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Characterization of domains of self-adjoint ordinary differential operators of any order even or odd

Xiaoling Hao[1], **Maozhu Zhang**[2], **Jiong Sun**[1] and **Anton Zettl**✉[3]

[1]School of Mathematical Sciences, Inner Mongolia University, Hohhot, 010021, China
[2]College of Mathematics and Statistics, Taishan University, Taian, 271021, China
[3]Math. Dept., Northern Illinois University, DeKalb, Il. 60115, USA

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Abstract. We characterize the self-adjoint domains of very general ordinary differential operators of any order, even or odd, with complex coefficients and arbitrary deficiency index. This characterization is based on a new decomposition of the maximal domain in terms of LC solutions for real values of the spectral parameter in the Hilbert space of square-integrable functions. These LC solutions reduce to Weyl limit-circle solutions in the second order case.

Keywords: differential operators, deficiency index, self-adjoint domains, real-parameter solutions.

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1 Introduction

Given a symmetric (formally self-adjoint) differential expression M of order $n > 2$ and a positive weight function w, we characterize all self-adjoint realizations of the equation

$$My = \lambda w y \quad \text{on } J = (a,b), \qquad -\infty \leq a < b \leq \infty \tag{1.1}$$

in the Hilbert space $H = L^2(J,w)$. (For the case $n = 2$, see the book [42].) A self-adjoint realization of equation (1.1) is an operator S which satisfies

$$S_{\min} \subset S = S^* \subset S_{\max}, \tag{1.2}$$

where $S_{\min}$ and $S_{\max}$ are the minimal and maximal operators of (1.1). Clearly each such operator S is an extension of $S_{\min}$ and a restriction of $S_{\max}$. These operators S are generally referred to as self-adjoint extensions of the minimal operator $S_{\min}$ but are characterized as restrictions of the maximal operator $S_{\max}$. How many independent restrictions on $D_{\max}$ are required? What are these restrictions?

✉Corresponding author. Email: zettl@msn.com

The answer to the first question is well known and is given by the deficiency index d of the minimal operator $S_{\min}$.

An answer to the second question was found by Everitt and Markus in their monograph [8], see also [9]. They characterized self-adjoint boundary conditions in terms of Lagrangian subspaces of symplectic spaces using methods from symplectic algebra and geometry.

There are two other approaches to answering the second question. One of these uses the method of 'boundary triplets' to determine self-adjoint operators in the Hilbert space H. This approach has an extensive literature dating back to the middle of the 20th century but with some major recent developments. See the papers by V. I. and M. I. Gorbachuk [13], Derkach, Hassi, Malamud and de Snoo [5], Gorimov, Mikhailets and Pankrashkin [14], Kholkin [20], Mogilevskii [22–26] and the book by Rofe-Beketov and Kholkin [33] with its 941 references.

Our approach uses the GKN (Galzman–Krein–Naimark) theorem. This theorem was so named by Everitt and Zettl [11] in honor of the work of these authors for reasons given in Section 9 of [11]. This approach also has an extensive literature dating back to the middle of the 20th century and with some major recent developments. See the survey paper by Sun and Zettl [43].

In this paper we characterize the self-adjoint operators S in H satisfying (1.2). This characterization is based on LC solutions. These are solutions near an endpoint of equation (1.1) for some real value of λ. In the maximal deficiency case $d = n$, all solutions of (1.1) are in H for any λ, real or complex, and any solution basis for a real value of λ can be used to describe all self-adjoint domains. In the minimal deficiency case $d = 0$ the operator $S_{\min}$ is self-adjoint and has no proper self-adjoint extension.

In the much more difficult intermediate deficiency case, $0 < d < n$, it is not clear which solutions contribute to the determination of the singular self-adjoint domains and which ones do not. Here we identify those which do contribute and call them LC solutions in analogy with the case when $d = n$, particularly for $n = 2$ when we have the celebrated Weyl limit-circle case. Solutions which lie in H but do not contribute to the characterization of the self-adjoint domains are called LP solutions, again in analogy with the second order limit-point case when there is no boundary condition needed at a limit-point endpoint to determine a self-adjoint operator. However, in contrast to the second order case, for $n > 2$ a solution basis consists of three types of solutions: LC, LP and those not in H and only the LC solutions contribute to the characterization of the self-adjoint domains.

The construction of LC solutions is based on the assumption that for some real value of the spectral parameter λ there exist d linearly independent solutions of equation (1.1) which are square-integrable near each endpoint. It is well known that, if this assumption does hold, then the essential spectrum of every self-adjoint extension S covers the whole real line $(-\infty, \infty)$. In this case if there is an eigenvalue for some S, it is embedded in the essential spectrum. There seems to be little known, other than examples, about boundary conditions which produce such an eigenvalue, indeed these seem to be coincidental. Thus this is a 'mild' additional assumption.

Our characterization reduces to that previously found by Wang–Sun–Zettl [37] for the even order case with real coefficients and one regular endpoint and its extension by Hao–Sun–Wang–Zettl [15] to the case when both endpoints are singular. The characterization in both papers is given in terms of LC solutions. Such solutions were first constructed by Sun [34] (without the additional assumption) using complex values of the spectral parameter λ. In [15,37] real values of λ were used. This real λ characterization was achieved with a significant modification of Sun's method and led to obtaining information about the discrete, continuous, and essential spectrum of these operators [16,17,28,29,35]. It also led to general classification results for self-adjoint boundary conditions as separated, coupled, and mixed. For $n = 4$ canonical forms for regular and singular self-adjoint boundary conditions for all three types were found [18,19].

This classification clarified a point made by Everitt and Markus in [8,9] about nonreal boundary conditions for self-adjoint operators S. They state:

"We provide an affirmative answer [...] to a long standing open question concerning the existence of real differential expressions of even order ≥ 4, *for which there are non-real self-adjoint differential operators specified by strictly separated boundary conditions [...] This is somewhat surprising because it is well known that for order* $n = 2$ *strictly separated conditions can produce only real operators (that is, any given such complex conditions can always be replaced by corresponding real boundary conditions.)"*

It is clear from [38] that such conditions occur naturally and explicitly for regular and singular problems for all $n = 2k$, $k > 1$. *Furthermore, the analysis of Wang, Sun and Zettl [38] shows that it is not the order of the equation which is the relevant factor for the existence of non-real separated self-adjoint boundary conditions but the **number** of boundary conditions. If there is only one, regular or singular, separated boundary condition at a given endpoint, as must be the case for* $n = 2$, *then it can always be replaced by an equivalent real condition. On the other hand if there are two or more separated conditions at a given regular or singular endpoint, then some of these are not equivalent to real conditions.*

In [40] Yao–Sun–Zettl found a 1–1 correspondence between the EM symplectic geometry characterization [8] and the HSWZ Hilbert space characterization [15] thereby creating a 'bridge' for the study of differential operators using methods of symplectic algebra and geometry.

