = C \sin t\sqrt{\lambda_n},\ t \in [0, 1],$$

where C is a nonzero constant. By the ordinary method, we can show that any two eigenfunctions corresponding to the same eigenvalue $\frac{1}{\lambda_n^2}$ are merely nonzero constant multiples of each other. Consequently,

$$\dim \ker(\frac{1}{\lambda_n^2} I - L_1^2) = 1. \tag{2.8}$$

Now we show that

$$\ker(\frac{1}{\lambda_n^2} I - L_1^2) = \ker(\frac{1}{\lambda_n^2} I - L_1^2)^2. \tag{2.9}$$

Obviously, we need only show that

$$\ker(\frac{1}{\lambda_n^2} I - L_1^2)^2 \subseteq \ker(\frac{1}{\lambda_n^2} I - L_1^2). \tag{2.10}$$

For any $y \in \ker(\frac{1}{\lambda_n^2} I - L_1^2)^2$, $(I - \lambda_n^2 L_1^2)y$ is an eigenfunction of the linear operator L_1^2 corresponding to the eigenvalue $\frac{1}{\lambda_n^2}$ if $(I - \lambda_n^2 L_1^2)y \neq \theta$. Then there exists a nonzero constant γ such that

$$(I - \lambda_n^2 L_1^2)y = \gamma \sin t\sqrt{\lambda_n},\ t \in [0,\ 1].$$

By direct computation, we have

$$\begin{cases} y^{(4)} = \lambda_n^2 y + \gamma\lambda_n^2 \sin t\sqrt{\lambda_n}, \quad t \in [0,1]; \\ y(0) = 0, \quad y(1) = \sum\limits_{i=1}^{m-2} \alpha_i y(\eta_i); \\ y''(0) = 0, \quad y''(1) = \sum\limits_{i=1}^{m-2} \alpha_i y''(\eta_i). \end{cases} \tag{2.11}$$

It is easy to see that the general solution of (2.11) is of the form

$$y(t) = C_1 \cos t\sqrt{\lambda_n} + C_2 \sin t\sqrt{\lambda_n} + C_3 \exp(t\sqrt{\lambda_n}) + C_4 \exp(\sqrt{-t\lambda_n}) + \frac{\gamma\sqrt{\lambda_n}}{4} t \cos t\sqrt{\lambda_n}, \ t \in [0,\ 1]$$

where C_1, C_2, C_3, C_4 are nonzero constants.

Applying the boundary conditions, we obtain that

$$\begin{cases} C_1 = 0, \\ C_3 + C_4 = 0, \\ C_3 F + \frac{\gamma\sqrt{\lambda_n}}{4} G = 0, \\ C_3 \lambda_n F - \lambda_n \frac{\gamma\sqrt{\lambda_n}}{4} G = 0, \end{cases} \tag{2.12}$$

where

$$F = e^{\sqrt{\lambda_n}} - e^{-\sqrt{\lambda_n}} - \sum_{i=1}^{m-2} \alpha_i (\exp(\sqrt{\lambda_n \eta_i}) - \exp(-\sqrt{\lambda_n \eta_i})) > 0, \tag{2.13}$$

$$G = \cos\sqrt{\lambda_n} - \sum_{i=1}^{m-2} \alpha_i \eta_i \cos \eta_i \sqrt{\lambda_n}. \tag{2.14}$$

If $G \neq 0$, then $C_3 = \frac{\gamma\sqrt{\lambda_n}}{4} = 0$, which is a contradiction to $\gamma \neq 0$, and $y(t) = C_2 \sin t\sqrt{\lambda_n} \in \ker(\frac{1}{\lambda_n^2} I - L_1^2)$. So (2.10) holds, which means (2.9) also holds.

