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Instability for a class of second order delay differential equations

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Abstract. There exists a well-developed stability theory for many classes of functional-differential equations and only a few results on their instability. One of the aims of this paper is to fill this gap. Explicit tests for instability of linear delay differential equations of the second order with damping terms are obtained.

Keywords: instability, second order delay differential equations, damping terms.

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1 Introduction

The ordinary second order differential equation

$$x''(t) + p(t)x(t) = 0, \qquad t \in [0, \infty), \tag{1.1}$$

with a nonnegative coefficient $p(t)$ is one of the classical objects in a qualitative theory of linear differential equations. In spite of the quite simple shape of this equation it appears to provide a variety of different oscillatory and asymptotic properties to its solutions. Asymptotic properties of the solutions have been studied in the classical monographs by R. Bellman [5], G. Sansone [26], P. Hartman [13] and I. T. Kiguradze and T. A. Chanturia [17]. The latter states the current situation in the subject and at the same time encourages further investigation. Some of the most important results are various generalizations of oscillation and asymptotic results to equations with delayed argument.

The second order delay differential equation (DDE)

$$x''(t) + p(t)x(t - \tau(t)) = 0, \qquad t \in [0, +\infty), \tag{1.2}$$

has its own history. Oscillation and asymptotic properties of this equation were considered in the well known monographs by A. D. Myshkis [22], S. B. Norkin [24], G. S. Ladde, V. Lakshmikantham and B. G. Zhang [21], I. Győri and G. Ladas [16], L. N. Erbe, Q. Kong and B. G. Zhang [12].

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A delay equation is generally known to inherit oscillation properties of the corresponding ordinary equation. For example, it was proven by J. J. A. M. Brands [7] that for each bounded delay $\tau(t)$, equation (1.2) is oscillatory if and only if the corresponding ordinary differential equation (1.1) is oscillatory. Thus, the following question arises: are the asymptotic properties of an ordinary differential equation inherited by a delay equation? The answer is negative. A. D. Myshkis [22] proved that there exists an unbounded solution of the equation

$$x''(t) + px(t-\varepsilon) = 0, \qquad t \in [0, +\infty),$$

for each couple of positive constants p and ε. The problem of unboundedness of solutions in the case of non-constant coefficients was formulated in [22] as one that needs to be solved. The first results on this subject were obtained by A. Domoshnitsky in [10]: if there exists a positive constant ε such that $p(t) > \varepsilon, \tau(t) \geq \varepsilon$ for $t \geq 0$, then there exists an unbounded solution to equation (1.2). A necessary and sufficient condition for the boundedness of all solutions to equation (1.2) in the case of bounded and nondecreasing $p(t) \geq 0$ is the following [10]

$$\int^{\infty} \tau(t)dt < \infty.$$

It was proved in [23] that if $0 \leq \tau(t) \leq \tau$ and $tp(t)$ is an integrable function on the semi-axis then equation (1.2) is unstable.

In this paper we consider instability problems for DDE of the second order with damping terms. Various stability and instability results for *autonomous* DDE of the second order with damping terms were obtained in [9]. Other stability and instability results for this class of equations can be found in [8, 28] and also in monographs [18, 19]. All of these results were obtained by analysis of the roots of the characteristic equation. Concerning stability of second order equations, the recent papers [2] (equations with damping terms) and [11] (equations without damping term) can be noted. Results on oscillation and asymptotic properties for functional differential equations of the second order can be found in [3, 15, 20, 25, 29] (see also references therein).

There are many mathematical models described by delay differential equations of the second order with damping terms [19]. In the case of instability, one of the stabilizing methods is used. Before an application of such methods it is necessary to check that this equation is unstable. So stability/instability investigations are an important part of the theory and applications of DDE.

To the best of our knowledge the problem of instability for *non-autonomous* DDE of the second order with damping terms, which is the aim of this paper, has not been studied before. For such equations, the method of analysis of the characteristic equation does not work and researchers need other methods for analysis. We consider several methods, in particular perturbations by asymptotically small terms, applications of properties of non-oscillatory equations, reducing a differential equation of the second order to a system of two equations of the first order and some other approaches. Some of these methods are new and can be used to study instability for other classes of functional differential equations.

We discuss several open problems on instability in the last section of this paper.