Our proof is in the spirit of the proofs in [15,37] but there are some significant differences between even and odd order differential operators and real and complex coefficients. In particular, although our construction uses solutions for real λ these solutions cannot be chosen to be real valued in contrast to the even order case with real coefficients.

We believe our characterization will also yield information about the spectrum of these operators including the odd order ones. We plan to investigate this in a subsequent paper.

See the survey paper [43] for more information about self-adjoint ordinary differential operators in Hilbert space, additional references, historical comments, etc.

John von Neumann

"[...] when America's National Academy of Science asked shortly before his death what he thought were his three greatest achievements [...] Johnny replied to the academy that he considered his most important contributions to have been on the theory of self-adjoint operators in Hilbert space, and on the mathematical foundations of quantum theory and the ergotic theorem."

Macrae's biography of John von Neumann [31]

Applications

"From the point of view of applications, the most important single class of operators are the differential operators. The study of these operators is complicated by the fact that they are necessarily unbounded. Consequently, the problem of choosing a domain for a differential operator is by no means trivial; [...] for unbounded operators the choice of domains can be quite crucial".

Dunford–Schwartz, Vol. II ([6, p. 1278])

The organization of the paper is as follows: this introduction is followed by a brief discussion of the basic theory of first order systems of differential equations and their relationship to very general n-th order scalar equations in Section 2. Section 3 contains the statement of the characterization. The proof is given in Section 4 along with several other results, some of which we believe are of independent interest. In particular the decomposition of the maximal domain:

$$D_{\max}(a,b) = D_{\min}(a,b) \dotplus \operatorname{span}\{u_1,\ldots,u_{m_a}\} \dotplus \operatorname{span}\{v_1,\ldots,v_{m_b}\} \tag{1.3}$$

here $u_1,\ldots,u_{m_a}$, $v_1,\ldots,v_{m_b}$ are the LC solutions at a,b, respectively. This is the ode version of the abstract von Neumann formula for the adjoint of a symmetric operator in Hilbert space. It plays a critical role the proof of the characterization of self-adjoint operators and, we believe, will be useful in the study of other classes of operators in Hilbert space.

2 Preliminaries

In this section we summarize some basic facts about general symmetric quasi-differential equations of even and odd order with real or complex coefficients for the convenience of the reader. For a comprehensive discussion of these equations and their relationship to the classical symmetric (formally self-adjoint) case discussed in the well known books by Coddington and Levinson [4] and Dunford and Schwartz [6] as well as to the 'special' symmetric quasi-differential expressions studied in Naimark [30], as well as additional references, historical remarks and other comments, notation, definitions, etc., the reader is referred to the recent survey article by Sun and Zettl [43].

These expressions generate symmetric differential operators in the Hilbert space $L^2(J,w)$ and it is these operators and their self-adjoint extensions which are studied here.

Definition 2.1. Let $J = (a,b)$, $-\infty \le a < b \le \infty$. For $w \in L_{loc}(J,\mathbb{R})$, $w > 0$ a.e. in J, $L^2(J,w)$ denotes the Hilbert space of functions $f : J \to \mathbb{C}$ satisfying $\int_J |f|^2 w < \infty$ with inner product $(f,g)_w = \int_J f\,\overline{g}\,w$. Such a w is called a 'weight function'. Here $L_{\text{loc}}(J,\mathbb{R})$ denotes the real valued functions which are Lebesgue-integrable on every compact subinterval of J and $L(J,\mathbb{R})$ denotes the real valued functions which are Lebesgue-integrable on the whole interval J.

Notation 2.2. Let $\mathbb{R}$ denote the real numbers, $\mathbb{C}$ the complex numbers, $\mathbb{N} = \{1,2,3,\ldots\}$, $\mathbb{N}_0 = \{0,1,2,3,\ldots\}$, $\mathbb{N}_2 = \{2,3,4,\ldots\}$, $J = (a,b)$ for $-\infty \le a < b \le \infty$, $M_{nk}(X)$ the $n \times k$ matrices with entries from X, $M_n(X) = M_{nk}(X)$ when $n = k$, $M_{n1}(X)$ is also denoted by X^n; $L(J,\mathbb{R})$ and $L(J,\mathbb{C})$ the Lebesgue integrable real and complex valued functions on J, respectively, $L_{\text{loc}}(J,\mathbb{R})$ and $L_{\text{loc}}(J,\mathbb{C})$ the real and complex valued functions which are Lebesgue integrable on all compact subintervals of J, respectively. We also use $L_{\text{loc}}(J) = L_{\text{loc}}(J,\mathbb{C})$ and $L(J) = L(J,\mathbb{C})$. $AC_{\text{loc}}(J)$ denotes the complex valued functions which are absolutely continuous on compact subintervals of J and $AC(J)$ denotes the absolutely continuous functions on J, $C^j(J)$ denotes the complex functions on J which have j continuous derivatives. $D(A)$ denotes the domain of the operator A.

Let $J = (a,b)$ be an interval with $-\infty \le a < b \le \infty$ and let $n > 2$ be a positive integer (even or odd). Let

$$\begin{aligned} Z_n(J) := \{ Q = (q_{rs})_{r,s=1}^n,\ q_{r,r+1} &\neq 0 \text{ a.e. on } J,\ q_{r,r+1}^{-1} \in L_{\text{loc}}(J),\ 1 \le r \le n-1, \\ q_{rs} = 0 \text{ a.e. on } J,\ 2 \le r+1 < s \le n;\ & q_{rs} \in L_{\text{loc}}(J),\ s \neq r+1,\ 1 \le r \le n-1 \}. \end{aligned} \tag{2.1}$$