If $G = 0$, then

$$\cos\sqrt{\lambda_n} = \sum_{i=1}^{m-2} \alpha_i \eta_i \cos \eta_i \sqrt{\lambda_n}.$$

By the Schwarz inequality, we obtain

$$\begin{aligned} 1 - \sin^2\sqrt{\lambda_n} &= (\sum_{i=1}^{m-2} \alpha_i \eta_i \cos \eta_i \sqrt{\lambda_n})^2 \\ &\leq (\sum_{i=1}^{m-2} \eta_i^2)(\sum_{i=1}^{m-2} \alpha_i^2 \cos^2 \eta_i \sqrt{\lambda_n}) \\ &= (\sum_{i=1}^{m-2} \eta_i^2)(\sum_{i=1}^{m-2} \alpha_i^2) - (\sum_{i=1}^{m-2} \eta_i^2)(\sum_{i=1}^{m-2} \alpha_i^2 \sin^2 \eta_i \sqrt{\lambda_n}). \end{aligned}$$

Applying the condition $\sin\sqrt{\lambda_n} = \sum\limits_{i=1}^{m-2}\alpha_i \sin\eta_i\sqrt{\lambda_n}$, we obtain

$$\begin{aligned}
1 &\le (\sum_{i=1}^{m-2}\eta_i^2)(\sum_{i=1}^{m-2}\alpha_i^2) + (\sum_{i=1}^{m-2}\alpha_i\sin\eta_i\sqrt{\lambda_n})^2 - (\sum_{i=1}^{m-2}\eta_i^2)(\sum_{i=1}^{m-2}\alpha_i^2\sin^2\eta_i\sqrt{\lambda_n})\\
&= (\sum_{i=1}^{m-2}\eta_i^2)(\sum_{i=1}^{m-2}\alpha_i^2) + [1-(\sum_{i=1}^{m-2}\eta_i^2)](\sum_{i=1}^{m-2}\alpha_i^2\sin^2\eta_i\sqrt{\lambda_n})\\
&\quad + \sum_{i\neq j}\alpha_i\alpha_j\sin\eta_i\sqrt{\lambda_n}\sin\eta_j\sqrt{\lambda_n}\\
&\le (\sum_{i=1}^{m-2}\eta_i^2)(\sum_{i=1}^{m-2}\alpha_i^2) + [1-(\sum_{i=1}^{m-2}\eta_i^2)](\sum_{i=1}^{m-2}\alpha_i^2) + \sum_{i\neq j}\alpha_i\alpha_j\\
&= (\sum_{i=1}^{m-2}\alpha_i)^2,
\end{aligned}$$

which is a contradiction to $\sum\limits_{i=1}^{m-2}\alpha_i < 1$. Thus, (2.9) holds. It follows from (2.8) and (2.9) that the algebraic multiplicity of the eigenvalue $\frac{1}{\lambda_n^2}$ is 1.

Similarly, we can show that the sequence of all eigenvalues of the operator L_2^2 is $\{\frac{1}{s_n^2},\ n = 1, 2, \cdots\}$, and the algebraic multiplicity of $\frac{1}{s_n^2}$ is 1.

By the definition of the spectral radius, we have

$$r(L_1^2) = \sup_{\lambda\in\{\frac{1}{\lambda_n^2},\ n=1,2,\cdots\}} |\lambda| = \frac{1}{\lambda_1^2},$$

$$r(L_2^2) = \sup_{\lambda\in\{\frac{1}{s_n^2},\ n=1,2,\cdots\}} |\lambda| = \frac{1}{s_1^2}.$$

The proof of this Lemma is complete. □

Lemma 2.4. *Let A_i and L_i^2 $(i = 1,\ 2)$ be defined by (2.1) - (2.6). Then*
(i) $(A_i)'_P = \alpha L_i^2$ $(i = 1,\ 2)$ if f satisfies (H1);
(ii) $(A_i)'_{(-P)} = \beta L_i^2$ $(i = 1,\ 2)$ if f satisfies (H2);
(iii) $(A_i)'_\theta = \gamma L_i^2$ $(i = 1,\ 2)$ if f satisfies (H3).

