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Integral equations, transformations, and a Krasnoselskii–Schaefer type fixed point theorem

Dedicated to Professor Tibor Krisztin on the occasion of his 60th birthday

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Abstract. In this paper we extend the work begun in 1998 by the author and Kirk for integral equations in which we combined Krasnoselskii's fixed point theorem on the sum of two operators with Schaefer's fixed point theorem. Schaefer's theorem eliminates a difficult hypothesis in Krasnoselskii's theorem, but requires an *a priori* bound on solutions. Here, we simplify the work by means of a transformation which often reduces the *a priori* bound to a triviality.

Our work is focused on an integral equation in which the goal is to prove that there is a unique continuous positive solution on $[0, \infty)$. In addition to the transformation, there are two techniques which we would emphasize.

A technique is introduced yielding a lower bound on the solutions which enables us to deal with problems threatening non-uniqueness. The technique offers a solution to a classical problem and it seems entirely new.

We show that when the equation defines the sum of a contraction and a Lipschitz operator, then we first get existence on arbitrary intervals $[0, E]$ and then introduce a technique which we call a *progressive contraction* which allows us to prove uniqueness and then parlay the solution to $[0, \infty)$. The technique is well suited to integral equations.

Keywords: fixed points, Krasnoselskii–Schaefer fixed point theorem, integral equations, positive solutions, progressive contractions.

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1 Introduction

As all crafts people know, a main part of any task is finding the right tools. In the mid 1950s Krasnoselskii studied the inversion of a perturbed differential operator and arrived at the following working hypothesis.

The inversion of a perturbed differential operator yields the sum of a contraction and a compact map. See [13, p. 370], [16], [17, p. 31].

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This led to an appropriate set of tools consisting of fixed point theorems of Banach and Schauder. To this set he added his own theorem on the sum of a contraction and compact map. All of these have been used and modified many times. See, for example, the survey paper by Park [15]. Krasnoselskii's theorem may be stated as follows [17, p. 31].

Theorem 1.1 (Krasnoselskii). *Let M be a closed convex nonempty subset of a Banach space $(\mathcal{B}, \|\cdot\|)$. Suppose that C and D map M into $\mathcal{B}$ such that*

(i) if $x, y \in M$, then $Cx + Dy \in M$,

(ii) C is compact and continuous,

(iii) D is a contraction mapping.

Then there exists $y \in M$ with $y = Cy + Dy$.

Item (i) has proven to be very difficult to verify [3].

At about the same time Schaefer published an interesting variation of Schauder's theorem which allowed the investigator to avoid finding a self-mapping set. Schaefer's theorem [17, p. 29] can be stated as follows.

Theorem 1.2 (Schaefer). *Let $(\mathcal{B}, \|\cdot\|)$ be a normed space, P a continuous mapping of $\mathcal{B}$ into $\mathcal{B}$ which is compact on each bounded subset X of $\mathcal{B}$. Then either*

(i) the equation $x = \lambda Px$ has a solution for $\lambda = 1$, or

(ii) the set of all such solutions x, for $0 < \lambda < 1$, is unbounded.

We come now to the start of the present work. In 1998 Kirk and the author [6] combined Krasnoselskii's theorem with Schaefer's theorem as follows.

Theorem 1.3. *Let $(\mathcal{B}, \|\cdot\|)$ be a Banach space, C, $D : \mathcal{B} \to \mathcal{B}$, D a contraction with contraction constant $\alpha < 1$, and C continuous with C mapping bounded sets into compact sets. Either*

(i) $x = \lambda D(x/\lambda) + \lambda Cx$ has a solution in $\mathcal{B}$ for $\lambda = 1$, or

(ii) the set of all such solutions, $0 < \lambda < 1$, is unbounded.

Two results have appeared in the intervening years which radically simplified Schaefer's theorem.

First, for many integral equations of applied mathematics taking the form

$$x(t) = a(t) + g(t, x) - \int_0^t A(t-s) f(s, x(s)) ds$$

where A satisfies (A1)–(A3), below, it is now known that the integral maps bounded sets of continuous functions into equicontinuous sets [8] and [9]. Moreover, when we work on a finite interval $[0, E]$, then it is actually a compact map. The following result obtainable from [9, Thm. 2.2] covers the case here. Relative to Theorem 1.3 and this equation, the integral term is C the compact mapping, while contraction conditions on g will be given making $D = a(t) + g(t, x)$ the contraction.

