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Criteria for the existence of positive solutions to delayed functional differential equations

Dedicated to Professor Tibor Krisztin on the occasion of his 60th birthday

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Abstract. The paper is concerned with the large time behavior of solutions to functional delayed differential equations $\dot{y}(t) = f(t, y_t)$ where $f\colon \Omega_n \mapsto \mathbb{R}^n$ is a continuous map satisfying a local Lipschitz condition with respect to the second argument and Ω_n is an open subset in $\mathbb{R} \times \mathcal{C}_n$, $\mathcal{C}_n := C_n([-r, 0], \mathbb{R}^n)$, $r > 0$. Criteria on the existence of positive solutions (different from the well-known published results) and their estimates from above are derived. The results are illustrated by examples.

Keywords: positive solution, large time behavior, delayed differential equation.

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1 Introduction and the problems considered

Let $C_n([a, b], \mathbb{R}^n)$ where $a, b \in \mathbb{R}, a < b, \mathbb{R} = (-\infty, +\infty)$ be the Banach space of continuous functions mapping the interval $[a, b]$ into $\mathbb{R}^n$. If $a = -r < 0$ and $b = 0$, we denote this space by $\mathcal{C}_n$, that is, $\mathcal{C}_n := C_n([-r, 0], \mathbb{R}^n)$.

The paper is concerned with the large time behavior of solutions of functional delayed differential equations

$$\dot{y}(t) = f(t, y_t) \tag{1.1}$$

where $f\colon \Omega_n \mapsto \mathbb{R}^n$ is a continuous map that satisfies a local Lipschitz condition with respect to the second argument (these conditions are tacitly assumed throughout the paper), and Ω_n is an open subset in $\mathbb{R} \times \mathcal{C}_n$. The paper particularly considers the problem of the existence of solutions to systems of linear and nonlinear functional delayed differential equations (1.1) with positive coordinates when $t \to \infty$.

Let $\sigma \in \mathbb{R}$, $A \geq 0$ and $y \in C_n([\sigma - r, \sigma + A], \mathbb{R}^n)$. For each $t \in [\sigma, \sigma + A]$, we define $y_t \in \mathcal{C}_n$ by means of the relation $y_t(\theta) = y(t + \theta)$, $\theta \in [-r, 0]$. Whenever it is necessary, we will assume that the derivatives in (1.1) are right-sided.

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A function $y\colon [\sigma - r, \sigma + A) \to \mathbb{R}^n$, where $A > 0$, is called a solution (1.1) on $[\sigma - r, \sigma + A)$ if y is continuous on $[\sigma - r, \sigma + A)$, continuously differentiable on $[\sigma, \sigma + A)$, $(t, y_t) \in \Omega_n$, and satisfies (1.1) for every $t \in [\sigma, \sigma + A)$.

For a given $\sigma \in \mathbb{R}$ and $\varphi \in \mathcal{C}_n$, we say that $y(\sigma, \varphi)$ is a solution of (1.1) through $(\sigma, \varphi) \in \Omega_n$ if there is an $A > 0$ such that $y(\sigma, \varphi)$ is a solution of (1.1) on $[\sigma - r, \sigma + A)$ and $y_\sigma(\sigma, \varphi) \equiv \varphi$.

In view of the above conditions, each element $(\sigma, \varphi) \in \Omega_n$ determines a unique solution $y(\sigma, \varphi)$ of (1.1) through $(\sigma, \varphi) \in \Omega_n$ on its maximal interval of existence $I_{\sigma,\varphi} = [\sigma, a), \sigma < a \leq \infty$ and $y(\sigma, \varphi)$ depends continuously on initial data [15]. We say, that a solution $y(\sigma, \varphi)$ of (1.1) is positive if $y_i(\sigma, \varphi)(t) > 0$ on $[\sigma - r, \sigma] \cup I_{\sigma,\varphi}$ for each $i = 1, 2, \ldots, n$.

The problem of the existence of positive solutions to systems (1.1), or to more general systems, is a classical one. The results related to the existence of positive solutions and their properties are published, e.g., in the books [1, 2, 11, 12, 14, 19] and in numerous papers, e.g., in [4, 6–8, 10, 13, 16–18, 21].

In the present paper we prove the existence of positive solutions by an approach that, to the author's knowledge has not yet been published and is not a direct consequence of any known results.

Set $f(t, y_t) := -F_s(t, y_t)$ in (1.1) where $F_s(t, y_t) = (F_{s1}(t, y_t), \ldots, F_{sn}(t, y_t))$ and consider a system

$$\dot{y}(t) = -F_s(t, y_t). \tag{1.2}$$

Moreover, set $f(t, y_t) := -F_e(t, y_t)$, $n = 1$ in (1.1) and, along with system (1.2), consider a scalar equation

$$\dot{x}(t) = -F_e(t, x_t). \tag{1.3}$$

The paper is organized as follows. In part 2, the main results are formulated and accompanied by examples. Particularly, in part 2.1 we investigate the equivalence between the existence of a positive solution to (1.3) and the existence of a positive solution to (1.2). In part 2.2, given two different systems (1.2), a statement is proved on the existence of a positive solution to system if the other system has a positive solution. Part 2.3 applies derived results to particular systems to obtain some easily verifiable conditions. The proofs of the statements with the necessary auxiliary information are brought together in part 3.