For $Q \in Z_n(J)$ we define

$$V_0 := \{y : J \to \mathbb{C},\ y \text{ is measurable}\} \tag{2.2}$$

and

$$y^{[0]} := y \qquad (y \in V_0). \tag{2.3}$$

Inductively, for $r = 1, \ldots, n$, we define

$$V_r = \left\{ y \in V_{r-1} : y^{[r-1]} \in (AC_{\mathrm{loc}}(J)) \right\}, \tag{2.4}$$

$$y^{[r]} = q_{r,r+1}^{-1} \left\{ y^{[r-1]'} - \sum_{s=1}^{r} q_{rs} y^{[s-1]} \right\} \qquad (y \in V_r), \tag{2.5}$$

where $q_{n,n+1} := 1$, and $AC_{\mathrm{loc}}(J)$ denotes the set of complex valued functions which are absolutely continuous on all compact subintervals of J. Finally we set

$$My = M_Q y := \mathrm{i}^n y^{[n]} \quad \text{on } J, \qquad (y \in V_n,\ \mathrm{i} = \sqrt{-1}). \tag{2.6}$$

The expression $M = M_Q$ is called the quasi-differential expression associated with Q. For V_n we also use the notations $V(M)$ and $D(Q)$. The function $y^{[r]}$ $(0 \le r \le n)$ is called the r-th quasi-derivative of y. Since the quasi-derivative depends on Q, we sometimes write $y_Q^{[r]}$ instead of $y^{[r]}$.

Remark 2.3. *The operator $M : D(Q) \to L_{\mathrm{loc}}(J)$ is linear.*

Remark 2.4. Note that the differential expression M_Q in equation (2.6) requires only local integrability assumptions on the coefficients (2.1).

The initial value problem associated with $Y' = AY + F$ has a unique solution.

Proposition 2.5. *For each $F \in (L_{\mathrm{loc}}(J))^n$, each α in J and each $C \in \mathbb{C}^n$ there is a unique $Y \in (AC_{\mathrm{loc}}(J))^n$ such that*

$$Y' = AY + F \quad \textit{and} \quad Y(\alpha) = C. \tag{2.7}$$

Proof. See Chapter 1 in [42]. □

From Proposition 2.5, we immediately infer the following corollary.

Corollary 2.6. *For each $f \in L_{\mathrm{loc}}(I)$, each $\alpha \in J$ and $c_0, \ldots, c_{n-1} \in \mathbb{C}$ there is a unique $y \in D(Q)$ such that*

$$y^{[n]} = f \quad \textit{and} \quad y^{[r]}(\alpha) = c_r \qquad (r = 0, \ldots, n-1). \tag{2.8}$$

If $f \in L(J)$, J is bounded and all components of Q are in $L(J)$, then $y \in AC(J)$.

Definition 2.7 (Regular endpoints)**.** Let $Q \in Z_n(J)$, $J = (a, b)$. The expression $M = M_Q$ is said to be regular at a if for some c, $a < c < b$, we have

$$\begin{aligned} q_{r,r+1}^{-1} &\in L(a,c), \qquad r = 1, \ldots, n-1; \\ q_{rs} &\in L(a,c), \qquad 1 \le r, s \le n,\ s \ne r+1. \end{aligned}$$

Similarly the endpoint b is regular if for some c, $a < c < b$, we have

$$\begin{aligned} q_{r,r+1}^{-1} &\in L(c,b), \qquad r = 1, \ldots, n-1; \\ q_{rs} &\in L(c,b), \qquad 1 \le r,\ s \le n,\ s \ne r+1. \end{aligned}$$

Note that, from (2.1) it follows that if the above hold for some $c \in J$ then they hold for any $c \in J$. We say that M is regular on J, or just M is regular, if M is regular at both endpoints. Equation (1.1) is regular at a if M is regular at a and w is integrable at a, i.e. there is a $c \in (a,b)$ such that $w \in L(a,c)$. Similarly for the endpoint b. We say that equation (1.1) is regular if it is regular at both endpoints.

Next we give the definition of symmetric quasi-differential expressions and indicate how they are are constructed. For examples and illustrations see [43].

Definition 2.8. Let $Q \in Z_n(J)$ and let $M = M_Q$ be defined as (2.6). Assume that

$$Q = -E_n^{-1} Q^* E_n, \quad \text{where} \quad E_n = ((-1)^r \delta_{r,n+1-s})_{r,s=1}^n. \tag{2.9}$$

Then we call Q an L-symmetric matrix and $M = M_Q$ is called a symmetric differential expression.

Definition 2.9. The symplectic matrix

$$E_k = ((-1)^r \delta_{r,k+1-s})_{r,s=1}^k, \qquad k = 2,3,4,5,\ldots \tag{2.10}$$

plays an important role in the theory of self-adjoint differential operators.

Next we define the maximal and minimal differential operators.

Definition 2.10. Let $Q \in Z_n(J)$ satisfy (2.9) and let $M = M_Q$ be the corresponding differential symmetric differential expression.The maximal operator $S_{\max}$ generated by M is defined by

$$\begin{aligned} D_{\max} = \Big\{ & u \in L^2(J,w) : u^{[0]}, u^{[1]}, \ldots, u^{[n-1]} \text{ are absolutely continuous in } (a,b), \\ & \text{and } w^{-1}Mu \in L^2(J,w) \Big\}, \\ S_{\max} u = {} & w^{-1}Mu, u \in D_{\max}. \end{aligned}$$

The minimal operator $S_{\min}$ can be defined by

$$S_{\min} = S_{\max}^*.$$

The next lemma justifies this definition.

Lemma 2.11. *Let $S_{\min}$ and $S_{\max}$ be defined as above. Then $S_{\min}$ and $S_{\max}$ are closed, densely defined, symmetric operators in H. Furthermore $S_{\min}^* = S_{\max}$.*

Proof. See [39]. □

Lemma 2.12. *Suppose M is regular at c. Then for any $y \in D_{\max}$ the limits*

$$y^{[r]}(c) = \lim_{t \to c} y^{[r]}(c)$$

exist and are finite, $r = 0, \ldots, n-1$. In particular this holds at any regular endpoint and at each interior point of J. At an endpoint the limit is the appropriate one sided limit.

Proof. See [30] or [39]. Although this lemma is more general than the corresponding result in these references, the same method of proof can be used here. □

Notation 2.13. Let $a < c < b$. Below we will also consider equation (2.6) and the operators generated by it on the intervals (a,c) and (c,b). Note that if $Q \in Z_n(J)$, then it follows that $Q \in Z_n(a,c)$, $Q \in Z_n(c,b)$ and we can study equation (2.6) on (a,c) and (c,b) as well as on $J = (a,b)$. Also (2.9) holds on (a,c) and on (c,b). In particular the minimal and maximal operators are defined on these two subintervals and we can also study the operator theory generated by (2.6) in the Hilbert spaces $L^2((a,c),w)$ and $L^2((c,b),w)$. Below we will use the notation $S_{\min}(I)$, $S_{\max}(I)$ for the minimal and maximal operators on the interval I for $I = (a,c)$, $I = (c,b)$, $I = (a,b) = J$. The interval $J = (a,b)$ may be omitted when it is clear from the context. So we make the following definition.