Theorem 1.4. *Let $(\mathcal{B}, \|\cdot\|)$ be the Banach space of bounded continuous functions $\phi : [0, E] \to \Re$ with the supremum norm. Let A satisfy (A1)–(A3) and let M be a bounded subset of $\mathcal{B}$. If $W : M \to \mathcal{B}$ is defined by $\phi \in M \implies (W\phi)(t) = \int_0^t A(t-s)\phi(s) ds, \quad 0 \le t \le E < \infty$, then W maps M into a compact subset of $\mathcal{B}$.*

For our equation above with $f(s, x(s))$ and for a given bounded set L of continuous functions, f of the functions in L yields the set M of bounded continuous functions on $[0, E]$.

If solutions are unique then the resulting solutions on $[0, E]$ can be parlayed into solutions on $[0, \infty)$. The details are given in our final result.

Next, there is a transformation ([4], [1]) which has been shown in a list of papers to negate (ii) in Schaefer's theorem for a wide class of problems.

The purpose of this paper is to advance the transformation to Theorem 1.3 thereby negating (ii) in that theorem. It turns out that all of the advantages previously seen in Schaefer's theorem still obtain. We will give a theorem and and example illustrating this. And this also leads us to results on positive solutions, as well as a new idea on uniqueness of solutions. In this context, uniqueness is critical for obtaining a clean statement that there is a solution on $[0, \infty)$ without using a statement that an extension process can be carried out an infinite number of times to finally get a solution on all of $[0, \infty)$.

2 The transformation

We will restrict our attention to a set of kernels $A(t - s)$ discussed, for example, in Miller [14, pp. 209–213] which satisfy:

(A1) $A \in C(0, \infty) \cap L^1(0, 1)$;

(A2) $A(t)$ is positive and non-increasing for $t > 0$;

(A3) for each $T > 0$ the function $A(t)/A(t + T)$ is non-increasing in t for $0 < t < \infty$.

The integral equation on which we focus is the scalar equation

$$x(t) = a(t) + g(t, x(t)) - \int_0^t A(t - s) f(s, x(s)) ds \tag{2.1}$$

where $a : [0, \infty) \to [0, \infty)$, $g : [0, \infty) \times \Re \to \Re$, $f : [0, \infty) \times \Re \to \Re$, all of which are continuous. We turn now to Theorem 1.3 and add the parameter λ in (2.1) obtaining

$$x(t) = \lambda a(t) + \lambda g\left(t, \frac{x(t)}{\lambda}\right) - \int_0^t \lambda A(t - s) f(s, x(s)) ds. \tag{2.2}$$

We note that Theorem 1.4 is satisfied so that all we need prove when applying Theorem 1.3 is continuity and that (ii) does not hold. Concerning continuity, the natural mapping defined by (2.2) is continuous. First, the contraction g is certainly continuous. The integral part has been shown continuous in many places. See, for example, the open access journal [2, Lemma 4.5].

Let $[0, E]$ be an arbitrary closed interval and suppose that $x(t)$ is any fixed solution of (2.2) on $[0, E]$. For that fixed $x(t)$ and fixed λ write

$$p(t) = \lambda a(t) + \lambda g\left(t, \frac{x(t)}{\lambda}\right). \tag{2.3}$$

We will be using a nonlinear variation of parameters formula found in Miller [14, pp. 189–193]. Write (2.2) using (2.3) as

$$\begin{aligned} x(t) &= p(t) - \int_0^t \lambda A(t - s)[Jx(s) - Jx(s) + f(s, x(s))] ds \\ &= p(t) - \int_0^t \lambda J A(t - s) x(s) ds + \int_0^t \lambda J A(t - s) \left[x(s) - \frac{f(s, x(s))}{J}\right] ds. \end{aligned} \tag{2.4}$$

Define

$$U(t) = \lambda J A(t) \quad \text{and} \quad R(t) = U(t) - \int_0^t U(t-s)R(s)ds. \tag{2.5}$$

Here, R is called the resolvent and it satisfies

$$0 < R(t) \le \frac{U(t)}{1 + \int_0^t U(s)ds}, \qquad \int_0^\infty R(s)ds = 1,$$

as may be found in [11] and [14, pp. 212–213].