2 Main results

With $\mathbb{R}^n_{\geq 0}$ ($\mathbb{R}^n_{>0}$) we denote the set of all component-wise nonnegative (positive) vectors v in $\mathbb{R}^n$, i.e., $v = (v_1, \ldots, v_n)$ with $v_i \geq 0$ ($v_i > 0$) for $i = 1, \ldots, n$. For $u, v \in \mathbb{R}$, we define $u \leq v$ if $v - u \in \mathbb{R}^n_{\geq 0}$; $u \ll v$ if $v - u \in \mathbb{R}^n_{>0}$; $u < v$ if $u \leq v$ and $u \neq v$. By 0_n we denote the n-dimensional null vector $(0, \ldots, 0)$.

2.1 Criterion for the existence of positive solutions

A theorem formulated below states that, under given assumptions, the existence of a positive solution of (1.3) is equivalent to the existence of a positive solution of (1.2). Let $\Omega_n := [t_0, \infty) \times \mathcal{C}_n$ and $t^* \geq t_0$ be assumed in the following text.

Theorem 2.1. *Assume that*

$$F_e(t, \varphi) \equiv F_{si}(t, \varphi, \ldots, \varphi), \qquad , i = 1, \ldots, n \tag{2.1}$$

for every $(t,\varphi) \in \Omega_1$, *and*

$$F_e(t,\varphi) = F_{si}(t,\varphi,\dots,\varphi) = 0, \qquad i = 1,\dots,n \tag{2.2}$$

for every $(t,\varphi) \in \Omega_1$ *where* $\varphi(\theta) = 0$, $\theta \in [-r,0]$. *Let, moreover,*

$$0 < F_e(t,\varphi^*) < F_e(t,\psi^*) \tag{2.3}$$

for every $(t,\varphi^*) \in \Omega_1$, $(t,\psi^*) \in \Omega_1$ *such that* $0 < \varphi^*(\theta) < \psi^*(\theta)$, $\theta \in [-r,0)$, *and*

$$0_n \ll F_s(t,\varphi^*) \leq F_s(t,\psi^*) \tag{2.4}$$

for every $(t,\varphi^*) \in \Omega_n$, $(t,\psi^*) \in \Omega_n$ *such that* $0_n \ll \varphi^*(\theta) \leq \psi^*(\theta)$, $\theta \in [-r,0)$.

Then, the existence of a positive solution $y = y(t)$ *on* $[t^* - r,\infty)$ *of system* (1.2) *is equivalent with the existence of a positive solution* $x = x(t)$ *on* $[t^* - r,\infty)$ *of equation* (1.3). *Moreover, if a positive solution* $y = y(t)$ *on* $[t^* - r,\infty)$ *of system* (1.2) *exists, then there exist a positive solution* $x = x(t)$ *of equation* (1.3) *satisfying*

$$x(t) < \min\{y_1(t), y_2(t),\dots,y_n(t)\} \tag{2.5}$$

on $[t^* - r,\infty)$.

2.2 A comparison result

In this part, we formulate a comparison result. Put $f(t,y_t) := -F^*(t,y_t)$ and $f(t,y_t) := -F^{**}(t,y_t)$ in (1.1) where

$$F^*(t,y_t) = (F_1^*(t,y_t),\dots,F_n^*(t,y_t)),$$
$$F^{**}(t,y_t) = (F_1^{**}(t,y_t),\dots,F_n^{**}(t,y_t)).$$

Consider two systems

$$\dot{y}(t) = -F^*(t,y_t), \tag{2.6}$$
$$\dot{y}(t) = -F^{**}(t,y_t). \tag{2.7}$$

The following theorem is of a comparison type and provides conditions sufficient for the existence of a positive solution of a nonlinear system (2.6) if system (2.7) has a positive solution and some inequalities hold between their right-hand sides.

Theorem 2.2. *Let, for every* $(t,\varphi^*) \in \Omega_n$, $(t,\psi^*) \in \Omega_n$ *such that* $0_n \ll \varphi^*(\theta) \ll \psi^*(\theta)$, $\theta \in [-r,0)$, *we have*

$$0_n \ll F^*(t,\varphi^*) \ll F^*(t,\psi^*). \tag{2.8}$$

Let, moreover, system (2.7) *has a positive solution* $y = y^{**}(t)$ *on* $[t^* - r,\infty)$ *and, for every* $(t,\psi^*) \in \Omega_n$ *such that*

$$0 \ll \psi_s^*(\theta) < \min\{y_{1t}^{**}(\theta), y_{2t}^{**}(\theta),\dots,y_{nt}^{**}(\theta)\}, \qquad \theta \in [-r,0),\ s = 1,2,\dots,n,$$

we have

$$F_i^*(t,\psi^*) \leq F_j^{**}(t,\psi^*) \tag{2.9}$$

for every $i,j \in \{1,2,\dots,n\}$. *Then, system* (2.6) *has a positive solution* $y = y^*(t)$ *on* $[t^* - r,\infty)$ *satisfying*

$$y_i^*(t) < \min\{y_1^{**}(t), y_2^{**}(t),\dots,y_n^{**}(t)\}, \qquad i = 1,2,\dots,n. \tag{2.10}$$

Example 2.3. Let us consider a system (2.7)

$$\dot{y}_1(t) = -F_1^{**}(t, y_t) := -\frac{(t-r)^3}{2t^2} y_1^3(t-r) - \frac{(t-r)^3}{16t^2} y_2^3(t-r), \tag{2.11}$$

$$\dot{y}_2(t) = -F_2^{**}(t, y_t) := -\frac{(t-r)^3}{t^2} y_1^3(t-r) - \frac{(t-r)^3}{8t^2} y_2^3(t-r). \tag{2.12}$$

This system has a positive solution

$$y(t) = y^{**}(t) = (y_1^{**}(t), y_2^{**}(t)) = \left(t^{-1}, 2t^{-1}\right).$$