2 Preliminaries

We consider the following equation

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k(t)\dot{x}(g_k(t)) + \sum_{k=1}^{l} b_k(t)x(h_k(t)) = 0, \qquad t \geq 0, \tag{2.1}$$

where

(a1) a_k, b_k are measurable essentially bounded function on $[0, \infty)$, $g_k(t) \leq t, h_k(t) \leq t$, $t \geq 0$ are measurable delay functions and

(a2) $$\limsup_{t\to\infty}(t - g_k(t)) < \infty, \qquad \limsup_{t\to\infty}(t - h_k(t)) < \infty.$$

Together with (2.1) consider for each $t_0 \geq 0$ an initial value problem

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k(t)\dot{x}(g_k(t)) + \sum_{k=1}^{l} b_k(t)x(h_k(t)) = f(t), \qquad t > t_0, \tag{2.2}$$

$$x(t) = \varphi(t), \quad \dot{x}(t) = \psi(t), \quad t < t_0, \quad x(t_0) = x_0, \quad \dot{x}(t_0) = x_0^{'}. \tag{2.3}$$

We also assume that the following hypothesis holds

(a3) $f : [t_0, \infty) \to \mathbb{R}$ is a Lebesgue measurable locally essentially bounded function, $\varphi : (-\infty, t_0) \to \mathbb{R}, \psi : (-\infty, t_0) \to \mathbb{R}$ are continuous bounded functions, $x_0, x_0^{'}$ are real numbers.

Definition 2.1. Suppose a function $x : [t_0, \infty) \to R$ is differentiable and $\dot{x}$ is a locally absolutely continuous function. Let us extend the functions x and $\dot{x}$ for $t \leq t_0$ by the help of equalities (2.3). We say that so extended function x is a *a solution* of problem (2.2), (2.3) if it satisfies equation (2.2) for almost every $t \in (t_0, \infty)$.

Let functions x_1 and x_2 be solutions of (2.1) for $t \geq t_0$ with the conditions $x_1(t) = x_2(t) = 0$, $t < t_0$; $x_1(t_0) = 1$, $x_1^{'}(t_0) = 0$, $x_2(t_0) = 0$, $x_2^{'}(t_0) = 1$.

The fundamental solution $X(t,s)$ is the solution of (2.1) for $t \geq s \geq 0$ with the initial conditions

$$\dot{x}(t) = x(t) = 0, \qquad t < s; \; x(s) = 0, \; \dot{x}(s) = 1.$$

It is evident that $x_2(t) = X(t, t_0)$.

Lemma 2.2 ([4]). *Let (a1)–(a3) hold. Then there exists one and only one solution of problem* (2.2), (2.3) *that can be presented in the form*

$$\begin{aligned} x(t) &= x_1(t)x_0 + x_2(t)x_0^{'} + \int_{t_0}^{t} X(t,s)f(s)ds \\ &\quad - \int_{t_0}^{t} X(t,s) \left[\sum_{k=1}^{m} a_k(s)\psi(g_k(s)) + \sum_{k=1}^{l} b_k(s)\varphi(h_k(s)) \right] ds, \qquad t \geq t_0 \end{aligned}$$

where $\varphi(h_k(s)) = 0$ if $h_k(s) > t_0$ and $\psi(g_k(s))) = 0$ if $g_k(s) > t_0$.

Definition 2.3. Equation (2.1) is *uniformly exponentially stable, if there exist* $M > 0$, $\mu > 0$, *such that the solution of problem* (2.2), (2.3) *with* $f \equiv 0$ *has the estimate*

$$\max\{|\dot{x}(t)|, |x(t)|\} \le M\, e^{-\mu(t-t_0)} \left[\sup_{t<t_0} |\varphi(t)| + \sup_{t<t_0} |\psi(t)|\right], \qquad t \ge t_0,$$

where M *and* μ *do not depend on* t_0.

We say that the fundamental solution has an exponential estimate if there exist positive numbers $\lambda > 0, N > 0$ *such that*

$$|X(t,s)| \le Ne^{-\lambda(t-s)}, \qquad t \ge s \ge t_0.$$

It is known that for bounded delays (i.e. (a2) holds), the existence of an exponential estimate for the fundamental solution is equivalent to exponential stability and also it is equivalent to uniform asymptotic stability [14].

Definition 2.4. We will say, that equation (2.1) is not exponentially stable (exponentially unstable), if the fundamental solution of the equation has no exponential estimate.

