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Positive solutions for higher-order nonlinear fractional differential equation with integral boundary condition*

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Abstract: In this paper, we study a kind of higher-order nonlinear fractional differential equation with integral boundary condition. The fractional differential operator here is the Caputo's fractional derivative. By means of fixed point theorems, the existence and multiplicity results of positive solutions are obtained. Furthermore, some examples given here illustrate that the results are almost sharp.

Keywords: Fractional differential equation; Positive solution; Boundary value problem; Higher-order; Integral boundary condition.

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1. Introduction

We are interested in the following nonlinear fractional differential equation

$$D_{0+}^{\tau}u(t)-\sum_{i=1}^{n-1}a_iD_{0+}^{\tau-i}u(t)=f(t,u(t),u'(t)),\quad 0<t<1,\ n-1<\tau<n, \tag{1.1}$$

subject to the boundary conditions

$$u(1)-u(0)=\sum_{i=1}^{n-1}a_i[I_{0+}^{i}u(t)]_{t=1},\quad u^{(k)}(0)=b_k,\ k=1,2,\cdots,n-1, \tag{1.2}$$

where D_{0+}^{τ} is the the Caputo's fractional derivative of order τ, $n\in\mathbb{N}$, $n\geq 2$.

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(i) $a_i \geq 0$, $b_i \geq a_i \cdot \sum\limits_{k=1}^{n-i-1} \frac{i!}{(k+i)!} b_k$ for $i = 1, 2, \cdots, n-1$, and $0 < \sum\limits_{i=1}^{n-1} \frac{a_i}{(i-1)!} < 1$,

(ii) $f : [0,1] \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is continuous.

There has been a significant development in fractional differential equations (in short:FDEs) in recent years. The motivation for those works stems from both the development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, mechanics, chemistry, engineering, etc. For an extensive collection of such results, we refer the readers to the monographs by Samko et al [1], Podlubny [2], Miller and Ross [3] and Kilbas et al [4].

Some basic theory for the initial value problems of FDE involving Riemann-Liouville differential operator has been discussed by Lakshmikantham and Vatsala [5,6,7], A. M. A. El-Sayed et al [8,9], Kai Diethelm and Neville J. Ford [10] and C. Bai [11], etc. Also, there are some papers which deal with the existence and multiplicity of solutions for nonlinear FDE boundary value problems (in short:BVPs) by using techniques of topological degree theory (see [12-15,20,21] and the references therein). For example, Bai and L$\ddot{u}$ [12] obtained positive solutions of the two-point BVP of FDE

$$D_{0+}^{\alpha} u(t) = f(t, u(t)), \quad 0 < t < 1, \quad 1 < \alpha \leq 2, \tag{1.3}$$

$$u(0) = u(1) = 0 \tag{1.4}$$

by means of Krasnosel'skii fixed point theorem and Leggett-Williams fixed point theorem. D_{0+}^{α} is the standard Riemann-Liouville fractional derivative.

Zhang discussed the existence of solutions of the nonlinear FDE

$$^{c}D_{0+}^{\alpha} u(t) = f(t, u(t)), \quad 0 < t < 1, \quad 1 < \alpha \leq 2 \tag{1.5}$$

with the boundary conditions

$$u(0) = \nu \neq 0, \quad u(1) = \rho \neq 0 \tag{1.6}$$

and

$$u(0) + u'(0) = 0, \quad u(1) + u'(1) = 0 \tag{1.7}$$

in [13] and [14], respectively. Since conditions (1.6) and (1.7) are not zero boundary value, the Riemann-Liouville fractional derivative D_{0+}^{α} is not suitable. Therefore, the author investigated the BVPs (1.5)-(1.6) and (1.5)-(1.7) by involving in the Caputo's fractional derivative $^{c}D_{0+}^{\alpha}$.

In [15], M. Benchohra et al considered the following BVP:

$$\begin{cases} {}^cD^\alpha y(t) = f(t,y), \quad \text{for each } t \in [0,T],\ 1 < \alpha \leq 2, \\ y(0) - y'(0) = \int_0^T g(s,y)ds, \\ y(T) + y'(T) = \int_0^T h(s,y)ds, \end{cases} \tag{1.8}$$

where ${}^cD^\alpha$ is the Caputo's fractional derivative. By using a series of fixed point theorems, some existence results were given.