Consider a system (2.6)

$$\dot{y}_1(t) = -F_1^{*}(t, y_t) := -2(\sqrt{t}) y_1^3(t-r) - (\ln t) y_2^3(t-r), \tag{2.13}$$

$$\dot{y}_2(t) = -F_2^{*}(t, y_t) := -(\ln t)^2 y_1^3(t-r) - (\sqrt{t}) y_2^3(t-r). \tag{2.14}$$

Assume t^* sufficiently large. It is a trivial matter to see that properties (2.8) and (2.9) of Theorem 2.2 are fulfilled. Therefore, system (2.13), (2.14) has a positive solution

$$y(t) = y^*(t) = (y_1^*(t), y_2^*(t))$$

on $[t^* - r, \infty)$ satisfying (2.10), i.e.,

$$y_i^*(t) < \min\{y_1^{**}(t), y_2^{**}(t)\} = \min\{t^{-1}, 2t^{-1}\} = t^{-1}, \qquad i = 1, 2.$$

Example 2.4. Consider a system (2.6)

$$\dot{y}_1(t) = -F_1^{*}(t, y_t) = -F_{s1}(t, y_t) := -\frac{(t-r)^3}{2t^2} y_1^3(t-r) - \frac{(t-r)^3}{16t^2} y_2^3(t-r), \tag{2.15}$$

$$\dot{y}_2(t) = -F_2^{*}(t, y_t) = -F_{s2}(t, y_t) := -\frac{(t-r)^3}{2t^2} y_1^3(t-r) - \frac{(t-r)^3}{16t^2} y_2^3(t-r). \tag{2.16}$$

Assume t^* sufficiently large. Apply Theorem 2.2 to systems (2.15), (2.16) and (2.11), (2.12). As all assumptions are fulfilled, system (2.15), (2.16) has a positive solution

$$y(t) = y^*(t) = (y_1^*(t), y_2^*(t))$$

on $[t^* - r, \infty)$ satisfying (2.10), i.e.,

$$y_i^*(t) < \min\{y_1^{**}(t), y_2^{**}(t)\} = \min\{t^{-1}, 2t^{-1}\} = t^{-1}, \qquad i = 1, 2.$$

Moreover, between system (2.15), (2.16) and an equation

$$\dot{x}(t) = -F_e(t, x_t) = -\frac{9}{16} \frac{(t-r)^3}{t^2} x^3(t-r), \tag{2.17}$$

the following equality (2.1) holds

$$F_e(t, \varphi) = F_{s1}(t, \varphi, \varphi) = F_{s2}(t, \varphi, \varphi)$$

for every $(t, \varphi) \in \Omega_1$. Since not only this equality, but all the assumptions of Theorem 2.1 are fulfilled, equation (2.17) has a positive solution $x = x(t)$ on $[t^* - r, \infty)$ satisfying inequality (2.5), i.e.,

$$x(t) < \min\{y_1^*(t), y_2^*(t)\} < \min\{y_1^{**}(t), y_2^{**}(t)\} < \min\{t^{-1}, 2t^{-1}\} = t^{-1}.$$

2.3 Some consequences

In this part, we apply the criteria derived to achieve some results with easily verifiable assumptions.

2.3.1 A linear case

Consider a linear differential system with delay

$$\dot{y}(t) = -C(t)y(h(t)) \tag{2.18}$$

where $C(t) = \{c_{ij}(t)\}_{i,j=1}^{n}$ is an $n \times n$ continuous matrix defined on $[t_0, \infty)$. Assume that the elements $c_{ij}(t) \geq 0$, $i, j = 1, \dots, n$, the delay $h(t)$ is continuous on $[t_0, \infty)$ and

$$t - r \leq h(t) < t, \qquad t \in [t_0, \infty). \tag{2.19}$$

System (2.18) is a particular case of system (1.2) if

$$F_s(t, \varphi) := C(t)\varphi(h(t) - t). \tag{2.20}$$

Assume that $\sum_{j=1}^{n} c_{ij}(t) = \sum_{j=1}^{n} c_{sj}(t)$, $t \in [t_0, \infty)$, $i, s = 1, \dots, n$ and denote

$$c(t) := \sum_{j=1}^{n} c_{1j}(t).$$

Together with system (2.18), consider a scalar equation

$$\dot{x}(t) = -c(t)x(h(t)), \tag{2.21}$$

being a special case of equation (1.3) with

$$F_e(t, \varphi) := c(t)\varphi(h(t) - t). \tag{2.22}$$

Theorem 2.5. *Let $c_{ij}(t) \geq 0$, $i, j = 1, \dots, n$ be continuous functions on $[t_0, \infty)$, let the delay $h(t)$ be continuous on $[t_0, \infty)$ and satisfies (2.19). If, moreover, $c(t) > 0$, $t \in [t_0, \infty)$, then the existence of a positive solution $y = y(t)$ on $[t^* - r, \infty)$ of system (2.18) is equivalent with the existence of a positive solution $x = x(t)$ on $[t^* - r, \infty)$ of equation (2.21). Moreover, a positive solution $x = x(t)$ of equation (2.21), defined on $[t^* - r, \infty)$, satisfies*

$$x(t) < \min\{y_1(t), y_2(t), \dots, y_n(t)\}.$$

Example 2.6. Let us consider system

$$\begin{aligned} \dot{y}_1(t) &= -c_{11}(t)y_1(t-r) - c_{12}(t)y_2(t-r), \\ \dot{y}_2(t) &= -c_{21}(t)y_1(t-r) - c_{22}(t)y_2(t-r) \end{aligned}$$

