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# On oscillation of solutions of differential equations with distributed delay

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**Abstract.** We obtain sufficient conditions for oscillation of solutions to a linear differential equation with distributed delay. We construct examples showing that constants in the conditions are unimprovable.

**Keywords:** functional differential equation, distributed delay, oscillation.

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## 1 Introduction

The problem of definite-sign solutions and the opposite problem of oscillating solutions (having an unbounded sequence of zeros, from the right) for ordinary differential equations are well known and significant. These problems for functional differential equations are nontrivial even for first-order equations, whose solutions, as is known, can have zeros and oscillate.

In particular, the problem of conditions for the existence of oscillating solutions for the equation

$$\dot{x}(t) + a(t)x(h(t)) = 0, \qquad t \geqslant 0, \tag{1.1}$$

has been studied in detail. We cite the two most known and well-supplementing each other conditions for the oscillation of solutions to equation (1.1).

The first condition goes back to paper [17]. Later it was generalized in [12,14,21], and took the following complete form in [11].

**Condition 1.1.** *In (1.1), let $a(t) \geqslant 0$, $h(t) \leqslant t$, $\lim_{t\to\infty} h(t) = \infty$, and $\varliminf_{t\to\infty} \int_{h(t)}^{t} a(s)\,ds > 1/e$. Then every solution of (1.1) oscillates.*

The constant $1/e$ is unimprovable. If $a(t) \equiv a = \text{const}$, $h(t) = t - r$, where $r = \text{const}$, then the condition $ar > 1/e$ is necessary and sufficient for the oscillation of every solution of equation (1.1).

The first variants of the other condition were obtained in papers [13,22]. Its most general form, obtained in [7], is the following.

Denote $E(t) = \{s : h(s) \leqslant t \leqslant s\}$.

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**Condition 1.2.** *In* (1.1), *let* $a(t) \geqslant 0$, $h(t) \leqslant t$, $\lim_{t\to\infty} h(t) = \infty$ *and* $\overline{\lim}_{t\to\infty} \int_{E(t)} a(s)\,ds > 1$. *Then every solution of* (1.1) *oscillates.*

The constant 1 is sharp. There is an example in [22] showing that it is impossible to decrease the constant by an arbitrarily small value.

Conditions 1.1 and 1.2 were generalized for the case of several concentrated delays (see [5,7,8,10] and references therein). Similar conditions for equations with distributed delay are far less known. Yet, papers [3,4,18,20] should be noted. The aim of this paper is to obtain new conditions for the oscillation of solutions for equations with distributed delay.

## 2 Preliminaries

Consider a functional differential equation

$$(Lx)(t) \triangleq \dot{x}(t) + \int_{h(t)}^{g(t)} K(t,s)x(s)\,ds = 0, \qquad t \geqslant 0. \tag{2.1}$$

Here the functions $h$, $g$ are measurable, $h(t) \leqslant g(t) \leqslant t$, $\lim_{t\to\infty} h(t) = \infty$, the function $K$ is nonnegative, measurable with respect to the first argument, and locally summable with respect to the second argument.

Denote $\rho(t) = \int_{h(t)}^{g(t)} K(t,s)\,ds$ and assume that the function $\rho$ is locally summable and positive. In this case, as it has been shown in [1], for every given initial condition there exists a unique solution of equation (2.1) in the class of locally absolutely continuous functions.

**Definition 2.1.** We say that a continuous function defined on the real positive semiaxis is *oscillatory* if the function has an unbounded sequence of zeros, from the right.

**Definition 2.2.** We say that equation (2.1) is *oscillatory* if each of its solutions is oscillatory.

Since all solutions of equation (2.1) are continuous, it follows from definition 2.1 that a solution which is not oscillatory has definite sign everywhere to the right from some point. Such solutions are said to be definite-sign. Using the linearity of equation (2.1) we can say, without loss of generality, that a solution is definite-sign if it is positive starting from some point.

In order to obtain conditions for oscillation, we use a proposition known as the lemma on differential inequality. The lemma occurs in papers [2, p. 57, Lemma 2.4.3], [3,6,9] in different equivalent reformulations. Here we formulate it in the form suitable for us in connection with equation (2.1).

**Lemma 2.3.** *If there exists an absolutely continuous function $v$ and a number $T \geqslant 0$ such that $v(t) > 0$ and $(Lv)(t) \leqslant 0$ for all $t \geqslant T$, then equation* (2.1) *has a definite-sign solution.*

## 3 Autonomous equations

We begin obtaining oscillation conditions with autonomous equations. Suppose in equation (2.1) $h(t) = t - r$, $r = \text{const} > 0$, $g(t) = t - p$, $p = \text{const} > 0$, $r > p \geqslant 0$, $K(t,s) = k(t-s)$, where $k$ is a locally summable function. We get the equation

$$\dot{x}(t) + \int_{t-r}^{t-p} k(t-s)x(s)\,ds = 0, \qquad t \geqslant 0. \tag{3.1}$$

The function $F(\lambda) = -\lambda + \int_p^r k(t)e^{\lambda t}\,dt$, $F\colon \mathbb{C} \to \mathbb{C}$, is said to be the characteristic function of equation (3.1). It was shown in [19, Lemma 3.1] that $F$ is analytic everywhere in $\mathbb{C}$ and has a countable set of roots to the left from every vertical line $\operatorname{Re}\lambda = \text{const}$, the set of roots of $F$ in every vertical band being finite. Note that roots of $F$ depend on the parameters $p$ and $r$ continuously.

**Theorem 3.1.** *Equation (3.1) is oscillatory if and only if the function $F$ has no real roots.*

*Proof.* If the function $F$ has a real root $\lambda = \lambda_0$, then the function $x(t) = e^{\lambda_0 t}$ is a positive solution of equation (3.1). Hence, equation (3.1) is not oscillatory. Conversely, suppose that the function $F$ has no real roots. Then it has a finite number $n$ of roots whose real part is maximal. Denote them by $\lambda_j$, $j = 1, \dots, n$. Denote $\alpha_{\max} = \operatorname{Re}\lambda_j$, $\beta_j = \operatorname{Im}\lambda_j$.