From above works, we can see two facts: the first, although the BVPs of nonlinear FDE have been studied by some authors, to the best of our knowledge, higher-order fractional equations with integral boundary conditions are seldom considered; the second, the author in [15] studied the BVP with integral conditions, however, those results can't ensure the solutions to be positive. Since only positive solutions are useful for many applications, we investigate the existence and multiplicity of positive solutions for BVP (1.1)-(1.2) in this paper. In addition, two examples are given to demonstrate our results.

2. Preliminaries

For the convenience of the reader, we first recall some definitions and fundamental facts of fractional calculus theory, which can be found in the recent literatures [1-4].

Definition 2.1. The fractional integral of order $\tau > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0+}^\tau f(x) = \frac{1}{\Gamma(\tau)} \int_0^x \frac{f(t)}{(x-t)^{1-\tau}} dt, \quad x > 0, \tag{2.1}$$

provided that the integral exists, where $\Gamma(\tau)$ is the Euler gamma function defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad (z > 0), \tag{2.2}$$

for which, the reduction formula

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(1) = 1,\ \Gamma(\frac{1}{2}) = \sqrt{\pi} \tag{2.3}$$

and the Dirichlet formula

$$\int_0^1 t^{z-1}(1-t)^{\omega-1} dt = \frac{\Gamma(z)\Gamma(\omega)}{\Gamma(z+\omega)}, \quad (z, \omega \notin \mathbb{Z}_0^-) \tag{2.4}$$

hold.

Definition 2.2. The Caputo's fractional derivative of order $\tau > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ can be written as

$$D^{\tau}_{0+} f(x) = \frac{1}{\Gamma(n - \tau)} \int_0^x \frac{f^{(n)}(t)}{(x - t)^{\tau+1-n}} dt, \quad n = [\tau] + 1, \tag{2.5}$$

where $[\tau]$ denotes the integer part of τ, provided that the right side is pointwise defined on $(0, \infty)$.

Lemma 2.1. Let $\tau > 0$, then the differential equation

$$D^{\tau}_{0+} f(x) = 0$$

has solutions $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1}$. Moreover,

$$I^{\tau}_{0+} D^{\tau}_{0+} f(x) = f(x) - (c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1})$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \cdots, n - 1$.

We present the useful Lemmas which are fundamental in the proof of our main results below.

Lemma 2.2[16]**.** *Let C be a convex subset of a normed linear space E, and U be an open subset of C with $p^* \in U$. Then every compact continuous map $N : \overline{U} \to C$ has at least one of the following two properties:*

(A1) *N has a fixed point;*

(A2) *there is an $x \in \partial U$ with $x = (1 - \overline{\lambda})p^* + \overline{\lambda} N x$ for some $0 < \overline{\lambda} < 1$.*

Lemma 2.3[17]**.** *Let C be a closed convex nonempty subset of Banach space E. Suppose that A and B map C into E such that*

(A1) *$x, y \in C$ imply $Ax + By \in C$;*

(A2) *A is a contraction mapping;*

(A2) *B is compact and continuous.*

Then there exists $z \in C$ with $z = Az + Bz$.

Definition 2.3. The map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha(tx + (1 - t)y) \geq t\alpha(x) + (1 - t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Let α and β be nonnegative continuous convex functionals on the cone P, ψ be a nonnegative continuous concave functional on P. Then for positive real numbers $r > a$ and L, we define the following convex sets:

$$\begin{aligned}
&P(\alpha, r; \beta, L) = \{x \in P:\ \alpha(x) < r, \beta(x) < L\},\\
&\overline{P}(\alpha, r; \beta, L) = \{x \in P:\ \alpha(x) \le r, \beta(x) \le L\},\\
&P(\alpha, r; \beta, L; \psi, a) = \{x \in P:\ \alpha(x) < r, \beta(x) < L, \psi(x) > a\},\\
&\overline{P}(\alpha, r; \beta, L; \psi, a) = \{x \in P:\ \alpha(x) \le r, \beta(x) \le L, \psi(x) \ge a\}.
\end{aligned}$$

The assumptions below about the nonnegative continuous convex functionals α, β will be used:

(B1) there exists $M > 0$ such that $||x|| \le M \max\{\alpha(x),\ \beta(x)\}$ for all $x \in P$;

(B2) $P(\alpha, r; \beta, L) \ne \emptyset$ for all $r > 0,\ L > 0$.