In particular equation (2.1) is not exponentially stable if it has a solution which does not tend to zero as $t \to \infty$.

Lemma 2.5 ([1, 16])**.** *Assume that*

$$\int_{\max\{t_0, g(t)\}}^{t} \sum_{k=1}^{m} a_k^{+}(s)ds \le \frac{1}{e}, \qquad t \ge t_0 \ge 0, \tag{2.4}$$

where $g(t) = \min_k g_k(t), g_k(t) \le t, t \ge t_0$. *Then the fundamental solution* $Z(t,s)$ *of the equation*

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(g_k(t) = 0, \qquad t \ge t_0$$

is positive, i.e.

$$Z(t,s) > 0, \qquad t \ge s \ge t_0.$$

Lemma 2.6 ([1, Theorem 8.3])**.** *Suppose the fundamental solution* $X(t,s)$ *of equation* (2.1) *is non-negative and* $c_k(t) \le a_k(t)$, $d_k(t) \le b_k(t)$, $t \ge t_0$. *Then for the fundamental solution* $Y(t,s)$ *of the equation*

$$\ddot{y}(t) + \sum_{k=1}^{m} c_k(t)\dot{y}(g_k(t)) + \sum_{k=1}^{l} d_k(t)y(h_k(t)) = 0$$

the following inequality holds $Y(t,s) \ge X(t,s) \ge 0$, $t \ge s \ge t_0$.

3 Asymptotically small coefficients

Definition 3.1. We will say that a locally measurable scalar function a is *asymptotically small* if $\lim_{t\to\infty} \int_t^{t+1} |a(s)|ds = 0$. A matrix-function A is asymptotically small if all elements of the matrix are asymptotically small.

In particular, a function a is asymptotically small if at least one of the following conditions hold:

a) $\lim_{t\to\infty} a(t) = 0$,

b) $\int_{t_0}^{\infty} |a(s)|ds < \infty$.

Consider the following equation which is a perturbation of equation (2.1).

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k(t)\dot{x}(g_k(t)) + \sum_{k=1}^{l} b_k(t)x(h_k(t)) + \sum_{k=1}^{\tilde{m}} \tilde{a}_k(t)\dot{x}(\tilde{g}_k(t)) + \sum_{k=1}^{\tilde{l}} \tilde{b}_k(t)x(\tilde{h}_k(t)) = 0 \tag{3.1}$$

where for the parameters of (3.1) conditions (a1)–(a2) hold.

Theorem 3.2. *Assume equation (2.1) is not exponentially stable and $\tilde{a}_k$ and $\tilde{b}_k$ are asymptotically small. Then equation (3.1) is not exponentially stable.*

Proof. For a vector linear delay differential equation it is known [14] that exponential stability is preserved under asymptotically small perturbations. Hence exponential instability of this equation is also preserved under asymptotically small perturbations. Equations (2.1) and (3.1) can be written in a vector form. Hence exponential instability of equation (2.1) implies exponential instability of equation (3.1). □

Corollary 3.3. *If there exists*

$$\lim_{t\to\infty} a(t) = a \in \mathbb{R}, \qquad \lim_{t\to\infty} b(t) = b > 0$$

then the equation

$$\ddot{x}(t) + a(t)\dot{x}(t-\sigma) - b(t)x(t-\tau) = 0 \tag{3.2}$$

is not exponentially stable.

Proof. Consider the autonomous equation

$$\ddot{x}(t) + a\dot{x}(t-\sigma) - bx(t-\tau) = 0. \tag{3.3}$$

The characteristic equation of (3.3) is

$$f(\lambda) := \lambda^2 + a\lambda e^{-\lambda\sigma} - be^{-\lambda\tau} = 0.$$

We have $f(0) = -b < 0, \lim_{\lambda\to\infty} f(\lambda) = +\infty$. Hence equation (3.3) has an unbounded solution $x(t) = e^{\lambda t}, \lambda > 0$. Equation (3.2) can be rewritten in the form

$$\ddot{x}(t) + a\dot{x}(t-\sigma) - bx(t-\tau) + (a(t)-a)\dot{x}(t-\sigma) - (b(t)-b)x(t-\tau) = 0,$$

where

$$\lim_{t\to\infty}(a(t)-a) = \lim_{t\to\infty}(b(t)-b) = 0.$$

By Theorem 3.2 equation (3.2) is not exponentially stable. □

Corollary 3.4. *Assume that $b_k, k = 1,\ldots,l$ are asymptotically small. Then equation (2.1) is not exponentially stable.*

Proof. The equation

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k(t)\dot{x}(g_k(t)) = 0$$

has a constant solution $x(t) \equiv 1$, hence by Theorem 3.2 equation (2.1) is not exponentially stable. □

The equation

$$\ddot{x}(t) + \sum_{k=1}^{l} b_k(t)x(h_k(t)) = 0 \tag{3.4}$$

without damping term is not exponentially stable if any function b_k is asymptotically small. In the next corollary we will improve this statement.