Consider an arbitrary solution of (3.1). It is known (see [23]) that it has the form

$$x(t) = e^{\alpha_{\max} t}\sum_{j=1}^{n}(A_j(t)\cos\beta_j t + B_j(t)\sin\beta_j t) + z(t), \qquad t \in \mathbb{R}_+,$$

where $A_j(t)$ and $B_j(t)$ are polynomials, and $\lim_{t\to+\infty}|z(t)|e^{-\alpha_{\max} t} = 0$. Denote by $m_0$ the greatest degree of $A_j(t)$, $B_j(t)$, $j = 1, \dots, n$. Then we have

$$\frac{x(t)}{t^{m_0}e^{\alpha_{\max} t}} = w(t) + \varepsilon(t),$$

where $\lim_{t\to+\infty}\varepsilon(t) = 0$, $w(t) = \sum_{j=1}^{n} R_j\cos(\beta_j t - \varphi_j)$, $R_j > 0$, $\beta_j, \varphi_j \in \mathbb{R}$, $0 < \beta_1 < \cdots < \beta_n$.

Consider the function

$$y(t) = \sum_{j=1}^{n}\frac{R_j}{\beta_j^{4m}}\cos(\beta_j t - \varphi_j).$$

We have $\beta_1 < \beta_j$ for all $j \geqslant 2$. Therefore there exists a sufficiently large $m$ such that the inequality

$$\frac{R_1}{\beta_1^{4m}} > \sum_{j=2}^{n}\frac{R_j}{\beta_j^{4m}}.$$

holds. Take $\theta_l = \frac{\varphi_1 + 2\pi l}{\beta_1}$, $l \in \mathbb{N}$. Calculate

$$y(\theta_l) = \frac{R_1}{\beta_1^{4m}} + \sum_{j=2}^{n}\frac{R_j}{\beta_j^{4m}}\cos(\beta_j\theta_l - \varphi_j) \geqslant \frac{R_1}{\beta_1^{4m}} - \sum_{j=2}^{n}\frac{R_j}{\beta_j^{4m}} > 0,$$

$$y\left(\theta_l + \frac{\pi}{\beta_1}\right) \leqslant -\frac{R_1}{\beta_1^{4m}} + \sum_{j=2}^{n}\frac{R_j}{\beta_j^{4m}} < 0.$$

So, the function $y$ has an infinite set of roots and extrema in $\mathbb{R}_+$, with maxima and minima of $y$ bounded away from zero uniformly. By the mean value theorem, all the derivatives of $y$ possess these properties (and $w$ does, since $y^{(4m)}(t) = w(t)$). Therefore $x$ is oscillatory. □

On the basis of Theorem 3.1, we will find effective conditions for the oscillation of solutions for some classes of autonomous equations. Let $k(t) = \mu t^\alpha$, where $\mu > 0$, $\alpha > -1$, in equation (3.1). We have

$$\dot{x}(t) + \mu\int_{t-r}^{t-p}(t-s)^\alpha x(s)\,ds = 0, \qquad t \geqslant 0. \tag{3.2}$$

Denote

$$I(\zeta) = (\alpha+2)\int_q^1 s^\alpha e^{\zeta(s-1)}\,ds + e^{\zeta(q-1)}q^{\alpha+1}, \qquad \zeta \in \mathbb{R}.$$

Consider the equation $I(\zeta) = 1$ for a fixed $q \in [0,1)$. The function $I = I(\zeta)$ is continuous and, as $I'(\zeta) < 0$, decreases everywhere on the real axis. Since $I(0) = 1 + \frac{1-q^{\alpha+1}}{\alpha+1} > 1$ and $\lim_{\zeta\to\infty} I(\zeta) = 0$, it follows that equation $I(\zeta) = 1$ has a unique root, which is positive. Denote it by $\zeta_\alpha$. So, for every fixed $q \in [0,1)$ we will consider $\zeta_\alpha$ as a positive function of $\alpha$, defined on the set $(-1,\infty)$ .

**Theorem 3.2.** *Equation (3.2) is oscillatory if and only if*

$$\mu r^{\alpha+2} > \frac{(\alpha+2)\zeta_\alpha e^{-\zeta_\alpha}}{1-q^{\alpha+1}e^{(q-1)\zeta_\alpha}}, \tag{3.3}$$

*where $\zeta_\alpha$ is the root of the equation $I(\zeta) = 1$, and $q = p/r$.*

*Proof.* Consider the family of functions $f_u(\zeta) = -\zeta + u\int_q^1 s^\alpha e^{\zeta s}\,ds$ with the parameter $u > 0$, and the family of their derivatives $f_u'(\zeta) = -1 + u\int_q^1 s^{\alpha+1}e^{\zeta s}\,ds$. Clearly, $f_u''(\zeta) > 0$. Hence $f_u'$ increases from $-1$ to $+\infty$ and has a unique real zero, which is the minimum point of the function $f_u$.

Let us find $\zeta^*$ and $u^*$ such that $f_{u^*}(\zeta^*) = 0$ and $f_{u^*}'(\zeta^*) = 0$. It is obtained in a standard way that $\zeta^*$ is a root of the equation $\int_q^1 s^\alpha e^{\zeta s}\,ds = \zeta\int_q^1 s^{\alpha+1}e^{\zeta s}\,ds$, which is equivalent to the equation $I(\zeta) = 1$. By virtue of the properties of the function $I(\zeta)$, the equation has a unique solution $\zeta^* = \zeta_\alpha$, which corresponds to the unique $u^* = \frac{(\alpha+2)\zeta^* e^{-\zeta^*}}{1-q^{\alpha+1}e^{(q-1)\zeta^*}}$. Thus, $f_{u^*}(\zeta^*) = 0$, and for all $\zeta \neq \zeta^*$ we have $f_{u^*}(\zeta) > 0$.