Lemma 2.4[18]**.** *Let P be a cone in a real Banach space E, and $r_2 \ge d > b > r_1 > 0$, $L_2 \ge L_1 > 0$. Assume that α, β are nonnegative continuous convex functionals satisfying* (B1) *and* (B2)*, ψ is a nonnegative continuous concave functional on P such that $\psi(y) \le \alpha(y)$ for all $y \in \overline{P}(\alpha, r_1; \beta, L_1)$, and $T : \overline{P}(\alpha, r_2; \beta, L_2) \to \overline{P}(\alpha, r_2; \beta, L_2)$ is a completely continuous operator. Suppose*

(C1) $\{y \in P(\alpha, d; \beta, L_2; \psi, b) : \psi(y) > b\} \ne \emptyset$, *$\psi(Ty) > b$ for $y \in \overline{P}(\alpha, d; \beta, L_2; \psi, b)$;*

(C2) *$\alpha(Ty) < r_1$, $\beta(Ty) < L_1$ for all $y \in \overline{P}(\alpha, r_1; \beta, L_1)$;*

(C3) *$\psi(Ty) > b$ for all $y \in \overline{P}(\alpha, d; \beta, L_2; \psi, b)$ with $\alpha(Ty) > d$.*

Then T has at least three fixed points $y_1, y_2, y_3 \in \overline{P}(\alpha, r_2; \beta, L_2)$ with

$$\begin{aligned}
&y_1 \in P(\alpha, r_1; \beta, L_1),\\
&y_2 \in \{y \in \overline{P}(\alpha, r_2; \beta, L_2; \psi, b) : \psi(y) > b\},\\
&y_3 \in \overline{P}(\alpha, r_2; \beta, L_2) \setminus (\overline{P}(\alpha, r_2; \beta, L_2; \psi, b) \cup \overline{P}(\alpha, r_1; \beta, L_1)).
\end{aligned}$$

3. Maim Results

Let $X = C^1[0, 1]$ with the norm $||x|| = \max\{||x||_\infty,\ ||x'||_\infty\}$, where $||\cdot||_\infty$ is the standard sup norm of the space $C[0, 1]$. Obviously, X is a Banach space. Define the cone $P \subset X$ by

$$P = \{x \in X : x(t) \ge 0,\ t \in [0, 1]\}.$$

Lemma 3.1. *Assume that $\phi \in C[0,1]$. Then $u \in X$ is a solution of the BVP*

$$\begin{cases} D_{0+}^{\tau}u(t) - \sum\limits_{i=1}^{n-1} a_i D_{0+}^{\tau-i}u(t) = \phi(t), \quad 0 < t < 1, \\ u(1) - u(0) = \sum\limits_{i=1}^{n-1} a_i [I_{0+}^{i}u(t)]_{t=1}, \quad u^{(k)}(0) = b_k, \ k = 1, 2, \cdots, n-1 \end{cases} \tag{3.1}$$

if and only if u is a solution of the fractional integral equation

$$u(t) = \Upsilon(t) + \sum_{i=1}^{n-1} \frac{a_i}{(i-1)!} \int_0^t (t-s)^{i-1} u(s)ds + \int_0^1 G(t,s)\phi(s)ds \tag{3.2}$$

where

$$\Upsilon(t) = \sum_{i=1}^{n-1} \left(\frac{b_i t^i}{i!} - a_i \sum_{k=1}^{n-i-1} \frac{b_k t^{k+i}}{(k+i)!}\right) + B(t) \sum_{i=1}^{n-1} \left(\frac{b_i}{i!} - a_i \sum_{k=1}^{n-i-1} \frac{b_k}{(k+i)!}\right), \tag{3.3}$$