Write the linear part of (2.4) as

$$z(t) = p(t) - \int_0^t U(t-s)z(s)ds \tag{2.6}$$

so that using the linear variation of parameters formula we have

$$z(t) = p(t) - \int_0^t R(t-s)p(s)ds. \tag{2.7}$$

Rewrite (2.7) as

$$z(t) = p(t) - \int_0^t R(t-s)\lambda a(s)ds - \int_0^t R(t-s)\lambda g\left(s, \frac{x(s)}{\lambda}\right)ds. \tag{2.8}$$

Theorem 2.1. *Let (A1)–(A3) hold and suppose that the functions a, g, and f are continuous. Then* (2.2) *and*

$$\begin{aligned} x(t) = \lambda g\left(t, \frac{x(t)}{\lambda}\right) + \lambda a(t) - \int_0^t R(t-s)\lambda a(s)ds \\ + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J} - \lambda g\left(s, \frac{x(s)}{\lambda}\right)\right]ds \end{aligned} \tag{2.9}$$

share solutions where J is an arbitrary positive constant.

Proof. We are now ready to apply the nonlinear variation of parameters formula [14, p. 192] to (2.4) writing

$$x(t) = z(t) + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds. \tag{2.10}$$

One more step will complete the process. We take the last term in (2.8) and put it in the last integrand of (2.10) obtaining (2.9)

$$\begin{aligned} x(t) = \lambda g\left(t, \frac{x(t)}{\lambda}\right) + \lambda a(t) - \int_0^t R(t-s)\lambda a(s)ds \\ + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J} - \lambda g\left(s, \frac{x(s)}{\lambda}\right)\right]ds \end{aligned}$$

Miller notes that the process is reversible so (2.2) and (2.9) share solutions. □

In our work here we are usually considering continuous functions $\phi : [0, E] \to \Re$ and by $\|\phi\|$ we mean the supremum on the interval of definition. When that function ϕ is restricted to a subinterval $[a, b]$, then the supremum on that interval is denoted by $\|\phi\|^{[a,b]}$.

3 Positive solutions

We now prepare to use Theorem 2.1 and Theorem 1.3 to show that (2.1) has a positive solution on an arbitrary interval $[0, E]$. Thus, we suppose that

$$x > 0 \implies f(t, x) > 0, \qquad x > 0 \implies g(t, x) > 0. \tag{3.1}$$

Because of the negative sign in front of the integral in (2.1), we see that f and g are opposing each other. Also, suppose that there is an $\alpha \in (0, 1)$ such that $x, y \in \Re$, $0 \leq t < \infty$ implies

$$|g(t, x) - g(t, y)| \leq \alpha |x - y|, \qquad |g(t, x)| \leq \alpha |x|. \tag{3.2}$$

Finally, we will ask that

$$a(0) > 0 \quad \text{and} \quad a(t) - \int_0^t R(t-s)a(s)ds > 0. \tag{3.3}$$

This is a condition discussed in many places. See, for example, [12, pp. 259], [9], and [5]. It is readily satisfied if $a(t)$ is positive and non-decreasing.

Lemma 3.1. *The equation*

$$y = \lambda a(0) + \lambda g(0, y/\lambda)$$

has a unique positive solution for every $\lambda \in (0, 1]$.

Proof. The mapping $Q : \Re \to \Re$ defined by $y \in \Re$ implies

$$Qy = \lambda a(0) + \lambda g(0, y/\lambda)$$

is a contraction with unique fixed point $y(\lambda)$.

First, $y(\lambda) \neq 0$ since otherwise this gives $\lambda g(0, y/\lambda) = 0$ by (3.2), contradicting (3.3).

Next, suppose $y(\lambda) < 0$. Then

$$\lambda g(0, y(\lambda)/\lambda) < 0$$

because otherwise the right-hand side of our equation would be positive and the left-hand side negative.