Corollary 3.5. *If $\sum_{k=1}^{l} b_k$ is asymptotically small then equation (3.4) is not exponentially stable.*

Proof. Rewrite equation (3.4)

$$\ddot{x}(t) + \sum_{k=1}^{l} b_k(t)x(t) - \sum_{k=1}^{l} b_k(t)(x(t) - x(h_k(t))) = 0,$$

or

$$\ddot{x}(t) - \sum_{k=1}^{l} b_k(t) \int_{h_k(t)}^{t} \dot{x}(s)ds + \sum_{k=1}^{l} b_k(t)x(t) = 0. \tag{3.5}$$

The equation

$$\ddot{x}(t) - \sum_{k=1}^{l} b_k(t) \int_{h_k(t)}^{t} \dot{x}(s)ds = 0$$

has a constant solution $x(t) \equiv 1$ hence it is not exponentially stable. Then equation (3.5) is not exponentially stable. □

4 Application of positivity of the fundamental solution

Theorem 4.1. *Assume that the fundamental solution $Z(t,s)$ of the equation*

$$\dot{z}(t) + \sum_{k=1}^{m} a_k(t)z(g_k(t)) = 0 \tag{4.1}$$

is positive and $b_k(t) \le 0, t \ge t_0, k = 1,\dots,l$. Then (2.1) is not exponentially stable.

Proof. Consider the following problem

$$\ddot{y}(t) + \sum_{k=1}^{m} a_k(t)\dot{y}(g_k(t)) = 0, \qquad t > t_0, \tag{4.2}$$

$$y(t) = \dot{y}(t) = 0, \quad t < t_0; \quad y(t_0) = 0, \quad \dot{y}(t_0) = 1.$$

It is easy to see, that the solution y of this problem has the form

$$y(t) = \int_{t_0}^{t} Z(s,t_0)ds, \qquad t \ge t_0.$$

Hence $y(t) \ge 0, t \ge t_0$. But $y(t) = Y(t,t_0)$ where $Y(t,s)$ is the fundamental solution of (4.2). Hence the fundamental solution of (4.2) is nonnegative.

Since constants are solutions of (4.2) then this equation is not exponentially stable. Equation (4.2) one can write in the form

$$\ddot{y}(t) + \sum_{k=1}^{m} a_k(t)\dot{y}(g_k(t)) + \sum_{k=1}^{l} 0 \cdot y(h_k(t)) = 0. \tag{4.3}$$

Since $b_k(t) \le 0, t \ge t_0$ then by Lemma 2.6 $X(t,s) \ge Y(t,s) \ge 0, t \ge s \ge t_0$, where $X(t,s)$ is the fundamental solution of (2.1). Then (2.1) is not exponentially stable. □

Corollary 4.2. *Suppose condition* (2.4) *holds and* $b_k(t) \leq 0, t \geq t_0, k = 1, \dots, l$. *Then* (2.1) *is not exponentially stable.*

Proof. Lemma 2.5 implies that the fundamental solution of equation 4.1 is positive. ☐

Corollary 4.3. *Suppose* $b_k(t) \leq 0, t \geq t_0, k = 1, \dots, l$. *Then the equation*

$$\ddot{x}(t) + a(t)\dot{x}(t) + \sum_{k=1}^{l} b_k(t)x(h_k(t)) = 0 \tag{4.4}$$

is not exponentially stable.

Proof. The fundamental solution of the ordinary differential equation

$$\dot{z}(t) + a(t)z(t) = 0, \qquad t \geq t_0,$$

is positive for any function a. ☐

5 Equation with a negative damping term

In the previous section we obtained instability conditions of equation (2.1) where $b_k(t) \leq 0$, $t \geq t_0$. The asymptotic behavior of equation (2.1) may be very complicated when $a_k(t) \leq 0$, $t \geq t_0$. In [19], page 241, the following autonomous equation was studied

$$\ddot{x} + a\dot{x}(t-1) + bx(t) = 0.$$

It was shown that for any $a, |a| < \pi$ there exist infinitely many intervals $(0, b_1), (b_1, b_2), \dots$ for parameter b such that the equation switches stability to instability and vice versa.