where $0 < r < \ln 2$ and

$$c_{11}(t) = \frac{1}{\Delta(t)}\left[\frac{1}{2}e^{r} + \frac{1}{2}e^{-t+2r} - 2e^{-2t+2r}\right],$$

$$c_{12}(t) = \frac{1}{\Delta(t)}\left(2-\mathrm{e}^r\right)\mathrm{e}^r,$$

$$c_{21}(t) = \frac{1}{2\Delta(t)}\left(2-\mathrm{e}^r\right)\mathrm{e}^{-t+r},$$

$$c_{22}(t) = \frac{1}{\Delta(t)}\left[\frac{1}{2}\mathrm{e}^r + \mathrm{e}^{-t+r} - 2\mathrm{e}^{-2t+2r}\right],$$

$$\Delta(t) = \frac{1}{2}\mathrm{e}^{2r} + \frac{1}{2}\mathrm{e}^{-t+3r} - \mathrm{e}^{-2t+4r}.$$

This system has a positive solution $y(t) = (y_1(t), y_2(t)) = (\exp(-t), \exp(-2t))$. It is easy to see that $c_{ij}(t) > 0$, $i, j = 1, 2$, on $[t^*, \infty)$, if t^* is sufficiently large. Moreover,

$$c(t) = c_{11}(t) + c_{12}(t) = c_{21}(t) + c_{22}(t) = \frac{1}{\Delta(t)}\left[\frac{1}{2}\mathrm{e}^r + \left(2 - \frac{3}{2}e^r\right)\mathrm{e}^{-t+r} - 2\mathrm{e}^{-2t+2r}\right] > 0.$$

All the assumptions of Theorem 2.5 are valid and, therefore, there exists a positive solution of the equation

$$\dot{x}(t) = -c(t)x(t-r).$$

satisfying

$$x(t) < \min\{y_1(t), y_2(t)\} = \min\{\exp(-t), \exp(-2t)\} = \exp(-2t)$$

if t^* is sufficiently large.

2.3.2 Criterion of positivity by a critical constant

It is well-known that a scalar differential equation with delay

$$\dot{x}(t) = -\frac{1}{\mathrm{e}r}\,x(t-r) \tag{2.23}$$

has two positive linearly independent solutions

$$x_1(t) = \exp\left(-t/r\right), \qquad x_2(t) = t\exp\left(-t/r\right), \tag{2.24}$$

and, in addition to this, equation

$$\dot{x}(t) = -cx(t-r) \tag{2.25}$$

with a positive coefficient $c = \text{const}$ has positive solutions if and only if $c \le 1/(\mathrm{e}r)$ since, for $c > 1/(\mathrm{e}r)$, all solutions of (2.25) are oscillating. Therefore, the value $c = 1/(\mathrm{e}r)$ is, in a sense, the best possible constant separating the case of the existence of positive solutions from the case of all the solutions being oscillating (i.e., any solution has infinitely many zero points with co-ordinates greater than any previously given number). Often, it is called a critical constant.

We utilize equation (2.23) to give a comparison criterion for the existence of positive solutions to systems of nonlinear equations.

Set $f(t, y_t) := -F_s(t, y_t)$ in (1.1) and consider a system (1.2), i.e,

$$\dot{y}(t) = -F_s(t, y_t).$$

Moreover, set $f(t, y_t) := -L_s(t, y_t)$ in (1.1) where $L_s(t, y_t)$ is a linear functional with respect to the second argument y_t and consider a linear system

$$\dot{y}(t) = -L_s(t, y_t). \tag{2.26}$$

Specify $L_s(t,y_t) = (L_{s1}(t,y_t),\dots,L_{sn}(t,y_t))$ as

$$L_{si}(t,y_t) := \sum_{j=1}^{n} c_{ij}(t) y_j(t-r), \qquad i = 1,2,\dots,n.$$

Theorem 2.7. *Let, for every* $(t,\varphi) \in \Omega_n$, $(t,\psi) \in \Omega_n$ *such that* $0_n \ll \varphi(\theta) \ll \psi(\theta)$, $\theta \in [-r,0)$,

$$0_n \ll F_s(t,\varphi) \ll F_s(t,\psi). \tag{2.27}$$

Let $c_{ij}(t)$, $i,j = 1,\dots,n$ *be continuous functions on* $[t_0,\infty)$ *and*

$$\sum_{j=1}^{n} c_{ij}(t) = \frac{1}{\mathrm{e}r} \tag{2.28}$$

for every $i = 1,2,\dots,n$ *and let, for every* $(t,\psi) \in \Omega_n$ *such that*

$$0 < \psi_k(\theta) < \exp\left(-(t+\theta)/r\right), \qquad \theta \in [-r,0), \; k = 1,2,\dots,n, \tag{2.29}$$

$$F_{si}(t,\psi) \le L_{sj}(t,\psi) \tag{2.30}$$

for every $i,j \in \{1,2,\dots,n\}$. *Then, system* (1.2) *has a positive solution* $y = y(t)$ *on* $[t_0 - r,\infty)$ *and*

$$y_i(t) < \exp\left(-t/r\right), \qquad i = 1,2,\dots,n. \tag{2.31}$$

Example 2.8. Assume that t^* is sufficiently large such that the below inequalities are true. Let system (1.2) be given as

$$\dot{y}_1(t) = -F_{s1}(t,y_t) := -t^5 y_1^6(t-r) - \mathrm{e}^t y_2^3(t-r), \tag{2.32}$$

$$\dot{y}_2(t) = -F_{s2}(t,y_t) := -\mathrm{e}^{2t} y_1^4(t-r) - y_2^2(t-r). \tag{2.33}$$

where $0 < r$. Assume that the auxiliary linear system (2.26) is the following

$$\dot{y}_1(t) = -L_{s1}(t,y_t) := -\frac{1}{2\mathrm{e}r} y_1(t-r) - \frac{1}{2\mathrm{e}r} y_2(t-r), \tag{2.34}$$

$$\dot{y}_2(t) = -L_{s2}(t,y_t) := -\frac{1}{2\mathrm{e}r} y_1(t-r) - \frac{1}{2\mathrm{e}r} y_2(t-r). \tag{2.35}$$