$$G(t,s) = \frac{1}{\Gamma(\tau)} \begin{cases} B(t)(1-s)^{\tau-1} + (t-s)^{\tau-1}, & 0 \le s \le t \le 1, \\ B(t)(1-s)^{\tau-1}, & 0 \le t \le s \le 1, \end{cases} \tag{3.4}$$

here note

$$B(t) = \frac{1 - \sum\limits_{i=1}^{n-1} \frac{a_i t^i}{i!}}{\sum\limits_{i=1}^{n-1} \frac{a_i}{i!}}.$$

For simplicity, for $\theta \in (0, \frac{1}{2})$, let

$$\omega := \max\{\Upsilon(t) : t \in [0,1]\}, \qquad \omega^* := \max\{|\Upsilon'(t)| : t \in [0,1]\},$$

$$\omega^{**} := \min\{\Upsilon(t) : t \in [\theta, 1-\theta]\}, \qquad \sigma_1 := \max_{t \in [0,1]} \int_0^1 G(t,s)ds,$$

$$\sigma_2 := \min_{t \in [\theta, 1-\theta]} \int_0^1 G(t,s)ds, \qquad \sigma_3 := \max_{t \in [0,1]} \int_0^1 \left|\frac{\partial G(t,s)}{\partial t}\right| ds,$$

$$\sigma_4 := \min_{t \in [\theta, 1-\theta]} \int_\theta^{1-\theta} G(t,s)ds.$$

Theorem 3.1. *If there exist $g, h, l \in C([0,1], \mathbb{R}_+)$ satisfying*

$$||h||_\infty + ||l||_\infty < \min\left\{\frac{1 - \sum\limits_{i=1}^{n-1} \frac{a_i}{i!}}{\sigma_1}, \ \frac{1 - \sum\limits_{i=1}^{n-1} \frac{a_i}{(i-1)!}}{\sigma_3}\right\} \tag{3.5}$$

such that

$$f(t,x,y) \le g(t) + h(t)x + l(t)y.$$

Then the BVP (1.1)-(1.2) *has at least one positive solution.*

Proof. Consider the operator $T : P \to P$ defined by

$$(Tu)(t) = \Upsilon(t) + \sum_{i=1}^{n-1} \frac{a_i}{(i-1)!} \int_0^t (t-s)^{i-1} u(s)ds + \int_0^1 G(t,s) f(s, u(s), u'(s))ds. \tag{3.6}$$

Clearly, the fixed point of T is a positive solution of the BVP (1.1)-(1.2). In order to apply Lemma 2.2, we shall first show T is completely continuous.

T is continuous on P follows from the Lebesgue dominated convergence theorem, which is valid due to the continuity of the function f.

Now, we will show that T is relatively compact. For any given bounded set $U \subset P$, there exists $M > 0$ such that $||u|| \le M$ for all $u \in U$. We take

$$\kappa = \max\{|f(t,u,v)| : t \in [0,1], |u| \le M, |v| \le M\}.$$

For any $u \in U$,

$$\begin{aligned}
||Tu||_\infty &= \max_{t\in[0,1]} |\Upsilon(t) + \sum_{i=1}^{n-1} \frac{a_i}{(i-1)!} \int_0^t (t-s)^{i-1} u(s)ds + \int_0^1 G(t,s) f(s,u(s),u'(s))ds| \\
&\le \omega + ||u||_\infty \sum_{i=1}^{n-1} \frac{a_i}{i!} + \max_{(t,x,y)\in[0,1]\times[0,M]\times[-M,M]} f(t,x,y) \cdot \max_{t\in[0,1]} \int_0^1 G(t,s)ds \\
&\le \omega + M + \kappa\sigma_1 < \infty,
\end{aligned}$$