The characteristic function of equation (3.2) has the form $F(\lambda) = -\lambda + \mu\int_p^r s^\alpha e^{\lambda s}\,ds$. Setting $\lambda = \zeta/r$, we get $rF(\lambda) = rF(\zeta/r) = -\zeta + \mu r^{\alpha+2}\int_q^1 s^\alpha e^{\zeta s}\,ds = f_{u^*}(\zeta) + (\mu r^{\alpha+2} - u^*)\int_q^1 s^\alpha e^{\zeta s}\,ds$. Suppose the inequality (3.3) holds. Then $\mu r^{\alpha+2} > u^*$. Therefore $F(\lambda) > 0$ for all $\lambda \in \mathbb{R}$, i.e., the characteristic function has no real roots. Conversely, if $\mu r^{\alpha+2} \leqslant u^*$ then $rF(\zeta^*/r) = f_{u^*}(\zeta^*) + (\mu r^{\alpha+2} - u^*)\int_q^1 s^\alpha e^{\zeta^* s}\,ds = (\mu r^{\alpha+2} - u^*)\int_q^1 s^\alpha e^{\zeta^* s}\,ds \leqslant 0$. However, $F(0) > 0$. Thus, the characteristic function of equation (3.2) has a real root. □

Let $v \in (-1,\infty)$, $w \in [0,1)$. As is noted above, in this case the equation

$$(v+2)\int_w^1 s^v e^{\zeta(s-1)}\,ds + e^{\zeta(w-1)}w^{v+1} = 1 \tag{3.4}$$

has a unique solution $\zeta > 0$. So, one can suppose that (3.4) defines the function $\zeta = \zeta(v,w)$. Denote

$$\psi(v,w) = \frac{(v+2)\zeta(v,w)e^{-\zeta(v,w)}}{1-w^{v+1}e^{(w-1)\zeta(v,w)}}.$$

The function $u = \psi(u,v)$ can be interpreted as a surface, in the space $Ouvw$, which is the boundary of the region of oscillation. Its graph created by a computer is presented on Fig. 3.1. Theorem 3.2 now obtains a geometric sense: equation (3.2) is oscillatory if and only if the point $(\mu r^{\alpha+2}, \alpha, q)$ is above the surface $u = \psi(v,w)$.

Putting $p = 0$ in (3.2), we get the equation

$$\dot{x}(t) + \mu\int_{t-r}^t (t-s)^\alpha x(s)\,ds = 0, \qquad t \geqslant 0. \tag{3.5}$$

Evidently, $q = 0$ for (3.5). Therefore the criterion of oscillation is simplified.

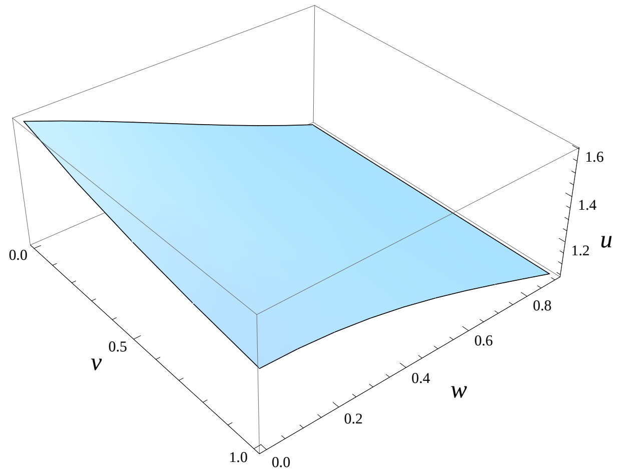


Figure 3.1: Boundary of the region of oscillation for equation (3.2).

**Corollary 3.3.** *Equation (3.5) is oscillatory if and only if $\mu r^{\alpha+2} > (\alpha+2)\zeta_\alpha e^{-\zeta_\alpha}$, where $\zeta_\alpha$ is the root of the equation*

$$(\alpha+2)\int_0^1 s^\alpha e^{\zeta(s-1)}\,ds = 1. \tag{3.6}$$

It is easy to calculate the roots of equation (3.6) approximately. The values of some roots $\zeta_\alpha$ and the corresponding oscillation conditions are represented in Table 3.1 for $\alpha$ chosen arbitrarily.

| $\alpha$ | $\zeta_\alpha$ | **Criterion of oscillation** |
|---|---|---|
| 0 | 1.59362 | $\mu r^2 > 0.64762$ |
| 0.5 | 1.44713 | $\mu r^{2.5} > 0.85108$ |
| 1 | 1.36078 | $\mu r^3 > 1.04696$ |
| 1.6 | 1.29391 | $\mu r^{3.6} > 1.27723$ |
| 2 | 1.26191 | $\mu r^4 > 1.42906$ |
| $e$ | 1.21935 | $\mu r^{2+e} > 1.69964$ |
| 3 | 1.20627 | $\mu r^5 > 1.80526$ |
| 10 | 1.08384 | $\mu r^{12} > 4.39988$ |
| 100 | 1.00980 | $\mu r^{102} > 37.52192$ |

Table 3.1: Criteria of the oscillation of solutions for equation (3.5).

Setting $\alpha = 0$ in (3.2), we get another equation

$$\dot{x}(t) + \mu\int_{t-r}^{t-p} x(s)\,ds = 0, \qquad t \geqslant 0. \tag{3.7}$$

The region of oscillation for equation (3.7) is described as a region in the parameter space with its boundary given analytically.

Consider the auxiliary function $f(\zeta) = -\zeta + u\int_v^{v+1} e^{\zeta s}\,ds$, where $u > 0$, $v \geqslant 0$, $\zeta \in \mathbb{R}$, and the derivative $f'(\zeta) = -1 + u\int_v^{v+1} se^{\zeta s}\,ds$. By the mentioned above restrictions on parameters,

$f''(\zeta) > 0$ for all $\zeta \in \mathbb{R}$. Hence $f'$ increases from $-1$ to $+\infty$ on the real axis and has a unique zero, which is the minimum point of $f$.

Note that the equalities $f(\zeta) = 0$ and $f'(\zeta) = 0$ are both true if and only if the point $(v, u)$ lies on the curve $u = \varphi(v)$ defined by the parametric equations

$$v = \frac{2}{\zeta} - \frac{1}{1 - e^{-\zeta}}, \qquad u = \frac{\zeta^2}{e^{\zeta} - 1} e^{-2 + \frac{\zeta}{1 - e^{-\zeta}}}.$$

Let us find the sharp range of the parameter $\zeta$. Since $f(\zeta) > 0$ for $\zeta \leqslant 0$, it follows that $\zeta > 0$. Denote by $\zeta_0$ the positive root of the equation $1 - \frac{\zeta}{2} = e^{-\zeta}$. It is clear that $v(\zeta_0) = 0$, $u(\zeta_0) = 2\zeta_0 e^{-\zeta_0}$, $\lim_{\zeta \to +0} v(\zeta) = +\infty$, and $\lim_{\zeta \to +0} u(\zeta) = +0$. Since $\frac{dv}{d\zeta} < 0$ and $\frac{du}{d\zeta} > 0$, it is the variation of $\zeta$ in the interval $(0, \zeta_0]$ that correspond to the inequalities $u > 0$, $v \geqslant 0$.