Example 5.1. The equation

$$\ddot{x}(t) - \frac{\pi}{2}\dot{x}(t-1) + \pi x(t) = 0$$

is unstable, but the equation

$$\ddot{x}(t) - \frac{\pi}{2}\dot{x}(t-1) + \pi^2 x(t-1) = 0$$

is exponentially stable.

We consider first the following equation

$$\ddot{x}(t) - a(t)\dot{x}(t) + \sum_{k=1}^{m} b_k(t)x(h_k(t)) = 0 \tag{5.1}$$

where $a(t) \geq 0,\ t \geq t_0$.

Theorem 5.2. *Let there exist an absolutely continuous function u such that $0 \leq u(t) \leq a(t),\ t \geq t_0$ and*

$$-\dot{u}(t) + a(t)u(t) - u^2(t) - \sum_{k=1}^{m} b_k^+(t)e^{-\int_{h_k(t)}^{t} u(s)ds} \geq 0 \tag{5.2}$$

for $t \geq t_0$. Then equation (5.1) *is not exponentially stable.*

Proof. Suppose that equation (5.1) is exponentially stable. By condition (a2) the delay functions $t - h_k(t)$ are bounded hence exponential stability is equivalent (for some $N > 0,\ \lambda > 0$) to the following inequality

$$|X(t,s| \le Ne^{-\lambda(t-s)}, \qquad t \ge s \ge t_0, \tag{5.3}$$

where $X(t,s)$ is the fundamental function of equation (5.1).

Consider the initial value problem

$$\begin{aligned} \ddot{x}(t) - a(t)\dot{x}(t) + \sum_{k=1}^{m} b_k(t)x(h_k(t)) = f(t), \qquad & t > t_0, \\ \dot{x}(t) = x(t) = 0, \qquad & t \le t_0, \end{aligned} \tag{5.4}$$

where f is a bounded function on $[t_0, \infty)$ and greater than some positive number ϵ. Lemma 2.2 implies that for the solution x of the problem we have

$$x(t) = \int_{t_0}^{t} X(t,s)f(s)ds.$$

Inequality (5.3) and the boundedness of f imply that solution x of problem (5.4) is a bounded function on $[t_0, \infty)$.

Suppose $u(t), a(t) \ge u(t) \ge 0$ is a solution of (5.2) and $z(t) = \dot{x}(t) - u(t)x(t),\ z(t_0) = 0$. Hence

$$x(t) = \int_{t_0}^{t} e^{\int_s^t u(\xi)d\xi} z(s)ds, \qquad \dot{x} = z + ux, \ddot{x} = \dot{z} + uz + (\dot{u} + u^2)x.$$

We have after substitution of $x, \dot{x}, \ddot{x}$ in (5.4):

$$\begin{aligned} &\dot{z}(t) - (a(t) - u(t))z(t) \\ &\quad = \left[-\dot{u}(t) + a(t)u(t) - u^2(t) - \sum_{k=1}^{m} b_k^+(t)e^{-\int_{h_k(t)}^t u(s)ds} \right] \int_{t_0}^{t} e^{\int_s^t u(\tau)d\tau} z(s)ds \\ &\qquad + \sum_{k=1}^{m} b_k^+(t) \int_{h_k(t)}^{t} e^{\int_{h_k(t)}^t u(\tau)d\tau} z(s)ds + \sum_{k=1}^{m} b_k^-(t) \int_{t_0}^{h_k(t)} e^{\int_s^{h_k(t)} u(\tau)d\tau} z(s)ds + f(t). \end{aligned}$$

Inequality (5.2) implies that $\dot{z}(t) - (a(t) - u(t))z(t) \ge f(t)$. We have $f(t) \ge \epsilon, t \ge t_0$. Then

$$z(t) \ge \epsilon \int_{t_0}^{t} e^{\int_s^t (a(\tau) - u(\tau))d\tau} ds, \qquad t \ge t_0,$$

and therefore

$$x(t) \ge \epsilon \int_{t_0}^{t} e^{\int_s^t u(\tau)d\tau} \int_{t_0}^{s} e^{\int_\xi^s (a(\tau) - u(\tau))d\tau} d\xi ds \ge \epsilon \int_{t_0}^{t} \int_{t_0}^{s} d\xi ds = \epsilon \frac{(t - t_0)^2}{2}.$$