Definition 2.14. Let $a < c < b$. Let d_a^+, d_b^+ denote the dimension of the solution space of $My = \mathrm{i}\,wy$ lying in $L^2(a,c,w)$ and $L^2(c,b,w)$, respectively, and let d_a^-, d_b^- denote the dimension of the solution space of $My = -\mathrm{i}\,wy$ lying in $L^2(a,c,w)$ and $L^2(c,b,w)$, respectively. Then d_a^+ and d_a^- are called the positive deficiency index and the negative deficiency index of $S_{\min}(a,c)$, respectively. Similarly for d_b^+ and d_b^- . Also d^+, d^- denote the deficiency indices of $S_{\min}(a,b)$; these are the dimensions of the solution spaces of $My = \mathrm{i}\,wy$, $My = -\mathrm{i}\,wy$ lying in $L^2(a,b,w)$. If $d_a^+ = d_a^-$, then the common value is denoted by d_a and is called the deficiency index of $S_{\min}(a,c)$, or the deficiency index at a. Similarly for d_b. Note that d_a, d_b are independent of c. If $d^+ = d^-$, then we denote the common value by d and call it the deficiency index of $S_{\min}(a,b)$ or of $S_{\min}$.

The relationships between d_a, d_b and d are well known and given in the next lemma along with some additional information.

Lemma 2.15. *For $d_a^+, d_b^+, d_a^-, d_b^-, d^+, d^-, d_a, d_b$ defined as Definition 2.14, we have*

(1) $d^+ = d_a^+ + d_b^+ - n,\ d^- = d_a^- + d_b^- - n$;

(2) if $d_a^+ = d_a^- = d_a, d_b^+ = d_b^- = d_b$, *then* $[\frac{n+1}{2}] \leq d_a, d_b \leq n$;

(3) the minimal operator $S_{\min}$ has self-adjoint extensions in H if and only if $d^+ = d^-$, in this case we let $d = d^+ = d^-$. In the even order case, if d has the minimum value, then $S_{\min}$ is self-adjoint with no proper self-adjoint extension; in all other cases $S_{\min}$ has an uncountable number of self-adjoint extensions, i.e. there are an uncountable number of operators S in H satisfying

$$S_{\min} \subset S = S^* \subset S_{\max}.$$

These are the operators we characterize in this paper in terms of two-point boundary conditions.

Proof. This is well known, e.g. see the book [39]. □

Remark 2.16. Let $a < c < b$. Below we assume that $d_a^+ = d_a^- = d_a$, $d_b^+ = d_b^- = d_b$ and that for some $\lambda_a \in \mathbb{R}$ there exist d_a linearly independent solutions of (1.1) lying in $L^2(a,c,w)$ and that for some $\lambda_b \in \mathbb{R}$ there exist d_b linearly independent solutions of (1.1) lying in $L^2(c,b,w)$. (If this holds for some $a < c < b$ then it holds for every such c.) This is a weak assumption because if there is no such λ_a, then (it is well known that) the essential spectrum of S is $(-\infty,\infty)$ for every self-adjoint realization S. Similarly for λ_b. In this case if some self-adjoint realization S has an eigenvalue it is embedded in the essential spectrum. We believe that the boundary conditions determining such embedded eigenvalues are coincidental. Except for examples there seems to be little known about such embedded eigenvalues.

In the study of boundary value problems the Lagrange identity is fundamental. Next we define the Lagrange bracket.

Definition 2.17. Define

$$[y,z] = \mathrm{i}^n \sum_{r=0}^{n-1} (-1)^{n+1-r} \bar{z}^{[n-r-1]} y^{[r]} = -\mathrm{i}^n Z^* E_n Y,$$

where

$$Y = \begin{pmatrix} y \\ y^{[1]} \\ \vdots \\ y^{[n-1]} \end{pmatrix}, \qquad Z = \begin{pmatrix} z \\ z^{[1]} \\ \vdots \\ z^{[n-1]} \end{pmatrix}.$$

Then $[\cdot,\cdot]$ is called a Lagrange bracket.

Lemma 2.18 (The Lagrange identity). *Let $Q \in Z_n(J)$ and $E := ((-1)^r \delta_{r,n+1-s})_{r,s=1}^n$. Let $M = M_Q$ be the corresponding differential expression. Let the quasi-derivative $y, y^{[1]}, \dots, y^{[n-1]}$ be defined as above. Then for any $y, z \in D(Q)$, we have*

$$\bar{z}My - (\overline{Mz})y = [y,z]'. \tag{2.11}$$

Proof. See [29]. □

Lemma 2.19. *For any y, z in $D_{\max}$ we have*

$$\int_a^b \{\bar{z}My - y\overline{Mz}\} = [y,z](b) - [y,z](a),$$

where $[y,z](b) = \lim_{t\to b}[y,z](t)$, and $[y,z](a) = \lim_{t\to a}, t \in (a,b)$. Thus $[\cdot,\cdot](s)$ exists as a finite limit for $s = a, b$.

Proof. This follows by integrating (2.11). □

The finite limits guaranteed by Lemma 2.19 play a fundamental role in the characterization of the self-adjoint domains given below.

Corollary 2.20. *If $My = \lambda w y$ and $Mz = \bar{\lambda} w z$ on some interval (a,b), then $[y,z]$ is constant on (a,b). In particular, if λ is real and $My = \lambda w y$, $Mz = \lambda wz$ on some interval (a,b), then $[y,z]$ is constant on (a,b).*

Proof. This follows directly from (2.11). □

Remark 2.21. For real λ, the solutions of (1.1) are not, in general, real-valued. However, the Lagrange bracket of two linearly independent solutions of (1.1) for real λ is a constant. For n even and real coefficients, if there are d linearly independent solutions of (1.1) in H, then there are d linearly independent real-valued solutions in H. This is one of the important differences between the equation (1.1) studied here and the equations studied in [15,37].

Following Everitt and Zettl [10] we call the next lemma, the Naimark patching lemma or just the patching lemma. Our version of it is more general than that given by Naimark [30] but the method of proof is the same.