The curve $\varphi$ is shown on Fig. 3.2. From the above examination of the function $u = \varphi(v)$ it follows that $\frac{du}{dv} < 0$. Thus $\varphi$ is a continuous and decreasing function, the axis $Ov$ is an asymptote of its graph, which crosses the axis $Ou$ at the point of the ordinate $2\zeta_0 e^{-\zeta_0}$. Table 3.1 shows that $\zeta_0 \approx 1.59362$, $2\zeta_0 e^{-\zeta_0} \approx 0.64761$.

Denote $D = \{(v, u) : v \geqslant 0,\ u > 0,\ u > \varphi(v)\}$. In Fig. 3.2, the set $D$ is colored.

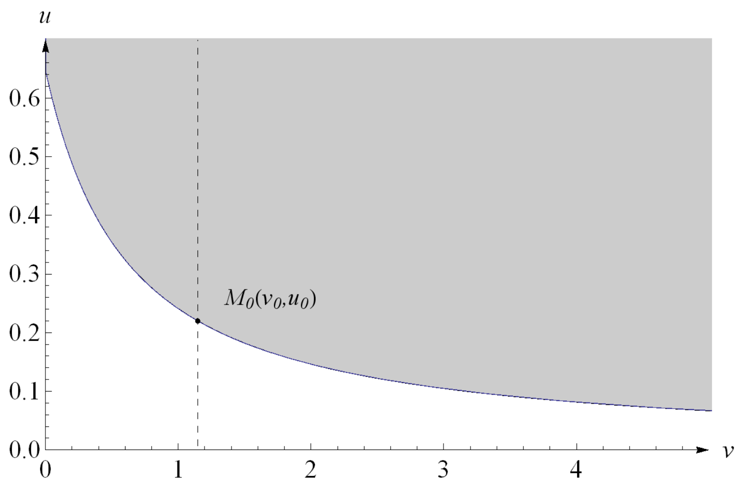


Figure 3.2: The function $u = \varphi(v)$.

**Lemma 3.4.** *The function $f = f(\zeta)$ has no real roots if and only if the point $(v, u)$ is in the set $D$.*

*Proof.* Let the point $M_0(v_0, u_0)$ lie on the curve $u = \varphi(v)$. By the above, it means that for the function $f_0(\zeta) = -\zeta + u_0 \int_{v_0}^{v_0+1} e^{\zeta s}\, ds$ there exists $\zeta^*$ such that $f_0(\zeta^*) = 0$ and $f_0(\zeta) > 0$ for all $\zeta \neq \zeta^*$.

Draw the line $v = v_0$ through the point $M_0$ (see Fig. 3.2). For every point $M(v_0, u)$ on the line and for all $\zeta \in \mathbb{R}$ we have

$$f(\zeta) = f_0(\zeta) + (u - u_0) \int_{v_0}^{v_0+1} e^{\zeta s}\, ds. \tag{3.8}$$

Suppose $M(v_0, u) \in D$. Then $u > u_0$ and $f(\zeta) > 0$ for all $\zeta \in \mathbb{R}$, i.e., the function $f$ has no real roots. Suppose $M(v_0, u) \notin D$. Then $u \leqslant u_0$. Using (3.8), we get $f(\zeta^*) \leqslant 0$. Since $\lim_{\zeta \to +0} f(\zeta) = u > 0$, we see that the function $f$ has a real root. Since the point $M_0$ on the curve $\varphi$ is taken arbitrarily, the lemma is proved. $\square$

**Theorem 3.5.** *The following conditions are equivalent.*

1. *Equation (3.7) is oscillatory.*
2. *The inequality $\mu r^2 > \frac{2\zeta_0 e^{-\zeta_0}}{1-qe^{(q-1)\zeta_0}}$ holds, where $\zeta_0$ is the positive root of the equation $(\zeta q - 2)e^{\zeta(q-1)} = \zeta - 2$.*
3. *The point $\left(\mu(r-p)^2, \frac{p}{r-p}\right)$ belongs to the set $D$.*

*Proof.* The equivalence of the conditions 1 and 2 follows from Theorem 3.2. Let us prove that the conditions 1 and 3 are also equivalent. The characteristic function of equation (3.7) has the form $F(\lambda) = -\lambda + \mu \int_p^r e^{\lambda s}\, ds$. It is easily shown that

$$(r-p)F\left(\frac{\zeta}{r-p}\right) = -\zeta + \mu(r-p)^2 \int_{\frac{p}{r-p}}^{\frac{r}{r-p}} e^{\zeta s}\, ds = f(\zeta)$$

for $u = \mu(r-p)^2$ and $v = \frac{p}{r-p}$. It remains to refer to Lemma 3.4. □

**Corollary 3.6.** *In equation (3.7), let $p = 0$. Then (3.7) is oscillatory if and only if $\mu r^2 > 2\zeta_0 e^{-\zeta_0}$, where $\zeta_0$ is the positive root of the equation $1 - \frac{\zeta}{2} = e^{-\zeta}$.*

Thus the fact that $\mu r^2$ is on the axis $Ou$ for $u > 2\zeta_0 e^{-\zeta_0}$ corresponds to the oscillation of equation (3.7) under the conditions of Corollary 3.6.

**Remark 3.7.** The set of oscillation for equation (3.7) is the complement to the set of positiveness for the fundamental solution. The common boundary of the sets is the curve $u = \varphi(v)$ obtained in paper [16], which is devoted to the study of the positiveness of the fundamental solution for equation (3.7).

## 4 Nonautonomous equations

Using Lemma 2.3 and the results of Section 3, we can obtain oscillation conditions for some classes of nonautonomous equations with distributed delay.