It is evident that x is an unbounded function. We have a contradiction with the assumption, hence equation (5.1) is not exponentially stable. □

Corollary 5.3. *Let $b_k^+(t) \le \tilde{b}_k,\ a(t) \ge \tilde{a} > 0,\ t - h_k(t) \ge \tau_k,\ t \ge t_0$, and*

$$\sum_{k=1}^{m} \tilde{b}_k e^{-\frac{\tau_k \tilde{a}}{2}} \le \frac{\tilde{a}^2}{2}.$$

Then equation (5.1) is not exponentially stable.

Proof. $u(t) \equiv \frac{\tilde{a}}{2}$ is a solution of inequality (5.2). □

Consider now an equation with delay in the damping term

$$\ddot{x}(t) - a(t)\dot{x}(g(t)) + b(t)x(h(t)) = 0, \tag{5.5}$$

where

$$a(t) \geq \tilde{a} > 0, \quad |b(t)| \leq \tilde{b}, \quad t - g(t) \leq \sigma, \quad \sigma \geq 0, \quad t - h(t) \geq \tau \geq 0, \quad t \geq t_0.$$

Theorem 5.4. *Assume there exists $\lambda > 0$ such that the following inequality holds*

$$\lambda^2 - \lambda e^{-\lambda\sigma}\tilde{a} + e^{-\lambda\tau}\tilde{b} \leq 0. \tag{5.6}$$

Then equation (5.5) is exponentially unstable.

Proof. Suppose $\lambda > 0$ is a solution of (5.6). Then the function $x(t) = e^{\lambda t}$ is a solution of the following problem

$$\ddot{x}(t) - a(t)\dot{x}(g(t)) + b(t)x(h(t)) = f(t), \tag{5.7}$$

where

$$x(t) = e^{\lambda t}, \quad \dot{x}(t) = \lambda e^{\lambda t}, \quad t \leq t_0, \quad x_0 = x(t_0) = e^{\lambda t_0}, \quad x_0' = \dot{x}(t_0) = \lambda e^{\lambda t_0},$$
$$f(t) = e^{\lambda t}(\lambda^2 - \lambda e^{-\lambda(t-g(t))}a(t) + e^{-\lambda(t-h(t))}b(t)).$$

By Lemma 2.2

$$\begin{aligned} e^{\lambda t} &= x_1(t)x_0 + x_2(t)x_0' + \int_{t_0}^t X(t,s)f(s)ds \\ &- \int_{t_0}^t X(t,s)[a(s)\psi(g(s)) + b(s)\varphi(h(s))]ds, \qquad t \geq t_0, \end{aligned} \tag{5.8}$$

where $X(t,s)$ is the fundamental solution of equation (5.5). Suppose that equation (5.5) is exponentially stable and $|X(t,s)| \leq Ne^{-\mu(t-s)}, t \geq s \geq t_0, N > 0, \mu > 0$. Then $\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = 0$ and the second integral in (5.8) also tends to zero since $\psi(g(s)) = \varphi(h(s)) = 0, t > t_0 + \max\{\sigma, \tau\}$.

For the function $f(t)$ we have

$$f(t) \leq e^{\lambda t}\left(\lambda^2 - \lambda e^{-\lambda\sigma}\tilde{a} + e^{-\lambda\tau}\tilde{b}\right), \qquad t \geq t_0.$$

Hence for the first integral in (5.8) we have

$$\int_{t_0}^t X(t,s)f(s)ds \leq e^{\lambda t}\left(\lambda^2 - \lambda e^{-\lambda\sigma}\tilde{a} + e^{-\lambda\tau}\tilde{b}\right)\frac{N}{\lambda+\mu} \leq 0.$$

We have a contradiction, since the left-hand side of (5.8) is a unbounded positive function, but the right-hand side is non-positive for t sufficiently large.

Thus equation (5.5) is exponentially unstable. □

Corollary 5.5. *Assume the following inequality holds*

$$\frac{\tilde{a}}{\sigma e} \geq \frac{1}{\sigma^2} + \tilde{b}e^{-\frac{\tau}{\sigma}}.$$

Then equation (5.5) is exponentially unstable.