$$\begin{aligned}
||(Tu)'||_\infty &= \max_{t\in[0,1]} |\Upsilon'(t) + a_1 u(t) + \sum_{i=2}^{n-1} \frac{a_i}{(i-2)!} \int_0^t (t-s)^{i-2} u(s)ds \\
&\quad + \int_0^1 \frac{\partial G(t,s)}{\partial t} f(s,u(s),u'(s))ds| \\
&\le \omega^* + ||u||_\infty \sum_{i=1}^{n-1} \frac{a_i}{(i-1)!} + \max_{(t,x,y)\in[0,1]\times[0,M]\times[-M,M]} f(t,x,y) \cdot \max_{t\in[0,1]} \int_0^1 |\frac{\partial G(t,s)}{\partial t}|ds \\
&\le \omega^* + M + \kappa\sigma_3 < \infty,
\end{aligned}$$

that is, TU is uniformly bounded. For $u \in U$, let $t_1, t_2 \in [0,1]$ be such that $t_1 < t_2$, we have

$$|Tu(t_2) - Tu(t_1)| = |\int_{t_1}^{t_2} (Tu)'(s)ds| \le (\omega^* + M + \kappa\sigma_3)|t_2 - t_1| \to 0, \quad \text{as } t_2 - t_1 \to 0.$$

Notice that

$$(Tu)'(t) = \Upsilon'(t) + a_1 u(t) + \sum_{i=2}^{n-1} \frac{a_i}{(i-2)!} \int_0^t (t-s)^{i-2} u(s)ds$$

$$+\frac{1}{\Gamma(\tau)}[-\frac{\sum\limits_{i=1}^{n-1}\frac{a_it^{i-1}}{(i-1)!}}{\sum\limits_{i=1}^{n-1}\frac{a_i}{i!}}\int_0^1\frac{f(s,u(s),u'(s))}{(1-s)^{1-\tau}}ds+(\tau-1)\int_0^t\frac{f(s,u(s),u'(s))}{(t-s)^{2-\tau}}ds],$$

we have

$$\begin{aligned}
&|(Tu)'(t_2)-(Tu)'(t_1)|\\
\leq\ &|\Upsilon'(t_2)-\Upsilon'(t_1)|+a_1|u(t_2)-u(t_1)|+\sum_{i=2}^{n-1}\frac{a_i}{(i-2)!}|\int_0^{t_2}(t_2-s)^{i-2}u(s)ds-\int_0^{t_1}(t_1-s)^{i-2}u(s)ds|\\
&+\frac{1}{\Gamma(\tau)}\frac{\sum\limits_{i=2}^{n-1}a_i|t_1^{i-1}-t_2^{i-1}|}{\sum\limits_{i=1}^{n-1}\frac{a_i}{i!}}\int_0^1(1-s)^{\tau-1}f(s,u(s),u'(s))ds\\
&+\frac{1}{\Gamma(\tau-1)}|\int_0^{t_2}\frac{f(s,u(s),u'(s))}{(t_2-s)^{2-\tau}}ds-\int_0^{t_1}\frac{f(s,u(s),u'(s))}{(t_1-s)^{2-\tau}}ds|\\
\leq\ &|\Upsilon'(t_2)-\Upsilon'(t_1)|+a_1|u(t_2)-u(t_1)|+\sum_{i=2}^{n-1}\frac{a_i}{(i-2)!}[\int_0^{t_1}|(t_2-s)^{i-2}-(t_1-s)^{i-2}|u(s)ds\\
&+\int_{t_1}^{t_2}(t_2-s)^{i-2}u(s)ds]+\frac{1}{\Gamma(\tau)}\frac{\sum\limits_{i=2}^{n-1}a_i|t_1^{i-1}-t_2^{i-1}|}{\sum\limits_{i=1}^{n-1}\frac{a_i}{i!}}\int_0^1(1-s)^{\tau-1}f(s,u(s),u'(s))ds\\
&+\frac{1}{\Gamma(\tau-1)}[\int_{t_1}^{t_2}\frac{f(s,u(s),u'(s))}{(t_2-s)^{2-\tau}}ds+|\int_0^{t_1}(\frac{1}{(t_2-s)^{2-\tau}}-\frac{1}{(t_1-s)^{2-\tau}})f(s,u(s),u'(s))ds|],\\
\leq\ &|\Upsilon'(t_2)-\Upsilon'(t_1)|+a_1|u(t_2)-u(t_1)|+M\sum_{i=2}^{n-1}\frac{a_i}{(i-1)!}|t_2^{i-1}-t_1^{i-1}|\\
&+\frac{\kappa}{\Gamma(\tau+1)}\frac{\sum\limits_{i=2}^{n-1}a_i|t_1^{i-1}-t_2^{i-1}|}{\sum\limits_{i=1}^{n-1}\frac{a_i}{i!}}+\frac{\kappa}{\Gamma(\tau)}[2(t_2-t_1)^{\tau-1}+(t_2^{\tau-1}-t_1^{\tau-1})]\to 0,\quad\text{as } t_2-t_1\to 0.
\end{aligned}$$