**Theorem 4.1.** *For equation (2.1), suppose that $K(t,s) \geqslant k(t-s) \geqslant 0$, where $k$ is a locally summable function, $\underline{\lim}_{t\to\infty}(t - h(t)) = r$, $\overline{\lim}_{t\to\infty}(t - g(t)) = p$, $r > p \geqslant 0$, and the function $F(\lambda) = -\lambda + \int_p^r k(t)e^{\lambda t}\, dt$ has no real roots. Then equation (2.1) is oscillatory.*

*Proof.* Assume that there exists a solution $v = v(t)$ of equation (2.1) that is positive starting from some point $T$. Then

$$\dot{v}(t) + \int_{t-r}^{t-p} k(t-s)v(s)\, ds \leqslant \dot{v}(t) + \int_{h(t)}^{g(t)} K(t,s)v(s)\, ds = 0, \qquad t \geqslant T + r.$$

By Lemma 2.3, equation (3.1) has a definite-sign solution. But then it follows from Theorem 3.1 that the function $F$, which is the characteristic function of (3.1), has a real root. □

**Corollary 4.2.** *For equation (2.1), suppose that $K(t,s) \geqslant \mu(t-s)^\alpha$, $\underline{\lim}_{t\to\infty}(t - h(t)) = r$, $\overline{\lim}_{t\to\infty}(t - g(t)) = p$, $r > p \geqslant 0$, and inequality (3.3) holds for the parameters $\mu$, $\alpha$, $p$ and $r$. Then equation (2.1) is oscillatory.*

If the function $K(t,s)$ is bounded below by a nonzero constant, then oscillation conditions for equation (2.1) can be conveniently formulated in terms of the belonging of a given point to the set $D$.

**Corollary 4.3.** *For (2.1), suppose that $K(t,s) \geqslant \mu$, $\underline{\lim}_{t\to\infty}(t-h(t)) = r$, $\overline{\lim}_{t\to\infty}(t-g(t)) = p$, $r > p \geqslant 0$, and the point $\left(\mu(r-p)^2, \frac{p}{r-p}\right)$ belongs to $D$. Then equation (2.1) is oscillatory.*

Consider equation (2.1) in the special case that $g(t) = t$. We have

$$\dot{x}(t) + \int_{h(t)}^{t} K(t,s)x(s)\,ds = 0, \qquad t \geqslant 0. \tag{4.1}$$

In this case Corollary 4.3 is simplified.

**Corollary 4.4.** *For (4.1), suppose that $K(t,s) \geqslant \mu$, $\underline{\lim}_{t\to\infty}(t-h(t)) = r$, and $\mu r^2 > 2\zeta_0 e^{-\zeta_0}$, where $\zeta_0$ is the positive root of the equation $1 - \frac{\zeta}{2} = e^{-\zeta}$. Then equation (4.1) is oscillatory.*

The nearer equation (2.1) is to the autonomous equation (3.1), the sharper Corollaries 4.2–4.4 are. For equation (3.1) sufficient oscillation conditions become necessary and sufficient.

The three propositions stated below (Theorem 4.5, Theorem 4.8, Condition 4.11) can be regarded as different variants of Condition 1.1. Each of them has its area of application.

**Theorem 4.5.** *For equation (2.1), suppose that $K(t,s) \geqslant a(t)a(s) > 0$, where the function $a$ is locally summable, $\overline{\lim}_{t\to\infty} \int_{g(t)}^{t} a(s)\,ds = p$, $\underline{\lim}_{t\to\infty} \int_{h(t)}^{t} a(s)\,ds = r$, and the point $\left((r-p)^2, \frac{p}{r-p}\right)$ belongs to the set $D$. Then equation (2.1) is oscillatory.*

*Proof.* First let us prove that $\int_0^\infty a(s)\,ds = \infty$. We have $\lim_{t\to\infty} h(t) = \infty$ and $g(t) \geqslant h(t)$, hence $\lim_{t\to\infty} g(t) = \infty$. Therefore, if the function $a$ is summable on the real positive semiaxis, then $p = \overline{\lim}_{t\to\infty} \int_{g(t)}^{t} a(s)\,ds = 0$. But this is impossible, since the axis $Ov$ is not included in $D$.

Denote $\varphi(t) = \int_0^t a(s)\,ds$. The function $\varphi$ is a continuous and increasing $\mathbb{R}_+$-onto-$\mathbb{R}_+$ map. Hence there exists the inverse function $\varphi^{-1}$ defined on $\mathbb{R}_+$. By the change of variables (analogous to that applied in [15]) $\tau = \varphi(t)$, $\zeta = \varphi(s)$, $x\left(\varphi^{-1}(\tau)\right) = y(\tau)$, equation (2.1) is reduced to the form

$$y'(\tau) + \int_{H(\tau)}^{G(\tau)} \frac{K\left(\varphi^{-1}(\tau), \varphi^{-1}(\zeta)\right)}{a\left(\varphi^{-1}(\tau)\right) a\left(\varphi^{-1}(\zeta)\right)} y(\zeta)\,d\zeta = 0, \qquad \tau \geqslant 0, \tag{4.2}$$

where

$$G(\tau) = \varphi\left(g\left(\varphi^{-1}(\tau)\right)\right) = \tau - \int_{g(\varphi^{-1}(\tau))}^{\varphi^{-1}(\tau)} a(s)\,ds,$$
$$H(\tau) = \varphi\left(h\left(\varphi^{-1}(\tau)\right)\right) = \tau - \int_{h(\varphi^{-1}(\tau))}^{\varphi^{-1}(\tau)} a(s)\,ds.$$

Since

$$\underline{\lim}_{\tau\to\infty} \int_{h(\varphi^{-1}(\tau))}^{\varphi^{-1}(\tau)} a(s)\,ds = \underline{\lim}_{t\to\infty} \int_{h(t)}^{t} a(s)\,ds = r,$$
$$\overline{\lim}_{\tau\to\infty} \int_{g(\varphi^{-1}(\tau))}^{\varphi^{-1}(\tau)} a(s)\,ds = \overline{\lim}_{t\to\infty} \int_{g(t)}^{t} a(s)\,ds = p, \quad \text{and}$$
$$\frac{K\left(\varphi^{-1}(\tau), \varphi^{-1}(\zeta)\right)}{a\left(\varphi^{-1}(\tau)\right) a\left(\varphi^{-1}(\zeta)\right)} \geqslant 1,$$

Corollary 4.3 can be applied to equation (4.2). This implies that every solution of equation (4.2) oscillates. We have $x(t) = y(\varphi(t))$, so every solution of equation (2.1) also oscillates. $\square$

**Remark 4.6.** The oscillation region $D$ defined by Theorem 4.5 is sharp, since Theorem 4.5 coincides with Theorem 3.5 in the case of constant coefficients and delays.