However,

$$|\lambda g(0, y(\lambda)/\lambda| \leq \alpha\lambda |y(\lambda)|/\lambda = \alpha |y(\lambda)|.$$

Thus,

$$\lambda g(0, y(\lambda)/\lambda) > y(\lambda).$$

Therefore

$$y(\lambda) < \lambda g(0, y(\lambda)/\lambda) + \lambda a(0),$$

a contradiction. □

Lemma 3.2. *Let (3.1) and (3.2) hold. If $x(t)$ is a positive solution of (2.2) on an interval $[0, E^*]$ with*

$$\frac{\max_{0 \leq t \leq E^*} |a(t)|}{1 - \alpha} =: a^*$$

then

$$0 \leq t \leq E^* \implies 0 < x(t) \leq a^*.$$

Proof. If $x(t)$ is positive on $[0, E^*]$ then by (2.2), (3.1), and (3.2) we see that

$$x(t) \leq \lambda a(t) + \lambda g\left(t, \frac{x}{\lambda}\right) \leq a(t) + \alpha x(t).$$

Thus,

$$x(t)(1-\alpha) \leq a(t)$$

from which the result follows. □

Now, for $x > 0$ we see that

$$m(s, x(s)) := x(s) - \lambda g\left(s, \frac{x(s)}{\lambda}\right) \geq (1-\alpha)x(s). \tag{3.4}$$

Relative to (3.5) and the condition on f below, see Remark 3.5.

Theorem 3.3. *Assume that (A1)–(A3) hold, that a, g, and f are continuous, and that (3.1)–(3.3) hold. If $E > 0$ is given and if for each $E^* \in (0, E]$ there is a $J > 0$ so that*

$$0 \leq t \leq E^*, \quad 0 < \lambda < 1, \quad 0 < x \leq a^* \implies 0 < \frac{f(t,x)}{J} < (1-\alpha)x \tag{3.5}$$

then (2.2) has a positive solution on $[0, E]$ for $\lambda = 1$.

Proof. Here is a sketch of how the proof will proceed. Theorem 3.3 seeks to use Theorem 1.3 by way of Theorem 2.1 to prove that there is a solution of (2.2) with $\lambda = 1$ and that it is positive. To use Theorem 2.1 we must see that there is a bound on any possible solution of (2.9) so that we can say that there is a bound on any possible solution of (2.2). That will rule out (ii) in Theorem 1.3, leaving us with (i). And that is what we wanted.

Fix $\lambda \in (0, 1]$. If $x(t)$ is a solution of (2.2) then by Lemma 3.1 $x(0) > 0$ and, hence, $x(t) > 0$ on a maximal interval $[0, E^*)$ for some $E^* > 0$. If $E^* > E$ there is nothing to prove. So assume that $E^* \leq E$ and $x(E^*) = 0$.

Note that by (3.1) and (3.3) we have

$$\lambda\left[a(t) - \int_0^t R(t-s)a(s)ds\right] + \lambda g(t, x(t)/\lambda) > 0$$

on $[0, E^*]$.

Moreover, by (3.4) and (3.5) we see that the last integral in (2.9) is positive on $[0, E^*]$ since the integrand is positive on $[0, E^*)$. The last two sentences contradict $x(E^*) = 0$

We have seen that if $x(t)$ is a solution of (2.2) on $[0, E]$ then it must be positive; hence by Lemma 3.2 it must be bounded above by a^* with E^* replaced by E in Lemma 3.2. So (ii) in Theorem 1.3 fails and there is a solution for $\lambda = 1$. But all solutions of (2.2) for $0 < \lambda \leq 1$ satisfy $0 < x(t) \leq a^*$ and that completes the proof. □

In Theorem 3.3 we had assumed that $\lambda g(t, x(t)/\lambda) > 0$ when $x > 0$. We will now get the same conclusion if it is negative when $x > 0$.