That is, TU is an equicontinuous set. Thus, T is relatively compact. By means of the Arzela-Ascoli theorem, $T:P\to P$ is completely continuous.

In the following, let

$$Q>\max\{\frac{\omega+\sigma_1||g||_\infty}{1-\sum\limits_{i=1}^{n-1}\frac{a_i}{i!}-\sigma_1(||h||_\infty+||l||_\infty)},\ \frac{\omega^*+\sigma_3||g||_\infty}{1-\sum\limits_{i=1}^{n-1}\frac{a_i}{(i-1)!}-\sigma_3(||h||_\infty+||l||_\infty)}\}.$$

Define $\Omega = \{u \in P : ||u|| < Q\}$, then $||u||_\infty \le Q$ and $||u'||_\infty \le Q$ for $u \in \partial\Omega$.

$$\begin{aligned}||Tu||_\infty &= \max_{t\in[0,1]} |\Upsilon(t) + \sum_{i=1}^{n-1} \frac{a_i}{(i-1)!}\int_0^t (t-s)^{i-1}u(s)ds + \int_0^1 G(t,s)f(s,u(s),u'(s))ds| \\ &\le \omega + Q\sum_{i=1}^{n-1}\frac{a_i}{i!} + \sigma_1[||g||_\infty + Q(||h||_\infty + ||l||_\infty)] < Q\end{aligned}$$

and

$$\begin{aligned}||(Tu)'||_\infty &= \max_{t\in[0,1]} |\Upsilon'(t) + a_1u(t) + \sum_{i=2}^{n-1} \frac{a_i}{(i-2)!}\int_0^t (t-s)^{i-2}u(s)ds \\ &\quad + \int_0^1 \frac{\partial G(t,s)}{\partial t} f(s,u(s),u'(s))ds| \\ &\le \omega^* + Q\sum_{i=1}^{n-1}\frac{a_i}{(i-1)!} + \sigma_3[||g||_\infty + Q(||h||_\infty + ||l||_\infty)] < Q\end{aligned}$$

indicate that $||Tu|| < Q$ for $u \in \partial\Omega$. Take $p^* = 0$ in Lemma 2.2, then $u = \overline{\lambda}Tu$ $(0 < \overline{\lambda} < 1)$ for any $x \in \partial\Omega$ dose not hold. Hence, the operator T has at least a fixed point, i.e. the BVP (1.1)-(1.2) has at least one positive solution. ☐

Theorem 3.2. *Under the assumptions* (i) *and* (ii)*, the BVP* (1.1)-(1.2) *has a positive solution.*
Proof. Define the functions $T_1, T_2 : P \to P$ by

$$\begin{aligned}&(T_1u)(t) = \Upsilon(t) + \sum_{i=1}^{n-1}\frac{a_i}{(i-1)!}\int_0^t (t-s)^{i-1}u(s)ds, \\ &(T_2u)(t) = \int_0^1 G(t,s)f(s,u(s),u'(s))ds.\end{aligned}$$

For $x, y \in P$, it is easy to see that $(T_1x)(t) + (T_2y)(t) \ge 0$, i.e. $T_1x + T_2y \in P$.