Proof. We only prove the conclusion (i), the proofs of conclusions (ii) and (iii) are similar. Suppose that f satisfies (H1). Then there exists $R > 0$ such that for a given $\varepsilon > 0$,

$$| f(t,\ x) - \alpha x | \le \varepsilon x,\ t \in [0,\ 1],\ x > R.$$

Set $M_R = \max\limits_{t\in[0,\ 1],\ 0\le x\le R} | f(t,\ x) |$. Then

$$\begin{aligned}
| (Fx)(t) - \alpha x(t) | &= | f(t,\ x(t)) - \alpha x(t) |\\
&\le M_R + \alpha R + \varepsilon \| x \|,\ \forall x \in P,\ \| x \| \ge R,
\end{aligned}$$

and hence

$$\begin{aligned}
\| A_i x - \alpha L_i^2 x \| &= \| L_i^2(Fx) - \alpha L_i^2 x \|\\
&\le \| L_i^2 \| (M_R + \alpha R + \varepsilon \| x \|),\ (i = 1,\ 2),
\end{aligned}$$

which means
$$\liminf_{x\in P,\ \|x\|\to\infty} \frac{\| A_i x-\alpha L_i^2 x \|}{\| x \|} \le \varepsilon \| L_i^2 \|,\ (i=1,\ 2).$$
Therefore, $(A_i)'_P=\alpha L_i^2\ (i=1,\ 2)$. □

3 Main Results

In order to consider the existence of multiple solutions for the BVP (1.1) and (1.2) and BVP (1.1) and (1.3), let us introduce another ordered Banach space.

Let $e_{1,i}$ be the first normalized eigenfunction of L_i^2 corresponding to its first eigenvalue; then $e_{1,i}(t)>0,\ \forall t\in(0,\ 1)$ and $\| e_{1,i} \|=1,\ (i=1,\ 2)$.

Let
$$E_{e,i}=\{x\in E:\exists\mu>0,\ -\mu e_{1,i}(t)\le x(t)\le \mu e_{1,i}(t),\ t\in[0,\ 1]\},\ (i=1,\ 2).$$
By [1] and [3], we know that $E_{e,i}$ is an ordered Banach space, $P_{e,i}=P\bigcap E_{e,i}\ (i=1,\ 2)$ is a normal solid cone in $E_{e,i}$, and $L_i^2:E\to E_{e,i}$ is a linear completely continuous operator satisfying $L_i^2(P\setminus\{\theta\})\subseteq intP_{e,i}=\{x\in P_{e,i}:\exists\alpha>0,\ \beta>0,\ \alpha e_{1,i}(t)\le x(t)\le\beta e_{1,i}(t),\ t\in[0,\ 1]\},\ (i=1,\ 2)$.

Now we are ready to give our main results.

Theorem 3.1. *Suppose that f satisfies (H1) -(H3). In addition, suppose*

(i) $f(t,\ x)$ is strictly increasing in x;

(ii) there exist an even number n_1 and a positive integer n_2, such that
$$\lambda_{n_1}^2<\alpha<\lambda_{n_1+1}^2,\ \lambda_{n_2}^2<\beta<\lambda_{n_2+1}^2; \tag{3.1}$$
(iii) $0<\gamma<\lambda_1^2$.

Then BVP (1.1) and (1.2) has at least three nontrivial solutions containing a sign-changing solution.

Proof. From Lemma 2.4 we know that
$$(A_1)'_\theta=\gamma L_1^2,\ (A_1)'_P=\alpha L_1^2,\ (A_1)'_{(-P)}=\beta L_1^2.$$
Notice that $P_{e,1}=P\bigcap E_{e,1}\subseteq P$ implies $(A_1)'_{P_{e,1}}=\alpha L_1^2$ and $(A_1)'_{(-P_{e,1})}=\beta L_1^2$.