Lemma 2.22 (Naimark patching lemma)**.** *Let $Q \in Z_n(J)$ and assume that M is regular on J. Let $\alpha_0, \dots, \alpha_{n-1}, \beta_0, \dots, \beta_{n-1} \in \mathbb{C}$. Then there is a function $y \in D_{\max}$ such that*

$$y^{[r]}(a) = \alpha_r, \quad y^{[r]}(b) = \beta_r \qquad (r = 0, \dots, n-1).$$

Corollary 2.23. *Let $a < c < d < b$ and $\alpha_0, \dots, \alpha_{n-1}, \beta_0, \dots, \beta_{n-1} \in \mathbb{C}$. Then there is a $y \in D_{\max}$ such that y has compact support in J and satisfies :*

$$y^{[r]}(c) = \alpha_r, \quad y^{[r]}(d) = \beta_r \qquad (r = 0, \dots, n-1).$$

Proof. The patching lemma gives a function y_1 on $[c, d]$ with the desired properties. Let c_1, d_1 with $a < c_1 < c < d < d_1 < b$. Then use the patching lemma again to find y_2 on (c_1, c) and y_3 on (d, d_1) such that

$$y_2^{[r]}(c_1) = 0, \quad y_2^{[r]}(c) = \alpha_r, \quad y_3^{[r]}(d) = \beta_r, \quad y_3^{[r]}(d_1) = 0 \qquad (r = 0, \dots, n-1).$$

Now set

$$y(x) := \begin{cases} y_1(x) & \text{for } x \in [c, d] \\ y_2(x) & \text{for } x \in (c_1, c) \\ y_3(x) & \text{for } x \in (d, d_1) \\ 0 & \text{for } x \in I \setminus (c_1, d_1). \end{cases}$$

Clearly y has compact support in J. Since the quasi-derivatives at c_1, c, d, d_1 coincide on both sides, $y \in D_{\max}$ follows. □

Corollary 2.24. *Let $a_1 < \cdots < a_k \in J$, where a_1 and a_k can also be regular endpoints. Let $\alpha_{jr} \in \mathbb{C}$, $(j = 1, \dots, k;\ r = 0, \dots, n-1)$. Then there is a $y \in D_{\max}$ such that*

$$y^{[r]}(a_j) = \alpha_{jr} \qquad (j = 1, \dots, k;\ r = 0, \dots, n-1).$$

Proof. This follows from repeated applications of the previous corollary. □

3 Self-adjoint domains

The next theorem characterizes the domains $D(S)$ for all S satisfying (1.2).

Theorem 3.1. *Let $Q \in Z_n(J)$, $J = (a, b)$, $-\infty \le a < b \le \infty$, $n > 2$, $a < c < b$. Suppose Q satisfies (2.9) and let $M = M_Q$ be constructed as above. Suppose $d_a = d_a^+ = d_a^-$, $d_b = d_b^+ = d_b^-$ and let $m_b = 2d_b - n$; $m_a = 2d_a - n$. Then*

(1) M is a symmetric differential expression.

(2) $d = d_a + d_b - n$.

(3) Assume there exists a $\lambda_b \in \mathbb{R}$ such that (1.1) has d_b linearly independent solutions lying in $H_b = L^2((c, b), w)$. Then there exist solutions v_j, $j = 1, \dots, m_b$, of (1.1) with $\lambda = \lambda_b$ lying in H_b such that the $m_b \times m_b$ matrix

$$V = ([v_i, v_j](b)), \quad 1 \le i, j \le m_b$$
$$= -i^n \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}_{m_b \times m_b} = -i^n E_{m_b} \tag{3.1}$$

is nonsingular and

$$[v_j, y](b) = 0, \qquad j = m_b + 1, \dots, d_b. \tag{3.2}$$

for all $y \in D_{\max}(c, b)$.

(4) *Assume there exists a $\lambda_a \in \mathbb{R}$ such that (1.1) has d_a linearly independent solutions lying in $H_a = L^2((a, c), w)$. Then there exist solutions u_j, $j = 1, \dots, m_a$, of (1.1) with $\lambda = \lambda_a$ lying in H_a such that the $m_a \times m_a$ matrix*

$$\begin{aligned} U &= [u_i, u_j](a),\ 1 \le i, j \le m_a \\ &= -\mathrm{i}^n \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}_{m_a \times m_a} = -\mathrm{i}^n E_{m_a} \end{aligned} \tag{3.3}$$

is nonsingular and

$$[u_j, y](a) = 0, \qquad j = m_a + 1, \dots, d_a. \tag{3.4}$$

for all $y \in D_{\max}(a, c)$.

(5) *The solutions $u_1, u_2, \dots, u_{d_a}$ can be extended to (a, b) such that the extended functions, also denoted by $u_1, \dots, u_{d_a}$, satisfy $u_j \in D_{\max}(a, b)$ and u_j is identically zero in a left neighborhood of b, $j = 1, \dots, d_a$.*

(6) *The solutions $v_1, v_2, \dots, v_{d_b}$ can be extended to (a, b) such that the extended functions, also denoted by $v_1, \dots, v_{d_b}$, satisfy $v_j \in D_{\max}(a, b)$ and v_j is identically zero in a right neighborhood of a, $j = 1, \dots, d_b$.*

(7) *A linear submanifold $D(S)$ of $D_{\max}(a, b)$ is the domain of a self-adjoint extension S satisfying (1.2) if and only if there exists a complex $d \times m_a$ matrix A and a complex $d \times m_b$ matrix B such that the following three conditions hold:*

(8)
$$\operatorname{rank}[A : B] = d; \tag{3.5}$$

(9)
$$A E_{m_a} A^* = B E_{m_b} B^*; \tag{3.6}$$

(10)
$$D(S) = \left\{ y \in D_{\max} : A \begin{pmatrix} [y,u_1](a) \\ \vdots \\ [y,u_{m_a}](a) \end{pmatrix} + B \begin{pmatrix} [y,v_1](b) \\ \vdots \\ [y,v_{m_b}](b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}. \tag{3.7}$$

Recall that by Lemma 2.19 the brackets $[y, u_j](a)$, $j = 1, \dots, m_a$; $[y, v_j](b)$, $j = 1, \dots, m_b$ exist as finite limits.

Proof. This will be given in Section 4. □

Although Theorem 3.1 is stated for the case when both endpoints are singular it reduces to the cases when one or both endpoints are regular. The proofs of these corollaries are similar to the proofs given in [37] and [15] for the even order case with real coefficients and therefore omitted.