Assumption (2.27) is obviously true. Assumption (2.28) holds as well since $c_{ij}(t) = 1/(2\mathrm{e}r)$, $i,j = 1,2$. For functions described by (2.29) we conclude that (2.30) holds. System (2.34), (2.35) has a positive solution

$$y(t) = (y_1(t), y_2(t)) = (\exp(-t/r), \exp(-t/r))$$

as suggested by the first formula (for the solution $x_1(t)$ of equation (2.23)) in (2.24). Theorem 2.7 is applicable and system (2.32), (2.33) has a positive solution $y = y(t)$ on $[t_0 - r,\infty)$ satisfying (2.31), i.e.,

$$y_i(t) < \exp(-t/r), \qquad i = 1,2.$$

3 Proofs and additional material

This part contains proofs of the statements formulated above and the necessary auxiliary results and material.

In the proofs, we make use of the following result. Let us define vectors

$$\rho(t) = (\rho_1(t), \rho_2(t), \dots, \rho_n(t)),$$
$$\delta(t) = (\delta_1(t), \delta_2(t), \dots, \delta_n(t)),$$

continuous on $[t^* - r, \infty)$, where $t^* \in \mathbb{R}$ is fixed, and such that $\rho(t) \ll \delta(t)$. Let us, moreover, define the set

$$\omega := \{(t, y) \colon t \geq t^* - r,\ \rho(t) \ll y \ll \delta(t)\}.$$

Below, $\overline{\omega}$ denotes the closure of ω, $\partial\omega$ its boundary and $\operatorname{int}\omega$ its interior.

Lemma 3.1. *Assume that, for all $i = 1, 2, \dots, n$ and all $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n) \in \mathcal{C}_n$ for which*

$$(t + \theta, \varphi(\theta)) \in \omega, \qquad \theta \in [-r, 0) \tag{3.1}$$

and either

$$\varphi_i(0) = \delta_i(t) \tag{3.2}$$

or

$$\varphi_i(0) = \rho_i(t), \tag{3.3}$$

we have

$$(s, y(t, \varphi)(s)) \notin \overline{\omega} \tag{3.4}$$

for all $s \in (t, t + \varepsilon)$ where $\varepsilon = \varepsilon(t, \varphi)$ is a sufficiently small positive number. Then, there exists a solution $y = y(t)$ of the system (1.1) on $[t^* - r, \infty)$ such that

$$\rho(t) \ll y(t) \ll \delta(t) \tag{3.5}$$

holds.

Proof of Lemma 3.1. First, let us define a retract and a retraction [20, p. 97].

Definition 3.2. If $\mathcal{A} \subset \mathcal{B}$ are any two sets of a topological space and $\pi \colon \mathcal{B} \to \mathcal{A}$ is a continuous mapping from $\mathcal{B}$ onto $\mathcal{A}$ such that $\pi(p) = p$ for every $p \in \mathcal{A}$, then π is called a retraction of $\mathcal{B}$ onto $\mathcal{A}$. If there exists a retraction of $\mathcal{B}$ onto $\mathcal{A}$, $\mathcal{A}$ is called a retract of $\mathcal{B}$.

Next, let us define a system of initial functions [5, Definition 4].

Definition 3.3. A system of initial functions $p_{A,\omega}$ with respect to the nonempty sets A and ω where $A \subset \overline{\omega} \subset \mathbb{R} \times \mathbb{R}^n$ is defined as a continuous mapping $p \colon A \to \mathcal{C}_n$ such that (α) and (β) below hold.

(α) If $z = (t, y) \in A \cap \operatorname{int}\omega$, then $(t + \theta, p(z)(\theta)) \in \omega$ for $\theta \in [-r, 0]$.

(α) If $z = (t, y) \in A \cap \partial\omega$, then $(t + \theta, p(z)(\theta)) \in \omega$ for $\theta \in [-r, 0)$ and $(t, p(z)(0)) = z$.

The proof of the lemma is based on the well-known fact that the boundary of an n-dimensional ball is not its retract (see e.g. [3]). Assuming that a solution $y = y(t)$ of the system (1.1) on $[t^* - r, \infty)$ satisfying (3.5) does not exist we show that the set

$$\mathcal{A} := \{(t, y) \colon t = t^*, y \in \mathbb{R}^n\} \cap \partial\omega$$

is a retract of the set

$$\mathcal{B} := \{(t, y) \colon t = t^*, y \in \mathbb{R}^n\} \cap \overline{\omega},$$

which is a contradiction to above-mentioned classical topology statement (note that $\mathcal{A}$ is homeomorphic to the boundary of an n-dimensional ball and $\mathcal{B}$ is homeomorphic to an n-dimensional ball).