**Lemma 4.7.** *If $\int_{h(t)}^{g(t)} K(t,s)\,ds = 1$ and $\underline{\lim}_{t\to\infty}(t - g(t)) = m > 1/e$, then equation (2.1) is oscillatory.*

*Proof.* Assume that there exists a definite-sign solution $v = v(t)$ of equation (2.1). Thus, by virtue of the equation, there exists $T > 0$ such that for all $t \geqslant T$ the inequalities $v(t) > 0$ and $\dot{v}(t) \leqslant 0$ hold. Hence

$$\dot{v}(t) + v(t-m) = \dot{v}(t) + \int_{h(t)}^{g(t)} K(t,s)v(t-m)\,ds \leqslant \dot{v}(t) + \int_{h(t)}^{g(t)} K(t,s)v(s)\,ds = 0, \quad t \geqslant T + m.$$

By Lemma 2.3, the equation $\dot{x}(t) + x(t-m) = 0$ has a definite-sign solution. Therefore $m \leqslant 1/e$. This contradiction completes the proof. □

**Theorem 4.8.** *If in equation* (2.1)

$$\underline{\lim}_{t\to\infty} \int_{g(t)}^{t} \int_{h(s)}^{g(s)} K(s,\zeta)\,d\zeta ds > \frac{1}{e}, \tag{4.3}$$

*then equation* (2.1) *is oscillatory.*

*Proof.* Let us prove that $\int_0^\infty \rho(s)\,ds = \infty$ under the conditions of Theorem 4.8. Since $\lim_{t\to\infty} h(t) = \infty$ and $g(t) \geqslant h(t)$, we obtain $\lim_{t\to\infty} g(t) = \infty$. Assume that $\int_0^\infty \rho(s)\,ds < \infty$. Then $\int_{g(t)}^t \rho(s)\,ds \leqslant \int_{g(t)}^\infty \rho(s)\,ds \to 0$ as $t \to \infty$. But from (4.3) we get $\int_{g(t)}^t \rho(s)\,ds > \frac{1}{e}$ for sufficiently large $t$. Hence the assumption is not true.

Denote $\varphi(t) = \int_0^t \rho(s)\,ds$. By the above, the function $\varphi$ is a continuous and increasing $\mathbb{R}_+$-onto-$\mathbb{R}_+$ map. Hence there exists the inverse function $\varphi^{-1}$ defined on $\mathbb{R}_+$. By the change of variables $\tau = \varphi(t)$, $\zeta = \varphi(s)$, $x\left(\varphi^{-1}(\tau)\right) = y(\tau)$, equation (2.1) is reduced to the form

$$y'(\tau) + \int_{H(\tau)}^{G(\tau)} K_0(\tau,\zeta)y(\zeta)\,d\zeta = 0, \qquad \tau \geqslant 0, \tag{4.4}$$

where $G(\tau) = \varphi\left(g\left(\varphi^{-1}(\tau)\right)\right) = \tau - \int_{g(\varphi^{-1}(\tau))}^{\varphi^{-1}(\tau)} \rho(s)\,ds$, $H(\tau) = \varphi\left(h\left(\varphi^{-1}(\tau)\right)\right)$, $K_0(\tau,\zeta) = \frac{K\left(\varphi^{-1}(\tau),\varphi^{-1}(\zeta)\right)}{\rho(\varphi^{-1}(\tau))\rho(\varphi^{-1}(\zeta))}$. Since

$$\begin{aligned}\int_{H(\tau)}^{G(\tau)} K_0(\tau,\zeta)\,d\zeta &= \frac{1}{\rho\left(\varphi^{-1}(\tau)\right)} \int_{H(\tau)}^{G(\tau)} \frac{K\left(\varphi^{-1}(\tau),\varphi^{-1}(\zeta)\right)}{\rho\left(\varphi^{-1}(\zeta)\right)}\,d\zeta = \\ &= \frac{1}{\rho\left(\varphi^{-1}(\tau)\right)} \int_{h(\varphi^{-1}(\tau))}^{g\left(\varphi^{-1}(\tau)\right)} K\left(\varphi^{-1}(\tau),s\right)\,ds = \frac{\rho\left(\varphi^{-1}(\tau)\right)}{\rho\left(\varphi^{-1}(\tau)\right)} = 1,\end{aligned}$$

and

$$\underline{\lim}_{\tau\to\infty}(\tau - G(\tau)) = \underline{\lim}_{\tau\to\infty} \int_{g(\varphi^{-1}(\tau))}^{\varphi^{-1}(\tau)} \rho(s)\,ds = \underline{\lim}_{t\to\infty} \int_{g(t)}^{t} \int_{h(s)}^{g(s)} K(s,\zeta)\,d\zeta ds > \frac{1}{e},$$

Lemma 4.7 can be applied to equation (4.4). So, (4.4) is oscillatory. Since $x(t) = y(\varphi(t))$, and $\varphi$ corresponds $\mathbb{R}_+$ to $\mathbb{R}_+$ bijectively, equation (2.1) is also oscillatory. □

Let us show that the constant $1/e$ is sharp in the inequality (4.3) .

**Example 4.9.** Consider the equation

$$\dot{x}(t) + \frac{1}{e(e^{\varepsilon(t)} - 1)} \int_{t-1-\varepsilon(t)}^{t-1} x(s)\, ds = 0, \qquad t \geqslant 0, \tag{4.5}$$

where $\varepsilon$ is a positive bounded function, and $\int_0^\infty \varepsilon(t)\, dt < \infty$.