Corollary 3.2. *Let the hypotheses and notation of Theorem 3.1 hold and assume the endpoint a is regular. Then $d_a = n$, assumption (4) holds and*

$$D(S) = \left\{ y \in D_{\max} : A \begin{pmatrix} y^{[0]}(a) \\ \vdots \\ y^{[n-1]}(a) \end{pmatrix} + B \begin{pmatrix} [y,v_1](b) \\ \vdots \\ [y,v_{m_b}](b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}. \tag{3.8}$$

Corollary 3.3. *Let the hypotheses and notation of Theorem 3.1 hold and assume the endpoint b is regular. Then $d_b = n$, assumption (3) holds and*

$$D(S) = \left\{ y \in D_{\max} : A \begin{pmatrix} [y,u_1](a) \\ \vdots \\ [y,u_{m_a}](a) \end{pmatrix} + B \begin{pmatrix} y^{[0]}(b) \\ \vdots \\ y^{[n-1]}(b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}. \tag{3.9}$$

Corollary 3.4. *Let the hypotheses and notation of Theorem 3.1 hold and assume that both endpoints are regular. Then $d_a = d_b = n$, assumptions (3) and (4) hold and*

$$D(S) = \left\{ y \in D_{\max} : A \begin{pmatrix} y^{[0]}(a) \\ \vdots \\ y^{[n-1]}(a) \end{pmatrix} + B \begin{pmatrix} y^{[0]}(b) \\ \vdots \\ y^{[n-1]}(b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}. \tag{3.10}$$

Corollaries 3.2 and 3.3 were proven by Wang–Sun–Zettl [37] for the case when $n = 2k$, $k > 1$, and real coefficients. Also the construction (and definition) of LC solutions is given in this paper.

Theorem 3.1 was proven by Hao–Sun–Wang–Zettl in [15] for the case when $n = 2k$, $k > 1$, and real coefficients.

Corollary 3.4 can be found in Naimark's book [30] for the case when $n = 2k$, $k > 1$, the coefficients are real, and Q has the special form

$$Q = \begin{bmatrix} & & 1 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 \\ & & & & q_{3,4} & 0 & 0 \\ & & & q_{43} & & 1 & 0 \\ & & q_{52} & & & & 1 \\ & q_{61} & & & & & \end{bmatrix} \tag{3.11}$$

when $n = 6$ and similar forms for $n = 4, 6, 8, 10, \ldots$; all entries not shown are 0.

Remark 3.5. Although the general appearance of the self-adjoint boundary conditions (3.5), (3.6), (3.7) is the same for n even and odd there are some major differences in the self-adjoint operators and their spectrum for these two cases. For example in the odd order case the the minimal operator $S_{\min}$, and therefore all of its extensions, is unbounded above and below. In the even order case when both endpoints are regular and the leading coefficient is positive $S_{\min}$ is bounded below and unbounded above. In the singular even order case with positive leading coefficient $S_{\min}$ is unbounded above and may or may not be bounded below. See [28], [29], [21]; also see [43].

Definition 3.6. We call the solutions $u_1, \ldots, u_{m_a}$ and $v_1, \ldots, v_{m_b}$ LC solutions at a and b, respectively. The solutions $u_{m_{a+1}}, \ldots, u_{d_a}$ and $v_{m_b+1}, \ldots, v_{d_b}$ are called LP solutions at a and b, respectively. The other solutions from a solution basis of $My = \lambda_a wy$ on (a, c) are not in $L^2((a, c), w)$ and have no role in the characterization. Similar remarks apply for the endpoint b. Thus by Theorem 3.1 the LC solutions contribute to the determination of the self-adjoint boundary conditions and the LP solutions do not contribute due to (3.2) and (3.4). (The solutions not in $L^2(J, w)$ do not play any role in the maximal domain decomposition nor in the characterization of the self-adjoint domains.)

4 Proof and other results

In this section we prove Theorem 3.1. This proof uses the well known GKN Theorem, which we state next for the convenience of the reader, and a decomposition of the maximal domain which we believe is of independent interest.

Theorem 4.1 (GKN). *Let $Q \in Z_n(J)$, $J = (a,b)$, $-\infty \leq a < b \leq \infty$, $n > 2$, $a < c < b$. Assume Q satisfies (2.9) and let $M = M_Q$ be constructed as above. Let $S_{\min}$ and $S_{\max}$ be defined as above. Then a linear submanifold $D(S)$ of $D_{\max}$ is the domain of a self-adjoint extension S of $S_{\min}$ if and only if there exist functions $w_1, w_2, \ldots, w_d$ in $D_{\max}$ satisfying the following conditions:*

(i) $w_1, w_2, \ldots, w_d$ *are linearly independent modulo* $D_{\min}$*;*

(ii) $[w_i, w_j](b) - [w_i, w_j](a) = 0,\ i, j = 1, \ldots, d;$

(iii) $D(S) = \{y \in D_{\max} : [y, w_j](b) - [y, w_j](a) = 0,\ j = 1, \ldots, d\}.$

Here $[\cdot,\cdot]$ *denotes the Lagrange bracket associated with (1.1) and d is the deficiency index of $S_{\min}$.*

Proof. This is well known, see [43]. □

The GKN characterization depends on the maximal domain functions w_j, $j = 1, \ldots, d$. These functions depend on the coefficients of the differential equation and this dependence is implicit and complicated.

Our construction of LC solutions $u_1, \ldots, u_{m_a}$, $v_1, \ldots, v_{m_b}$ leads to a new decomposition of the maximal domain.

Theorem 4.2. *Let the hypotheses and notation of Theorem 3.1 hold. Let u_j, $j = 1, \ldots, m_a$ and v_j, $j = 1, \ldots, m_b$ be LC solutions given by Theorem 3.1. Then*

$$D_{\max}(a,b) = D_{\min}(a,b) \dotplus \operatorname{span}\{u_1, \ldots, u_{m_a}\} \dotplus \operatorname{span}\{v_1, \ldots, v_{m_b}\}. \tag{4.1}$$

Proof. By Von Neumann's formula, $\dim D_{\max}(a,b)/D_{\min}(a,b) \leq 2d$. From Theorem 3.1 and the observation that the matrices U and V are nonsingular it follows that $u_1, \ldots, u_{m_a}$ and $v_1, \ldots, v_{m_b}$ are linearly independent $\operatorname{mod}(D_{\min}(a,b))$, since $m_a + m_b = 2(d_a + d_b - n) = 2d$, therefore $\dim D_{\max}(a,b)/\ D_{\min}(a,b) \geq 2d$, completing the proof. □

In view of the wide interest in the case when endpoint is regular we give the decomposition (4.1) for that case as a corollary.