Let us construct such a retract. First, we consider a system of initial functions $p_{\mathcal{A},\omega}$ defined by Definition 3.3 (with $A := \mathcal{A}$) and assume, following the outlined scheme of the proof, that every solution $y(t^*, p)$ defined by an initial function $p \in p_{\mathcal{A},\omega}$ leaves the set ω. Let the first point of the intersection of $y(t^*, p)(t)$ with $\partial\omega$ be a point $t = t^{**}$. Then, either $y_i(t^*, p)(t^{**}) = \delta_i(t^{**})$ or $y_i(t^*, p)(t^{**}) = \rho_i(t^{**})$ for an index $i \in \{1, 2, \dots, n\}$ and, according to (3.4), $(t, y(t^*, p)(t)) \notin \overline{\omega}$ for $t \in (t^{**}, t^{**} + \varepsilon)$ where ε is a positive number. Due to the continuous dependence of solutions on the initial data, we state that the mapping

$$\mathcal{M} \colon (t^*, y(t^*, p)(t^*)) \mapsto (t^{**}, y(t^*, p)(t^{**})) \in \partial\omega$$

is continuous and, moreover, the points $(t^*, y(t^*, p)(t^*)) \in \partial\omega$ are fixed points of $\mathcal{M}$.

Now we show that there exists a continuous mapping $\mathcal{N} \colon \partial\omega \mapsto \mathcal{A}$ such that the points of $\mathcal{A}$ are fixed points of $\mathcal{N}$. Let $(t^0, y^0) = (t, y_1^0, y_2^0, \dots, y_n^0) \in \partial\omega$. Then, there exists an index $i \in \{1, 2, \dots, n\}$ such that either $y_i^0 = \rho_i(t^0)$ or $y_i^0 = \delta_i(t^0)$. Define

$$\mathcal{N} := (t^0, y^0) \mapsto (t^*, y^{00}) = (t^*, y_1^{00}, y_2^{00}, \dots, y_n^{00}) \in \mathcal{A}$$

where

$$y_i^{00} := \rho_i(t^*) + \frac{\delta_i(t^*) - \rho_i(t^*)}{\delta_i(t^0) - \rho_i(t^0)} \cdot (y_i^0 - \rho_i(t^0)), \qquad i = 1, 2, \dots, n.$$

It is easy to see that all the above properties of $\mathcal{N}$ hold. If the property $y_i^0 = \rho_i(t^0)$ or $y_i^0 = \delta_i(t^0)$ is true for two different indices, the construction of $\mathcal{N}$ remains the same. We finish the proof with a conclusion that the composite mapping

$$\pi := \mathcal{N} \circ \mathcal{M} \colon \mathcal{B} \mapsto \mathcal{A}$$

is the desired retraction of $\mathcal{B}$ onto $\mathcal{A}$ and our assumption of the non-existence of a solution $y = y(t)$ of the system (1.1) on $[t^* - r, \infty)$ satisfying (3.5) is not true. □

Remark 3.4. The idea of the proof of Lemma 3.1 goes back to Ważewski [24] (see [22, 23] as well). In utilizing Lemma 3.1, it is necessary to know how the property (3.4) can be verified. We give sufficient conditions for the verification when the vectors $\rho(t)$ and $\delta(t)$ are continuously differentiable on $[t^*, \infty)$. Let (3.2) hold. We show that condition (3.4) is satisfied if

$$\delta_i'(t) < f_i(t, \varphi) \tag{3.6}$$

and if (3.3) is true, then for (3.4) to be true, it is sufficient that

$$\rho_i'(t) > f_i(t, \varphi). \tag{3.7}$$

Let, e.g., (3.6) hold. Then,

$$\delta_i'(t) < f_i(t, y_t(t,\varphi)) = y_i'(t,\varphi)(t)$$

and, integrating this inequality over the interval $[t,s]$ where $t < s < t + \varepsilon^*$, ε^* is a small positive number and taking into account (3.2), i.e., $\varphi_i(0) = y_i(t,\varphi)(t) = \delta_i(t)$, we have

$$\delta_i(t+s) < y_i(t,\varphi)(t+s).$$

Similarly, one can prove that, if (3.7) and (3.3) hold, then (3.4) holds as well.

3.1 Proof of Theorem 2.1

a) Let $x = x(t)$ be a positive solution of equation (1.3) on $[t^* - r, \infty)$. Then, the existence on $[t^* - r, \infty)$ of a positive solution $y = y(t)$ of system (1.2) is an obvious consequence of (2.1) because

$$F_e(t, x_t) \equiv F_{si}(t, x_t, \dots, x_t), \qquad i = 1, \dots, n$$

on $[t^*, \infty)$ and

$$y(t) = (x(t), \dots, x(t)), \qquad t \in [t^* - r, \infty)$$

is a positive solution of system (1.2) on $[t^* - r, \infty)$.

b) Let $y = y(t)$ be a positive solution of system (1.2) on $[t^* - r, \infty)$. To prove that there exists a positive solution $x = x(t)$ of equation (1.3) on $[t^* - r, \infty)$, we need Lemma 3.1. Set $n = 1$ (then $i = 1$), $f_1(t,\varphi) := -F_e(t,\varphi)$, $\rho_1(t) \equiv 0$ and

$$\delta_1(t) := \min\{y_1(t), y_2(t), \dots, y_n(t)\}.$$

Then, for this setting,

$$\omega := \{(t,y) \colon t \ge t^* - r,\ 0 < y < \min\{y_1(t), y_2(t), \dots, y_n(t)\}\}.$$

Verifying (3.7), we get

$$\rho_1'(t) - f_1(t,\varphi) = -f_1(t,\varphi) = F_e(t,\varphi).$$

By (3.1), we have $\varphi(\theta) > 0$ for every $\theta \in [-r, 0)$. Therefore, by (2.3) with $\varphi^* \equiv 0$, $\psi^* = \varphi$, and by (2.2), we have

$$F_e(t,\psi^*) = F_e(t,\varphi) > F_e(t,\varphi^*) = F_e(t,0) = 0$$

and (3.7) holds.