Theorem 3.4. *Assume that (A1)–(A3) hold, that $0 < \lambda \leq 1$, that a, g, f are continuous, and that $x\lambda g(t, x/\lambda) < 0$ if $x \neq 0$. Let (3.2) and (3.3) hold. Finally, suppose that for each $E > 0$ if $0 < \lambda \leq 1$ and $0 \leq t \leq E$ then for $0 < x(t) \leq a(t)$ there is a $J > 0$ so that $0 < f(t,x)/(Jx) < 1$. Then (2.2) has a positive solution on $[0, E]$ for $\lambda = 1$.*

Proof. A critical difference between Theorem 3.3 and Theorem 3.4 is that we cannot see by inspection that any solution, x, for any $\lambda \in (0,1]$ and any $J > 0$ satisfies $x(0) > 0$. Thus, for each of these λ we consider the complete metric space of real numbers with the usual distance function $(\Re, |\cdot|)$ and define a mapping $P : \Re \to \Re$ by $y \in \Re$ implies that

$$Py = \lambda a(0) + \lambda g(0, y.\lambda).$$

It is a contraction with a unique fixed point y. In particular if

$$x(0) = \lambda(0) + \lambda g(0, x(0)/\lambda)$$

then $x(0)$ is that fixed point, which, of course varies with λ.

We now show that $y > 0$. If $y = 0$, so is $\lambda g(0, y/\lambda)$, contradicting $a(0) > 0$. If $y < 0$, then $\lambda g(0, y/\lambda) > 0$ so the right-hand side is positive and the left-hand side is negative. Thus, the fixed point is positive so $y > 0$, meaning that $x(0) > 0$.

To see that the solution is positive, we proceed by way of contradiction and suppose that the solution for some λ is positive on a maximal interval $[0, E)$ with $x(E) = 0$. From (2.2) as $x > 0$ we have $\lambda g(t, x(t)/\lambda) < 0$ and $f(s, x(s)) > 0$. This yields $x(t) \leq \lambda a(t) \leq a(t)$. In (2.9) so long as $x(t) > 0$, $\lambda g(t, x(t)/\lambda) < 0$, but at $t = E$ we have this term equal to zero. However, there is a $\mu > 0$ such that on $[0, E]$ we have

$$\lambda a(t) - \int_0^t R(t-s)\lambda a(s)ds \geq \mu.$$

Moreover, on $[0, E)$ we have $x(s) - \lambda g(s, x(s)/\lambda) > 0$ and since $0 < f(t,x)/(Jx) < 1$ we see that the last integrand in (2.9) is positive so that integral at E is positive. Thus, at E there is a contradiction in (2.9) because the left-hand side is zero. This shows that any solution satisfies $0 < x(t) \leq a(t)$ which is an *a priori* bound on any finite interval. This means that (ii) is eliminated in Theorem 1.3 and so there is a solution of (2.2) for $\lambda = 1$ and it, too, satisfies $0 < x(t) \leq a(t)$. □

Remark 3.5. If in Theorem 3.3 we could show that

$$w(t) := a(t) - \int_0^t R(t-s)a(s)ds \geq D > 0 \tag{3.6}$$

when $g(t,x) > 0$ for $x > 0$ we would then know that every solution of (2.8) satisfies $x(t) \geq \lambda D$. Thus, our solution of Theorem 3.3 is bounded below by D. This can be critical in showing uniqueness of solutions when $f(t,x)$ is not Lipschitz, such as $f(t,x) = x^{1/3}$. See Section 5 for the application.

It takes more than expected to get (3.6). Note first that if $a(t) = c$, a positive constant, then

$$c - \int_0^t R(t-s)cds = c\left[1 - \int_0^t R(s)ds\right] = c\int_t^\infty R(s)ds \tag{3.7}$$

and that tends to zero.

Next, note that if $a(t)$ is non-decreasing we have

$$w(t) := a(t) - \int_0^t R(t-s)a(s)ds \geq a(t)\int_t^\infty R(s)ds \tag{3.8}$$

so we have learned that $a(t)$ may need to increase rapidly to make $w(t) \geq \alpha$, a given positive constant. In fact, for the common kernel $a(t) = t^{q-1}, 0 < q < 1$, it is known that [7, Lemma 4.3]

$$\int_t^\infty R(s)ds \in L^n[0,\infty) \iff n > 1/q. \tag{3.9}$$

As a very rough approximation, we think of $\int_t^\infty R(s)ds$ being approximately $1/(t+1)$ so we are led to ask that

$$a(t)\frac{1}{t+1} > \alpha \quad \text{or} \quad a(t) > \alpha(t+1).$$

This is very close to the conclusion we will offer. This problem is studied throughout the literature (cf. [12, p. 263]), but this idea seems entirely new.