Corollary 4.3. *Let the hypotheses and notation of Theorem 3.1 hold and assume the endpoint a is regular, $a < c < b$, $\lambda \in \mathbb{R}$. Then*

$$D_{\max} = D_{\min} \dotplus \operatorname{span}\{z_1, \ldots, z_n\} \dotplus \operatorname{span}\{v_1, \ldots, v_{m_b}\}. \tag{4.2}$$

where $z_j \in D_{\max}(a,b)$, $j = 1,\dots,n$ *such that* $z_j(t) = 0$ *for* $t \geq c$, $j = 1,\dots,n$ *and*

$$\begin{pmatrix} [z_1,z_1](a) & [z_2,z_1](a) & \cdots & [z_{n-1},z_1](a) & [z_n,z_1](a) \\ [z_1,z_2](a) & [z_2,z_2](a) & \cdots & [z_{n-1},z_2](a) & [z_n,z_2](a) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ [z_1,z_n](a) & [z_2,z_n](a) & \cdots & [z_{n-1},z_n](a) & [z_n,z_n](a) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & -1 & 0 & \cdots & 0 \\ (-1)^n & 0 & \cdots & 0 & 0 \end{pmatrix} = E_n.$$

Such functions z_j *exist by the Patching Lemma and the fact that for each* $i = 1,\dots,n$ *the values* $z_i^{[j]}(a)$ *can be assigned arbitrarily.*

A similar result holds if the endpoint b *is regular.*

Before we prove Corollary 4.3, firstly, we state the Sun decomposition theorem [34].

Theorem 4.4 (Sun). *Assume that the endpoint* a *is regular while the endpoint* b *maybe singular. Let* $a < c < b$. *Let* $m_b = 2d_b - n$ *and* $\lambda \in \mathbb{C}$, $\mathrm{Im}(\lambda) \neq 0$. *Then there exist solutions* ϕ_j, $j = 1,\dots,m_b$ *of* $My = \lambda wy$ *on* (c,b) *such that the* $m_b \times m_b$ *matrix* $[\phi_i,\phi_j](b)$, $1 \leq i, j \leq m_b$ *is nonsingular and there exist solutions* $z_1, z_2, \dots, z_n$ *on* (a,c) *such that*

$$D_{\max} = D_{\min} \dotplus \mathrm{span}\{z_1, z_2, \dots, z_n\} \dotplus \mathrm{span}\{\phi_1, \phi_2, \dots, \phi_{m_b}\}. \tag{4.3}$$

Proof. The proof given in [34] for a more restricted class of equations $My = \lambda wy$ can be easily adapted to the more general equations considered here. □

Next we give a proof of Corollary 4.3.

Proof. If $n = 2k$, although we do not assume that the coefficients are real, the proof given in [37] for real coefficients can readily be adapted to prove Corollary 4.3 in the even order case and is therefore omitted.

If $n = 2k+1$, we let $\theta_1, \dots, \theta_{d_b}$ be d_b linearly independent solutions of (1.1) for some real λ. By (4.1) there exist $y_j \in D_{\min}$ and $r_{is}, k_{ij} \in \mathbb{C}$ such that

$$\theta_i = y_i + \sum_{s=1}^{n} r_{is} z_s + \sum_{j=1}^{m_b} k_{ij}\phi_j, \qquad i = 1,\dots,d_b. \tag{4.4}$$

From this it follows that

$$\begin{aligned} ([\theta_h,\theta_l](b))_{1\leq h,l\leq d_b} &= \left(\left[\sum_{j=1}^{m_b} k_{hj}\phi_j, \sum_{j=1}^{m_b} k_{lj}\phi_j \right](b) \right) \\ &= F([\phi_i,\phi_j](b))_{1\leq i,j\leq m_b} F^*, \qquad F = (k_{ij})_{d_b\times m_b}. \end{aligned} \tag{4.5}$$

Hence

$$\mathrm{rank}([\theta_h,\theta_l](b))_{1\leq h,l\leq d_b} \leq m_b. \tag{4.6}$$

By Corollary 2.20 we know that,

$$([\theta_h,\theta_l](b))_{d_b\times d_b} = ([\theta_h,\theta_l](a))_{d_b\times d_b} = -\mathrm{i}^n G^* E_n G \tag{4.7}$$

where

$$G = \begin{pmatrix} \theta_1(a) & \cdots & \theta_{d_b}(a) \\ \vdots & \ddots & \vdots \\ \theta_1^{[n-1]}(a) & \cdots & \theta_{d_b}^{[n-1]}(a) \end{pmatrix}.$$

Since $\operatorname{rank} E_n = n$ and $\operatorname{rank} G = d$, we have

$$\begin{aligned} \operatorname{rank}([\theta_h,\theta_l](b))_{d_b\times d_b} &\geq \operatorname{rank} G^* + \operatorname{rank}(E_n G) - n \\ &= \operatorname{rank} G^* + \operatorname{rank} G - n \\ &= 2d_b - n = m_b. \end{aligned}$$

Hence

$$\operatorname{rank}([\theta_h,\theta_l](b))_{d_b\times d_b} = m_b.$$

By (4.7) we have

$$([\theta_h,\theta_l](b))^*_{1\leq h,l\leq d_b} = -([\theta_h,\theta_l](b))_{1\leq h,l\leq d_b}, \tag{4.8}$$

that is $([\theta_h,\theta_l](b))_{1\leq h,l\leq d_b}$ is a skew-Hermitian matrix.

Therefore there exists a nonsingular complex matrix $P = (p_{ij})_{d_b\times d_b}$ such that

$$P^*([\theta_h,\theta_l](b))_{1\leq h,l\leq d_b} P = -\mathrm{i}^n \begin{pmatrix} & & & -1 & \\ & & 1 & & \\ & \cdot^{\cdot^{\cdot}} & & & 0_{m_b\times(n-d_b)} \\ & 1 & & & \\ -1 & & & & \\ & & 0_{(n-d_b)\times m_b} & & 0_{(n-d_b)\times(n-d_b)} \end{pmatrix}, \tag{4.9}$$

Where $\mathrm{i} = \sqrt{-1}$.