As

$$\frac{\varepsilon(s)}{e^{\varepsilon(s)} - 1} = 1 + \frac{\varepsilon(s) + 1 - e^{\varepsilon(s)}}{e^{\varepsilon(s)} - 1},$$

we get

$$\lim_{t\to\infty} \int_{t-1}^{t} \frac{\varepsilon(s)}{e^{\varepsilon(s)} - 1} ds = 1 + \lim_{t\to\infty} \int_{t-1}^{t} \frac{\varepsilon(s) + 1 - e^{\varepsilon(s)}}{e^{\varepsilon(s)} - 1} ds.$$

However,

$$\left| \frac{e^{\varepsilon(s)} - 1 - \varepsilon(s)}{e^{\varepsilon(s)} - 1} \right| < \frac{\varepsilon(s)}{2}.$$

Therefore, taking account of properties of the function $\varepsilon$, we obtain that

$$\lim_{t\to\infty} \left| \int_{t-1}^{t} \frac{\varepsilon(s) + 1 - e^{\varepsilon(s)}}{e^{\varepsilon(s)} - 1} ds \right| \leqslant \lim_{t\to\infty} \int_{t-1}^{t} \frac{\varepsilon(s)}{2} ds = 0.$$

Hence for equation (4.5)

$$\varliminf_{t\to\infty} \int_{g(t)}^{t} \int_{h(s)}^{g(s)} K(s,\zeta)\, d\zeta ds = \frac{1}{e} \lim_{t\to\infty} \int_{t-1}^{t} \frac{\varepsilon(s)}{e^{\varepsilon(s)} - 1} ds = \frac{1}{e}.$$

The inequality (4.3) is violated, since the strict inequality is replaced by the nonstrict one. Now we apply Lemma 2.3 to equation (4.5). Let $v(t) = e^{-t} > 0$. Then

$$\dot{v}(t) + \frac{1}{e\left(e^{\varepsilon(t)} - 1\right)} \int_{t-1-\varepsilon(t)}^{t-1} v(s)\, ds = -e^{-t} - \frac{e^{-(t-1)} - e^{-(t-1-\varepsilon(t))}}{e\left(e^{\varepsilon(t)} - 1\right)} = e^{-t} \left( -1 + \frac{e^{\varepsilon(t)} - 1}{e^{\varepsilon(t)} - 1} \right) = 0.$$

Consequently, equation (4.5) has a positive solution.

**Remark 4.10.** Theorem 4.8 generalizes the following result by A. D. Myshkis [18, Theorem 49]: *if* $\varliminf_{\tau\to\infty} \rho(t) \varliminf_{\tau\to\infty} (t - g(t)) > 1/e$, *then equation* (2.1) *is oscillatory.* Inequality (4.3) gives the refined result, with the uniform estimation replaced by the integral one.

Theorem 4.8 is inapplicable if $g(t) = t$, which is the case for equation (4.1). In this case the other sufficient condition of oscillation is applicable, which was obtained in [3].

**Condition 4.11.** *If in equation* (2.1)

$$\varliminf_{t\to\infty} \int_{h(t)}^{t} K(t,s)(t-s)\, ds > \frac{1}{e}, \tag{4.6}$$

*then equation* (2.1) *is oscillatory.*

We will show that the constant $1/e$ in Condition 4.11 is also sharp.

**Example 4.12.** Consider equation (3.5) in the case $r = 1$, that is

$$\dot{x}(t) + \mu \int_{t-1}^{t} (t-s)^{\alpha} x(s)\, ds = 0, \qquad t \geqslant 0. \tag{4.7}$$

It follows from Corollary 3.3 that equation (4.7) is oscillatory if and only if $\mu > (\alpha+2)\zeta_\alpha e^{-\zeta_\alpha}$, where $\zeta_\alpha$ is the root of equation (3.6). From the equality $(\alpha+2)\int_0^1 s^{\alpha} e^{\zeta_\alpha(s-1)}\, ds = 1$, integrating by parts, we obtain $(\alpha+2)\int_0^1 s^{\alpha+1} e^{\zeta_\alpha(s-1)}\, ds = \frac{1}{\zeta_\alpha}$. Subtracting the second equality from the first one, we get

$$1 - \frac{1}{\zeta_\alpha} = (\alpha+2)\int_0^1 s^{\alpha}(1-s) e^{\zeta_\alpha(s-1)}\, ds. \tag{4.8}$$

Hence $\zeta_\alpha > 1$ for all $\alpha > -1$. Combining this with (4.8), we obtain

$$0 < 1 - \frac{1}{\zeta_\alpha} \leqslant (\alpha+2)\int_0^1 (s^{\alpha} - s^{\alpha+1}) ds = \frac{1}{\alpha+1}.$$

Therefore, $\lim_{\alpha\to\infty} \zeta_\alpha = 1$.

Applying Condition 4.11 to equation (4.7), we get the sufficient condition of oscillation $\frac{\mu}{\alpha+2} > \frac{1}{e}$; applying Corollary 3.3 we get the criterion $\frac{\mu}{\alpha+2} > \frac{\zeta_\alpha}{e^{\zeta_\alpha}}$. By the above, $\zeta_\alpha > 1$, hence $\zeta_\alpha e^{-\zeta_\alpha} < 1/e$. However, $\zeta_\alpha e^{-\zeta_\alpha} \to 1/e$. Thus the constant $1/e$ cannot be decreased.

A more refined construction is needed to prove that the strict inequality (4.6) cannot be replaced by the nonstrict one.