Theorem 3.6. *Let $a(t)$ be positive and non-decreasing. If there are $L > 0$ and $\beta > 0$ such that $t \geq 2L$ implies*

$$a(t) - a(t-L) \geq \beta$$

then there is a $\gamma > 0$ with $w(t) \geq \gamma$ for all $t \geq 0$.

Proof. Write

$$\begin{aligned} w(t) &= a(t) - \int_0^t R(t-s)a(s)ds \\ &= a(t) - \int_0^t R(t-s)a(t)ds + \int_0^t R(t-s)[a(t)-a(s)]ds \\ &= a(t)\left[1 - \int_0^t R(s)ds\right] + \int_0^t R(t-s)[a(t)-a(s)]ds \\ &= a(t)\int_t^\infty R(s)ds + \int_0^t R(t-s)[a(t)-a(s)]ds. \end{aligned}$$

We will find an $L > 0$ and show that for $0 \leq t \leq 2L$ the first term in the final line is bounded below by a $K > 0$, while for $2L \leq t < \infty$ the second term is bounded below by $K^* > 0$.

If $0 \leq t \leq 2L$ we have by the fact that $a(t)$ is non-decreasing

$$a(t)\int_t^\infty R(s)ds \geq a(0)\int_{2L}^\infty R(s)ds =: K > 0.$$

Next, suppose that $2L \leq t < \infty$ and note that

$$\begin{aligned} \int_0^t R(t-s)[a(t)-a(s)]ds &\geq \int_0^{t-L} R(t-s)[a(t)-a(s)]ds \\ &\geq \int_0^{t-L} R(t-s)[a(t)-a(t-L)]ds \\ &\geq \int_0^{t-L} R(t-s)\beta ds \\ &= \beta\int_L^t R(s)ds \\ &\geq \beta\int_L^{2L} R(s)ds =: K^*. \end{aligned}$$

Hence, for any $t \in [0,\infty)$ we have

$$w(t) \geq \min[K, K^*] =: \gamma.$$

□

As an example, take
$$a(t) = t + 1$$
and notice that for any $L > 0$ and for $t \geq 2L$ we have
$$a(t) - a(t - L) = t + 1 - (t - L + 1) = L$$
so that we may take $\beta = L$.

4 Contractions and the transformation

We saw in the last section how the transformation reduces the *a priori* bound to a triviality, mainly because it allows us to obtain a lower bound, while (2.1) itself yields a simple upper bound.

But there are other surprising properties in line with Krasnoselskii's conclusion. The work in Section 3 is simplified if $g(t, x) = 0$ making Lemma 3.1 trivial. And if we take $g(t, x) = 0$ then it covers a compact mapping problem. Here we can let $g(t, x) = 0$ and get an example of a contraction from the integrand alone. To say that the inversion of a perturbed differential operator yields the sum of a contraction and compact map means that there are the three cases to be considered.

The Caputo fractional differential equation
$${}^cD^q x(t) = -h(t, x(t)), \quad x(0) \in \Re, \quad 0 < q < 1,$$
inverts for all continuous h as
$$x(t) = x(0) - \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} h(s, x(s)) ds \tag{4.1}$$
where the q^{th} order fractional derivative of Caputo type is defined by [10, pp. 50, 86]
$${}^cD^q x(t) = \frac{1}{\Gamma(1 - q)} \frac{d}{dt} \int_0^t (t - s)^{-q} [x(s) - x(0)] ds$$
and Γ is the Euler Gamma function
$$\Gamma(x) = \int_{0^+}^{\infty} e^{-t} t^{x-1} dt$$
for $0 < x < \infty$.