Now we show that (3.6) holds as well. Assume first that δ_1 is continuously differentiable on $[t^*, \infty)$.

By (3.1), we have $\varphi(\theta) < \delta_1(t+\theta)$ for every $\theta \in [-r, 0)$. Therefore, by (2.3) with $\psi^*(\theta) = \delta_1(t+\theta)$, $\theta \in [-r, 0)$, $\varphi^* = \varphi$, we have

$$F_e(t,\varphi^*) = F_e(t,\varphi) < F_e(t,\psi^*) = F_e(t,\delta_{1t}).$$

Then,

$$\delta_1'(t) - f_1(t,\varphi) = \delta_1'(t) + F_e(t,\varphi) < \delta_1'(t) + F_e(t,\delta_{1t}). \tag{3.8}$$

Now we estimate the right-hand side of (3.8). Let, for a given $t \ge t^*$, there exist a unique value of index $j \in \{1, 2, \dots, n\}$ such that

$$\min\{y_1(t), y_2(t), \dots, y_n(t)\} = y_j(t).$$

Then, by (2.1) and by (2.4) with $\varphi^* = (y_{jt}, y_{jt}, \dots, y_{jt})$, $\psi^* = (y_{1t}, y_{2t}, \dots, y_{nt})$,

$$\begin{aligned}
\delta_1'(t) + F_e(t, \delta_{1t}) &= y_j'(t) + F_e(t, y_{jt}) \\
&= -F_{sj}(t, y_t) + F_e(t, y_{jt}) \\
&= -F_{sj}(t, y_{1t}, y_{2t}, \dots, y_{nt}) + F_e(t, y_{jt}) \\
&= -F_{sj}(t, y_{1t}, y_{2t}, \dots, y_{nt}) + F_{sj}(t, y_{jt}, y_{jt}, \dots, y_{jt}) \\
&\le -F_{sj}(t, y_{jt}, y_{jt}, \dots, y_{jt}) + F_{sj}(t, y_{jt}, y_{jt}, \dots, y_{jt}) = 0. \qquad (3.9)
\end{aligned}$$

Finally, from (3.8) and (3.9), we derive

$$\delta_1'(t) - f_1(t, \varphi) < 0,$$

i.e., (3.6) holds. Therefore, by Remark 3.4, property (3.4) is true.

Now assume that at a point $t \in [t^*, \infty)$, δ_1 is not continuously differentiable. Then, for at least two different indices $i = i^*$, $i = i^{**}$, $i^*, i^{**} \in \{1, 2, \dots, n\}$, we have

$$\min\{y_1(t), y_2(t), \dots, y_n(t)\} = y_{i^*}(t) = y_{i^{**}}(t) \qquad (3.10)$$

with $y_{i^*}'(t) \neq y_{i^{**}}'(t)$. Let (3.10) is valid exactly for two indices i^* and i^{**}. However, at the point t, both co-ordinates y_{i^*}, $y_{i^{**}}$ are continuously differentiable and, for both settings (at the given point t) $\delta_1(t) := y_{i^*}(t)$ and $\delta_1(t) := y_{i^{**}}(t)$, we can verify that (3.8) and (3.9) are valid. This means that property (3.4) holds again. Similarly we proceed if (3.10) holds for more than two indices.

From inequality (3.5) in Lemma 3.1, we conclude that there exists a positive solution $x = x(t)$ of equation (1.3) on $[t^* - r, \infty)$ satisfying

$$0 < x(t) < \min\{y_1(t), y_2(t), \dots, y_n(t)\},$$

i.e. (2.5) holds. □

3.2 Proof of Theorem 2.2

Let $y = y^{**}(t)$ be a positive solution of system (2.7) on $[t^* - r, \infty)$. To prove that there exists a positive solution $y = y^*(t)$ of system (2.6) on $[t^* - r, \infty)$, we use Lemma 3.1.

Set $\rho(t) \equiv 0_n$ and and $\delta(t) = (\delta_1(t), \delta_2(t), \dots, \delta_n(t))$ where

$$\delta_i(t) := \min\{y_1^{**}(t), y_2^{**}(t), \dots, y_n^{**}(t)\}, \qquad i = 1, 2, \dots, n.$$

Then,

$$\omega := \left\{(t, y) \colon t \ge t^* - r,\ 0 < y_i < \min\{y_1^{**}(t), y_2^{**}(t), \dots, y_n^{**}(t)\},\, i = 1, 2, \dots, n\right\}.$$

First, verifying (3.7), we obtain

$$\rho_i'(t) - f_i(t, \varphi) = -f_i(t, \varphi) = F_i^*(t, \varphi), \qquad i = 1, 2, \dots, n.$$

Using (2.8), we conclude that

$$F_i^*(t, \varphi) > 0, \qquad i = 1, 2, \dots, n$$

and (3.7) holds.

Now we show that (3.6) holds as well. Let $i \in \{1, 2, \ldots, n\}$ be fixed. Assume that δ_i is continuously differentiable on $[t^*, \infty)$.