Let

$$\begin{pmatrix} v_1 \\ \vdots \\ v_{d_b} \end{pmatrix} = P^* \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_{d_b} \end{pmatrix}. \tag{4.10}$$

Then $v_i, i = 1,\ldots,d_b$, are linearly independent solutions of (1.1) satisfying

$$([v_h,v_l](b))_{1\leq h,l\leq d_b} = -\mathrm{i}^n \begin{pmatrix} & & & -1 & \\ & & 1 & & \\ & \cdot^{\cdot^{\cdot}} & & & 0_{m_b\times(n-d_b)} \\ & 1 & & & \\ -1 & & & & \\ & & 0_{(n-d_b)\times m_b} & & 0_{(n-d_b)\times(n-d_b)} \end{pmatrix}. \tag{4.11}$$

By (4.10) and (4.5), we have

$$\begin{aligned} ([v_h,v_l](b))_{1\leq h,l\leq m_b} &= (P_1)([\theta_i,\theta_j](b))_{1\leq i,j\leq m_b} P_1^* \\ &= (P_1F)([\phi_i,\phi_j](b))_{1\leq i,j\leq m_b}(P_1F)^*,\ P_1 = (p_{ij})^*_{1\leq i\leq d_b, 1\leq j\leq m_b}. \end{aligned}$$

Hence $P_1F = M = (m_{ij})_{m_b\times m_b}$ is nonsingular.

By (4.4) and (4.10) , we get

$$\begin{aligned} v_j &= \sum_{i=1}^{d_b} \bar{p}_{ij}\theta_i \\ &= \sum_{i=1}^{d_b} \bar{p}_{ij}\left(y_i + \sum_{s=1}^{n} d_{is}z_s + \sum_{i=1}^{m_b} k_{is}\phi_s\right) \\ &= \sum_{i=1}^{d_b} \bar{p}_{ij}y_i + \sum_{i=1}^{d_b}\sum_{s=1}^{n} \bar{p}_{ij}d_{is}z_s + \sum_{i=1}^{d_b}\sum_{s=1}^{m_b} \bar{p}_{ij}k_{is}\phi_s \\ &= \sum_{i=1}^{d_b} \bar{p}_{ij}y_i + \sum_{i=1}^{d_b}\sum_{s=1}^{n} \bar{p}_{ij}d_{is}z_s + \sum_{s=1}^{m_b} m_{js}\phi_s, \qquad j = 1,\dots,m_b \end{aligned}$$

Therefore we have unique solutions

$$\phi_j = \tilde{y}_j + \sum_{i=1}^{n} \tilde{b}_{ji}z_i + \sum_{s=1}^{m_b} \tilde{c}_{js}v_s, \qquad j = 1,\dots,m_b, \tag{4.12}$$

where $\tilde{y}_j \in D_{\min}, \tilde{b}_{ji}, \tilde{c}_{js} \in \mathbb{C}$.

Substituting ϕ_j defined in Theorem 4.4 by (4.12), we conclude that

$$D_{\max} = D_{\min} \dot{+} \operatorname{span}\{z_1, z_2, \dots, z_n\} \dot{+} \operatorname{span}\{v_1, v_2, \dots, v_{m_b}\}. \qquad \square$$

Next we give the proof of Theorem 3.1.

Proof. Part (3) follows from (4.11). The proof of part (4) is similar.

Next we prove parts (7)–(10).

Sufficiency. Let the matrices A and B satisfy the conditions (3.5) and (3.6) of Theorem 3.1. We prove that $D(S)$ defined by the condition (3.7) is the domain of a self-adjoint extension S of $S_{\min}$ by showing that conditions (i), (ii), (iii) of the GKN Theorem are satisfied.

Let

$$A = -(\bar{a}_{ij})_{d\times m_a}, \qquad B = (\bar{b}_{ij})_{d\times m_b}.$$

$$w_i = \sum_{j=1}^{m_a} a_{ij}\,u_j + \sum_{j=1}^{m_b} b_{ij}v_j, \qquad i = 1,\dots,d. \tag{4.13}$$

By direct computation it follows that (iii) holds, i.e.,

$$[y, w_i](b) - [y, w_i](a) = 0, \qquad i = 1,\dots,d.$$

Note that

$$([w_i, w_j](a))^T_{d\times d} = AU^TA^* = \mathrm{i}^n AE_{m_a}A^*.$$

Similarly

$$(([w_i, w_j](b))^T_{d\times d} = \mathrm{i}^n BE_{m_b}B^*.$$

Therefore

$$([w_i, w_j]_a^b)^T = \mathrm{i}^n BE_{m_b}B^* - \mathrm{i}^n AE_{m_a}A^* = 0.$$

The proof that (i) and (ii) hold is similar to the proof of Theorem 5.1 in [15] and hence omitted.

Necessity. Let $D(S)$ be the domain of a self-adjoint extension S of $S_{\min}$. By the GKN Theorem there exist $w_1, \dots, w_d \in D_{\max}$ satisfying the conditions (i), (ii), (iii). From (4.1) we get

$$w_i = \widehat{y}_{i0} + \sum_{j=1}^{m_a} a_{ij} u_j + \sum_{j=1}^{m_b} b_{ij} v_j, \tag{4.14}$$

where $\widehat{y}_{i0} \in D_{\min}$, $a_{ij}, b_{ij} \in \mathbb{C}$.

Let

$$A = -(\bar{a}_{ij})_{d\times m_a}, \ B = (\bar{b}_{ij})_{d\times m_b},$$

Then

$$\begin{pmatrix} [y, w_1](a) \\ \vdots \\ [y, w_d](a) \end{pmatrix} = \begin{pmatrix} [y, \sum_{j=1}^{m_a} a_{1j} u_j](a) \\ \vdots \\ [y, \sum_{j=1}^{m_a} a_{dj} u_j](a) \end{pmatrix} = -A \begin{pmatrix} [y, u_1](a) \\ \vdots \\ [y, u_{m_a}](a) \end{pmatrix},$$

$$\begin{pmatrix} [y, w_1](b) \\ \vdots \\ [y, w_d](b) \end{pmatrix} = \begin{pmatrix} [y, \sum_{j=1}^{m_b} b_{1j} v_j](b) \\ \vdots \\ [y, \sum_{j=1}^{m_b} b_{dj} v_j](b) \end{pmatrix} = B \begin{pmatrix} [y, v_1](b) \\ \vdots \\ [y, v_{m_b}](b) \end{pmatrix}.$$

Hence condition (iii) of the GKN Theorem is equivalent to (3.10).

The proof that A, B satisfy (3.5), (3.6) of Theorem 3.1 is similar to the proof of Theorem 5.1 in [15] and hence omitted. □

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