Thus, the equation
$${}^cD^q x(t) = -\Gamma(q) x^{2n+1}, \qquad x(0) = 1,$$
and n a positive integer inverts as
$$x(t) = 1 - \int_0^t (t - s)^{q-1} x^{2n+1}(s) ds.$$
We seek a positive solution on any interval $[0, E]$. To that end, choose $(M, \|\cdot\|)$ to be the complete metric space of continuous functions $\phi : [0, E] \to [0, 1]$ and the natural mapping is $P : M \to M$ defined by $\phi \in M$ implies that
$$(P\phi)(t) = 1 - \int_0^t (t - s)^{q-1} \phi^{2n+1}(s) ds.$$

Lemma 4.1. *The mapping P does map M into M if and only if $E \leq q^{1/q}$.*

Proof. If $\phi \in M$ then $0 \leq \phi(t) \leq 1$ so $(P\phi)(t) \leq 1$ and the minimum of $(P\phi)(t)$ occurs at $\phi(t) \equiv 1$ so

$$\begin{aligned}(P\phi)(t) &= 1 - \int_0^t (t-s)^{q-1}\phi^{2n+1}(s)ds \\ &\geq 1 - \int_0^t (t-s)^{q-1}ds.\end{aligned}$$

Now this is non-negative if and only if

$$\int_0^t (t-s)^{q-1}ds = \int_0^t s^{q-1}ds = \frac{t^q}{q} \leq 1$$

or $t \leq q^{1/q}$. □

The fact that the integral of the kernel is unbounded presents real difficulties. Therefore, let us switch to the transformed equation

$$x(t) = 1 - \int_0^t R(s)ds + \int_0^t R(t-s)\left[x(s) - \frac{x^{2n+1}(s)}{J}\right]ds$$

and use the same M.

Theorem 4.2. *The natural mapping maps $M \to M$ and is a contraction on any interval $[0, E]$ and so (4.1) has a unique solution on that interval. Moreover, there is a strictly positive solution.*

Proof. For $J > 1$ we see that

$$(P\phi)(t) \leq 1 - \int_0^t R(s)ds + \int_0^t R(t-s)ds = 1,$$

while $(P\phi)(t) \geq 0$. Here, the contraction constant can be found by the mean value theorem and is bounded by the derivative of $f(x) = x - \frac{x^{2n+1}}{J}$ which is bounded by one if $J > 2n+1$. That is,

$$f(\phi(s)) - f(\psi(s)) = f'(\xi(s))[\phi(s) - \psi(s)]$$

where $\xi(s)$ is a point between $\psi(s)$ and $\phi(s)$. Thus, $\phi, \psi \in M$ implies that

$$\begin{aligned}|(P\phi)(t) - (P\psi)(t)| &\leq \int_0^t R(t-s)|\phi(s) - \psi(s)|ds \\ &\leq \|\phi - \psi\| \int_0^t R(s)ds.\end{aligned}$$

For a given $E > 0$ take

$$\alpha = \int_0^E R(s)ds < 1.$$

Then

$$\|P\phi - P\psi\| \leq \alpha\|\phi - \psi\|.$$

But our complete metric space includes the zero function and we want a positive solution. Because the solution is non-negative we see that the second integrand of our transformed equation is non-negative and, hence, the solution is strictly positive: $x(t) > 1 - \int_0^t R(s)ds > 0$ at every point of $[0, E]$. □

5 Uniqueness and continuation

Condition (3.5) would not hold for $f(t,x) = x^{1/3}$ and there would also be a question of uniqueness. Theorem 3.6 offered a way to get past such problems since the solution will reside above $z(t) \geq \gamma$. Thus, for a given $E > 0$ we would be arguing that the solution resides in the strip $0 < \gamma \leq x(t) \leq a(t)$ when $0 \leq t \leq E$.

Remark 5.1. In both Theorems 3.3 and 3.4 we are concerned about uniqueness of that positive solution. If the function $f(t,x)$ satisfies a condition

$$|f(t,x) - f(t,y)| \leq K|x-y|$$

for x and y in this strip and $0 \leq t \leq E$ then we can offer a uniqueness result without asking the usual condition that the α in (3.2) must satisfy $\alpha + K < 1$. In fact, we only need $\alpha < 1$ and K can be large. Since we are working on a finite interval, as we enlarge the interval then K can increase and actually tend to infinity as $E \to \infty$. Once we get uniqueness and existence on an arbitrary interval $[0,E]$ then we can parlay that into a solution on $[0,\infty)$ which may, indeed, be unbounded. By working on $[0,E]$ we avoid compactness requirements of many fixed point theorems on the entire interval $[0,\infty)$. We call the process a *progressive contraction*.