By (3.1), we have $\varphi_j(\theta) < \delta_j(t+\theta)$ for every $\theta \in [-r, 0)$ and every $j \in \{1, 2, \ldots, n\}$. Therefore, by (2.8) with $\psi^*(\theta) = \delta(t+\theta)$, $\theta \in [-r, 0)$, $\varphi^* = \varphi$, we have

$$F_i^*(t, \varphi^*) = F_i^*(t, \varphi) < F_i^*(t, \psi^*) = F_i^*(t, \delta_t).$$

Then,

$$\delta_i'(t) - f_i(t, \varphi) = \delta_i'(t) + F_i^*(t, \varphi) < \delta_i'(t) + F_i^*(t, \delta_t). \tag{3.11}$$

Now we estimate the right-hand side of (3.11). Let, for a given $t \geq t^*$, there exist a unique value of index $j \in \{1, 2, \ldots, n\}$ such that

$$\delta_i(t) = \min\{y_1^{**}(t), y_2^{**}(t), \ldots, y_n^{**}(t)\} = y_j^{**}(t).$$

Then, by (2.9) with $\psi^* = y_t^{**}$ and by (2.8) with $\varphi^* = \delta_t$, $\psi^* = y_t^{**}$,

$$\begin{aligned}
\delta_i'(t) + F_i^*(t, \delta_t) &= y_j'^{**}(t) + F_i^*(t, \delta_t) \\
&= -F_j^{**}(t, y_t^{**}) + F_i^*(t, \delta_t) \\
&\leq -F_i^*(t, y_t^{**}) + F_i^*(t, \delta_t) \\
&< -F_i^*(t, y_t^{**}) + F_i^*(t, y_t^{**}) = 0.
\end{aligned} \tag{3.12}$$

Finally, from (3.11) and (3.12), we derive

$$\delta_i'(t) - f_i(t, \varphi) < 0,$$

i.e., (3.6) holds. Therefore, by Remark 3.4, property (3.4) is valid.

Now assume that at a point $t \in [t^*, \infty)$, function δ is not continuously differentiable. Then, for at least two different indices $i = i_0$, $i = i_{00}$, $i_0, i_{00} \in \{1, 2, \ldots, n\}$, we have

$$\min\{y_1^{**}(t), y_2^{**}(t), \ldots, y_n^{**}(t)\} = y_{i_0}^{**}(t) = y_{i_{00}}^{**}(t)$$

with $y_{i_0}'(t) \neq y_{i_{00}}'(t)$. However, at the point t, both co-ordinates y_{i_0}, $y_{i_{00}}$ are continuously differentiable and we can proceed similarly to the proof of Theorem 2.1.

From inequality (3.5) in Lemma 3.1, we conclude that there exists a positive solution $y = y^*(t)$ of system (2.8) on $[t^* - r, \infty)$ satisfying

$$0 < y_i^*(t) < \min\{y_1^{**}(t), y_2^{**}(t), \ldots, y_n^{**}(t)\}, \qquad i = 1, 2, \ldots, n,$$

i.e. (2.10) holds. □

3.3 Proof of Theorem 2.5

Obviously, each solution $x = x(t)$ (not only a positive one) of (2.21) generates a solution $y = y(t) = (x(t), x(t), \ldots, x(t))$ of system (2.18). To get an inverse statement for a positive solution, we apply Theorem 2.1. We start to verify its conditions. The left-hand side of inequality (2.3) holds since, by (2.22),

$$F_e(t, \varphi) = c(t)\varphi(h(t) - t) > 0$$

for every $(t,\varphi) \in \Omega_1$ where $\varphi(\theta) > 0$, $\theta \in [-r,0)$. The left-hand side of inequality (2.4) is true as well because, by (2.20),

$$F_s(t,\varphi) = C(t)\varphi(h(t)-t) \gg 0_n$$

due to the non-positivity of entries of matrix $C(t)$ and the positivity of $c(t)$ for every $(t,\varphi) \in \Omega_n$ where $\varphi(\theta) \gg 0_n$, $\theta \in [-r,0)$. Condition (2.1) obviously holds since

$$F_e(t,\varphi) = c(t)\varphi(h(t)-t) = \sum_{j=1}^{n} c_{ij}(t)\varphi(h(t)-t) = F_{si}(t,\varphi,\dots,\varphi)$$

for every $(t,\varphi) \in \Omega_1$. The verification of (2.2) is trivial as well as the verification of the monotony properties (2.3) and (2.4).

Then, by Theorem 2.1, the existence of a positive solution $y = y(t)$ on $[t^* - r, \infty)$ of system (2.18) is equivalent to the existence of a positive solution $x = x(t)$ on $[t^* - r, \infty)$ of equation (2.21). □

3.4 Proof of Theorem 2.7

The proof is based on Theorem 2.2. To apply it we consider system (2.6) defined as

$$\dot{y}(t) = -F^*(t,y_t) := -F_s(t,y_t) \tag{3.13}$$

and system (2.7) defined as

$$\dot{y}(t) = -F^{**}(t,y_t) := -L_s(t,y_t). \tag{3.14}$$

Condition (2.8) holds due to (2.27). System (3.14) has a positive solution

$$y = y^{**}(t) = (y_1^{**}(t), y_2^{**}(t), \dots, y_n^{**}(t)) = (\exp(-t/r), \exp(-t/r), \dots, \exp(-t/r))$$

on $[t_0 - r, \infty)$ due to (2.28). Moreover, (2.9) holds due to (2.30). Theorem 2.2 is applicable and system (3.13) has a positive solution $y = y(t)$ on $[t_0 - r, \infty)$ such that (2.10) holds, i.e.,

$$y_i(t) := y_i^*(t) < \exp(-t/r), \qquad i = 1, 2, \dots, n.$$ □

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