**Example 4.13.** Consider the equation

$$\dot{x}(t) + \int_{t-1}^{t} K(t,s) x(s)\, ds = 0, \qquad t \geqslant 0, \tag{4.9}$$

where $K(t,s) = (n+2)e^{-\zeta_n}(t-s)^n$ for $t \in [n, n+1)$, $\zeta_n$ is the root of equation (3.6) for $\alpha = n$, $n \in \mathbb{N}_0$. We will prove that equation (4.9) has a positive root. Let $t \in [n, n+1)$ and $v(t) = e^{-t}$. By the inequality $\zeta_\alpha > 1$ proved above, we have

$$\begin{aligned}
\dot{v}(t) + (n+2)e^{-\zeta_n}\int_{t-1}^{t}(t-s)^n v(s)\, ds &= -e^{-t} + (n+2)e^{-\zeta_n}\int_{t-1}^{t}(t-s)^n e^{-s}\, ds \\
&= e^{-t}\left(-1 + (n+2)e^{-\zeta_n}\int_0^1 s^n e^s\, ds\right) \leqslant e^{-t}\left(-1 + (n+2)e^{-\zeta_n}\int_0^1 s^n e^{\zeta_n s}\, ds\right) \\
&= e^{-t}\left(-1 + (n+2)\int_0^1 s^n e^{\zeta_n(s-1)}\, ds\right) = 0.
\end{aligned}$$

By Lemma 2.3, it follows that equation (4.9) is not oscillatory. On the other hand,

$$\varliminf_{t\to\infty} \int_{t-1}^{t} K(t,s)(t-s)\, ds = \lim_{n\to\infty}(n+2)e^{-\zeta_n}\int_{t-1}^{t}(t-s)^{n+1}\, ds = \lim_{n\to\infty}(n+2)e^{-\zeta_n}\int_0^1 s^{n+1}\, ds = \frac{1}{e},$$

and the inequality turns into equality.

## 5 Analog of Condition 1.2 for equations with distributed delay

Let $t \in \mathbb{R}_+$. Define $E(t) = \{s : h(s) \leqslant t \leqslant g(s)\}$.

**Theorem 5.1.** *If* $\overline{\lim}_{t\to\infty}\int_{E(t)}\int_{h(s)}^{t} K(s,\zeta)\,d\zeta ds > 1$, *then equation* (2.1) *is oscillatory.*

*Proof.* Suppose equation (2.1) has a definite-sign solution $x = x(t)$. Then there exists a number $t_0 \geqslant 0$ such that $x(t) > 0$ and $\dot{x}(t) \leqslant 0$ for all $t \geqslant t_0$. Take $T$ such that $h(t) \geqslant t_0$ for all $t \geqslant T$. Clearly, $T \geqslant t_0$. From equation (2.1) we get

$$x(t) = x(T) - \int_T^t \int_{h(s)}^{g(s)} K(s,\zeta)x(\zeta)\,d\zeta ds > 0, \qquad t \geqslant T.$$

According to the inclusion $E(T) \subseteq [T,\infty)$ and the definition of the set $E(t)$, we have

$$x(T) > \int_T^\infty \int_{h(s)}^{g(s)} K(s,\zeta)x(\zeta)\,d\zeta ds \geqslant \int_{E(T)} \int_{h(s)}^{g(s)} K(s,\zeta)x(\zeta)\,d\zeta ds \geqslant \int_{E(T)} \int_{h(s)}^{T} K(s,\zeta)x(\zeta)\,d\zeta ds.$$

For $\zeta \in [h(s),T] \subseteq [t_0,T]$ the function $x(\zeta)$ is nonincreasing. Hence $x(\zeta) \geqslant x(T)$. Therefore,

$$x(T) \geqslant \int_{E(T)} \int_{h(s)}^{T} K(s,\zeta)x(\zeta)\,d\zeta ds \geqslant \left(\int_{E(T)} \int_{h(s)}^{T} K(s,\zeta)\,d\zeta ds\right) x(T) > x(T).$$

This contradiction completes the proof. □

**Corollary 5.2.** *Let $h$, $g$ be continuous and increasing functions, and*

$$\overline{\lim_{t\to\infty}} \int_{g^{-1}(t)}^{h^{-1}(t)} \int_{h(s)}^{t} K(s,\zeta)\,d\zeta ds > 1.$$

*Then equation* (2.1) *is oscillatory.*

*Proof.* Under the conditions given, we have $E(t) = \{s : g^{-1}(t) \leqslant s \leqslant h^{-1}(t)\}$. □

Note that the results obtained for equation (2.1) can be applied to equations with concentrated delay.

Consider the equation

$$\ddot{x}(t) + a(t)(x(g(t)) - x(h(t))) = 0, \qquad t \geqslant 0, \tag{5.1}$$

where $a$ is a locally summable function and the functions $g$ and $h$ satisfy the conditions imposed on equation (2.1). Rewrite (5.1) in the equivalent form,

$$\ddot{x}(t) + a(t)\int_{h(t)}^{g(t)} \dot{x}(s)\,ds = 0, \qquad t \geqslant 0.$$

Denote $\dot{x}(t) = y(t)$. Then we have an equation of the form (2.1), where $K(t,s) = a(t)$. Applying any of oscillation conditions represented above to this equation, we obtain conditions for all solutions of equation (5.1) to have oscillating derivatives. For example, by Theorem 5.1, we get the following result.

**Theorem 5.3.** *Let $a(t) \geqslant 0$ and $\overline{\lim}_{t\to\infty}\int_{E(t)} a(s)(t-h(s))\,ds > 1$. Then the derivatives of all solutions of equation* (5.1) *are oscillatory.*

Let us show that Theorem 5.3 implies the following result, first obtained in [13]. Consider the equation

$$\ddot{x}(t) + a(t)(x(t) - x(h(t))) = 0, \qquad t \geqslant 0. \tag{5.2}$$

**Corollary 5.4.** *Suppose $a = a(t)$ is a continuous nonnegative function, $h = h(t)$ is a continuously differentiable function such that $\dot{h}(t) > 0$ and $\lim_{t\to\infty} h(t) = \infty$. Let*

$$\varlimsup_{t\to\infty} \int_{h(t)}^{t} a(s)(h(t) - h(s))\, ds > 1.$$

*Then the derivatives of all solutions of equation (5.2) are oscillatory.*

*Proof.* For equation (5.2) we have $g(t) = t$, $E(t) = \left[t, h^{-1}(t)\right]$. Hence

$$\begin{aligned}\varlimsup_{t\to\infty} \int_{E(t)} a(s)(t - h(s))\, ds &= \varlimsup_{t\to\infty} \int_{t}^{h^{-1}(t)} a(s)(t - h(s))\, ds \\ &= \varlimsup_{t\to\infty} \int_{h(t)}^{t} a(s)(h(t) - h(s))\, ds > 1.\end{aligned}$$

Now the result follows from Theorem 5.3. □

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