Proof. $\lambda = \frac{1}{\sigma}$ is a solution of inequality (5.6). □

Example 5.6. By Corollary 5.5 the following equation

$$\ddot{x}(t) - 4\dot{x}(t-1) + bx(t-1) = 0$$

is exponentially unstable for $b < 4 - e \approx 1.28$. This result is supported numerically (MATLAB).

6 Reducing to a system of two first order equations

Consider the autonomous system

$$\dot{x}(t) = \sum_{k=1}^{m} A_k x(t-\tau_k), \tag{6.1}$$

where $A_k, k = 1, \dots, m$ are constant $n \times n$ matrices, $\tau_k \geq 0, k = 1, \dots, m$.

Lemma 6.1 ([27]). *If the determinant* $\det(-\sum_{k=1}^{m} A_k) < 0$*, then system* (6.1) *is unstable.*

To apply Lemma 6.1 consider the following autonomous equation

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k \dot{x}(t-\delta_k) + \sum_{k=1}^{l} b_k x(t-\tau_k) = 0, \tag{6.2}$$

where $\delta_k \geq 0, \tau_k \geq 0$.

Theorem 6.2. *If* $\sum_{k=1}^{l} b_k < 0$*, then equation* (6.2) *is unstable.*

Proof. Denote $\dot{x} = x_1,\ x = x_2,\ x = \{x_1, x_2\}^T$,

$$A_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad A_k = \begin{pmatrix} -a_k & 0 \\ 0 & 0 \end{pmatrix}, \qquad k = 1, \dots, m,$$

$$B_k = \begin{pmatrix} -b_k & 0 \\ 0 & 0 \end{pmatrix}, \qquad k = 1, \dots, l.$$

Then equation (6.2) can be written in the form form

$$\dot{x} = A_0 x + \sum_{k=1}^{m} A_k x(t-\delta_k) + \sum_{k=1}^{l} B_k x(t-\tau_k). \tag{6.3}$$

We have

$$\sum_{k=0}^{m} A_k + \sum_{k=1}^{l} B_k = \begin{pmatrix} -\sum_{k=1}^{m} a_k & -\sum_{k=1}^{l} b_k \\ 1 & 0 \end{pmatrix},$$

then

$$\det\left(-\left(\sum_{k=0}^{m} A_k + \sum_{k=1}^{l} B_k\right)\right) = \sum_{k=1}^{l} b_k < 0.$$

By Lemma 6.1 system (6.3) is unstable. Hence equation (6.2) is unstable. □

7 Discussion and topics for a future research

For instability we know very little even for the simple nonautonomous equation

$$\dot{x}(t) + b(t)x(h(t)) = 0. \tag{7.1}$$

We know that equation (7.1) is not exponentially stable if b is asymptotically small or $b(t) \leq 0$, $t \geq t_0$. We do not know if the last condition can be replaced by an integral one:

$$\limsup_{t\to\infty} \int_t^{t+1} b(s)ds \leq 0.$$

Also it is known that if $b(t) \geq b_0 > 0, t \geq t_0$ and

$$\limsup_{t\to\infty} \int_{h(t)}^t b(s)ds < \frac{3}{2}$$

then equation (7.1) is exponentially stable. But we do not know if there exists $b_0 > \frac{3}{2}$ such that a condition

$$\liminf_{t\to\infty} \int_{h(t)}^t b(s)ds > b_0$$

implies the instability of equation (7.1).

In this paper we obtain several instability conditions for linear delay differential equations of the second order assuming that all or part of the coefficients are asymptotically small and/or negative. We suppose that the results of the paper can be improved or extended for a more general class of equations. Bellow we present some of such problems for a future research.

1. Prove or disprove that equation 2.1 is exponentially unstable if $\sum_{k=1}^{l} b_k(t) \leq b_0 < 0$, $t \geq t_0$.

2. Prove or disprove that there exists $a_0 < 0$ such that the condition $\sum_{k=1}^{m} a_k(t) < a_0$, $t \geq t_0$ implies exponential instability of equation (2.1).

3. Derive sufficient conditions of asymptotic/exponential instability for nonlinear equations
$$\ddot{x}(t) + f(t, \dot{x}(g(t))) + p(t, x(h(t))) = 0.$$

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