Using a method similar to the one used in the proof of Lemma 2.4, it is not difficult to prove that $(A_1)'_\infty=\alpha L_1^2$.

By condition (i) and the fact that $L_1^2(P\setminus\{\theta\})\subseteq intP_{e,1}$, we know that A_1 is strongly increasing and hence the condition (i) of Lemma 2.1 is satisfied.

The condition (ii) shows that 1 is not an eigenvalue of $(A_1)'_P$ and $(A_1)'_{(-P)}$, and by Lemma 2.3, we have
$$r((A_1)'_P)=\frac{\alpha}{\lambda_1^2}>1,\ r((A_1)'_{(-P)})=\frac{\beta}{\lambda_1^2}>1.$$
Hence, condition (ii) of Lemma 2.1 is also satisfied.

Similarly, we can show that $(A_1)'_\theta$ is strongly positive, so condition (iii) of Lemma 2.1 is satisfied.

Since f satisfies (H1) and (H2), by condition (iii) we find that condition (iv) of Lemma 2.1 is satisfied.

Consequently, Lemmas 2.1 and 2.2 guarantee that the conclusion is valid. □

By the method used in the proof of Theorem 3.1, it is easy to show the following theorem.

Theorem 3.2. *Suppose that f satisfies (H1) -(H3). In addition, suppose*

(i) $f(t,\ x)$ is strictly increasing in x;

(ii) there exist an even number n_1 and a positive integer n_2, such that

$$s_{n_1}^2 < \alpha < s_{n_1+1}^2,\ s_{n_2}^2 < \beta < s_{n_2+1}^2; \tag{3.2}$$

(iii) $0 < \gamma < s_1^2$.

Then BVP (1.1) and (1.3) has at least three nontrivial solutions containing a sign-changing solution.

Example 3.1 Consider the following fourth order differential equation

$$\begin{cases} x^{(4)}(t) = f(t, x(t)), \quad t \in (0,1); \\ x'(0) = 0, \quad x(1) = \alpha_1 x(\eta_1); \\ \\ x'''(0) = 0, \quad x''(1) = \alpha_1 x''(\eta_1), \end{cases} \tag{3.3}$$

where

$$f(t,x) = \begin{cases} \inf_{t\in[0,1]}[400x(1+t)], & x \geq 10; \\ \frac{7998}{20-\pi}(x - \frac{\pi}{2}) + 1, & \frac{\pi}{2} \leq x < 10; \\ \sin x, & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}; \\ \frac{38}{20+\pi}(x + \frac{\pi}{2}) - 1, & -10 \leq x < -\frac{\pi}{2}; \\ \sup_{t\in[0,1]}[2tx], & x < -10, \end{cases}$$

and $\alpha_1 = \frac{\sqrt{3}}{3}$, $\eta_1 = \frac{1}{2}$. It is easy to see $f(t,x)$ is strictly increasing in x. By simple calculations, we can show $s_1 = \frac{\pi^2}{9}$, $s_2 = (2\pi - 2\arccos\frac{\sqrt{3}}{3})^2$, $s_3 = (2\pi + 2\arccos\frac{\sqrt{3}}{3})^2$, $s_4 = \frac{121\pi^2}{9}$, $\cdots$, $s_{4k+1} = (4k\pi + \frac{\pi}{3})^2$, $s_{4k+2} = (4k\pi + 2\pi - 2\arccos\frac{\sqrt{3}}{3})^2$, $s_{4k+3} = (4k\pi + 2\pi + 2\arccos\frac{\sqrt{3}}{3})^2$, $s_{4k+4} = (4k\pi + \frac{11\pi}{3})^2$, $k \in N$. And $\alpha = 400$, $\beta = 2$, $\gamma = 1$, so $s_2^2 < \alpha < s_3^2$, $s_1^2 < \beta < s_2^2$, $0 < \gamma < s_1^2$. Thus, by Theorem 3.2, BVP (3.3) has at least three nontrivial solutions containing a sign-changing solution.

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