Theorem 5.2. *Let A satisfy (A1)–(A3) and let (3.2) and the conditions with (2.1) hold. Suppose that there are $E > 0$, c_1, and c_2 such that when $c_1 \leq x \leq c_2$ and $0 \leq t \leq E$ then there is a $K > 0$ so that*

$$|f(t,x) - f(t,y)| \leq K|x-y|.$$

Then (2.1) has at most one solution, x, satisfying $c_1 \leq x(t) \leq c_2$ for $0 \leq t \leq E$.

Proof. By way of contradiction, suppose $x_1(t)$ and $x_2(t)$ are two solutions of (2.1) with the conditions on a, g, A, f holding. Let $[0,E]$ be given. Then

$$\begin{aligned}|x_1(t) - x_2(t)| &\leq \alpha|x_1(t) - x_2(t)| + \int_0^t A(t-s)|f(s,x_1(s)) - f(s,x_2(s))|ds \\ &\leq \alpha|x_1(t) - x_2(t)| + \int_0^t A(t-s)K|x_1(s) - x_2(s)|ds.\end{aligned}$$

Let

$$\eta = \frac{1-\alpha}{2}$$

and pick $T > 0$ so that

$$K\int_0^T A(s)ds < \eta.$$

Thus

$$\begin{aligned}\alpha + K\int_0^T A(s)ds &< \alpha + K\frac{1-\alpha}{2K} \\ &= \alpha + \frac{1-\alpha}{2} = \frac{2\alpha + 1 - \alpha}{2} \\ &= \frac{\alpha+1}{2} < 1.\end{aligned}$$

Divide $[0,E]$ into equal parts of length less than T with end points $0, T_1, T_2, \ldots, T_n = E$.

Taking norms on $[0, T_1]$ we have

$$\begin{aligned}\|x_1 - x_2\| &\le \alpha\|x_1 - x_2\| + K\int_0^{T_1} A(T_1 - s)ds\|x_1 - x_2\| \\ &\le \left[\alpha + \frac{1-\alpha}{2}\right]\|x_1 - x_2\| \\ &= \frac{2\alpha + 1 - \alpha}{2}\|x_1 - x_2\| \\ &= \frac{\alpha + 1}{2}\|x_1 - x_2\|\end{aligned}$$

which is a contradiction unless $x_1 = x_2$.

Taking norms on $[0, T_2]$ yields

$$\begin{aligned}|x_1(t) - x_2(t)| &\le \alpha|x_1(t) - x_2(t)| + \int_0^t A(t-s)K|x_1(s) - x_2(s)|ds \\ &\le \alpha\|x_1 - x_2\| + \int_{T_1}^t A(t-s)K|x_1(s) - x_2(s)|ds \\ &\le \alpha\|x_1 - x_2\| + K\|x_1 - x_2\| \int_{T_1}^t A(t-s)ds \\ &\le \alpha\|x_1 - x_2\| + K\|x_1 - x_2\| \int_0^{T_2 - T_1} A(s)ds \\ &\le \|x_1 - x_2\|\left[\alpha + \frac{1-\alpha}{2}\right] \\ &= \|x_1 - x_2\|^{[T_1, T_2]}\frac{2\alpha + 1 - \alpha}{2}\end{aligned}$$

Looking at the left side and taking into account that x_1 and x_2 are equal on the first segment, we see that we have arrived at

$$\|x_1 - x_2\|^{[T_1, T_2]} \le \frac{1+\alpha}{2}\|x_1 - x_2\|^{[T_1, T_2]}$$

which is a contradiction unless both sides are zero. This can be continued on each segment out to E.

□

Theorem 5.3. *If* (2.1) *has a unique continuous solution on any interval* $[0, E]$*, then it has a unique continuous solution on* $[0, \infty)$*.*

Proof. Define a sequence of uniformly continuous functions on $[0, \infty)$ by

$$x_n(t) = x(t), \qquad 0 \le t \le n$$

where n is a positive integer and $x(t)$ is the unique solution of (2.1) on $[0, n]$ and

$$x_n(t) = x_n(n), \qquad n \le t < \infty.$$

This sequence of uniformly continuous functions converges uniformly on compact intervals $[0, L]$ to a continuous function. Indeed, that function is a solution on $[0, \infty)$ because at every value of t in $[0, \infty)$ the function agrees with $x_n(t)$ for any $n > t$. □

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