Firstly, we show that T_1 is a contraction mapping. For any $u, v \in P$, we have

$$\begin{aligned}|(T_1u)(t) - (T_1v)(t)| &\le \sum_{i=1}^{n-1}\frac{a_i}{(i-1)!}\int_0^t (t-s)^{i-1}|u(s)-v(s)|ds \\ &\le \sum_{i=1}^{n-1}\frac{a_i}{i!}||u-v||_\infty \le \sum_{i=1}^{n-1}\frac{a_i}{i!}||u-v||\end{aligned}$$

and

$$|(T_1u)'(t) - (T_1v)'(t)| \le a_1|u(t)-v(t)| + \sum_{i=2}^{n-1}\frac{a_i}{(i-2)!}\int_0^t (t-s)^{i-2}|u(s)-v(s)|ds$$

$$\leq \sum_{i=1}^{n-1} \frac{a_i}{(i-1)!}||u-v||_\infty \leq \sum_{i=1}^{n-1} \frac{a_i}{(i-1)!}||u-v||.$$

Since $0 < \sum\limits_{i=1}^{n-1} \frac{a_i}{(i-1)!} < 1$ from the condition (i), T_1 is contractive.

Next, we shall verify that T_2 is completely continuous, which follows from the proof of Theorem 3.1. As a result, Lemma 2.3 implies that there exists an $x^* \in P$ such that $x^* = T_1x^* + T_2x^*$. In view of Lemma 3.1, x^* is a positive solution of the BVP (1.1)-(1.2). □

Let the nonnegative continuous convex functionals α, β and the nonnegative continuous concave functional ψ be defined on the cone P by

$$\alpha(x) = ||x||_\infty, \quad \beta(x) = ||x'||_\infty, \quad \psi(x) = \min_{\theta \leq t \leq 1-\theta} |x(t)|$$

Obviously, α and β satisfy (B1) and (B2), $\psi(x) \leq \alpha(x)$ for all $x \in P$.

Theorem 3.3. *Assume there exist constants $r_2 \geq \frac{b}{\theta} > b > r_1 > 0$, $L_2 \geq L_1 > 0$ such that $\omega < (1 - \sum\limits_{i=1}^{n-1} \frac{a_i}{i!})r_1$ and $\omega^* < L_j - r_j \sum\limits_{i=1}^{n-1} \frac{a_i}{(i-1)!}$, $j = 1, 2$. Suppose*

(H1) $f(t, u, v) \leq \min\{\frac{(1-\sum_{i=1}^{n-1} \frac{a_i}{i!})r_2 - \omega}{\sigma_1}, \frac{L_2 - \omega^* - r_2 \sum_{i=1}^{n-1} \frac{a_i}{(i-1)!}}{\sigma_3}\}$, $\quad (t, u, v) \in [0, 1] \times [0, r_2] \times [-L_2, L_2]$;

(H2) $f(t, u, v) > \frac{b - \omega^{**}}{\sigma_2}$, $\quad (t, u, v) \in [0, 1] \times [b, \frac{b}{\theta}] \times [-L_2, L_2]$;

(H3) $f(t, u, v) < \min\{\frac{(1-\sum_{i=1}^{n-1} \frac{a_i}{i!})r_1 - \omega}{\sigma_1}, \frac{L_1 - \omega^* - r_1 \sum_{i=1}^{n-1} \frac{a_i}{(i-1)!}}{\sigma_3}\}$, $\quad (t, u, v) \in [0, 1] \times [0, r_1] \times [-L_1, L_1]$;

(H4) $f(t, u, v) \geq \frac{\theta r_2 - \omega^{**}}{\sigma_4}$, $\quad (t, u, v) \in [\theta, 1-\theta] \times [b, r_2] \times [-L_2, L_2]$.

Then the BVP (1.1)-(1.2) *has at least three positive solutions x_1, x_2 and x_3 such that*

$$\begin{aligned}
&0 \leq x_i(t) \leq r_i, \quad ||x_i'||_\infty \leq L_i, \quad i = 1, 2, \\
&r_1 \leq x_3(t) \leq r_2, \quad -L_1 \leq x_3'(t) \leq L_2, \quad t \in [0, 1], \\
&x_2(t) > b, \quad x_3(t) \leq b, \quad t \in [\theta, 1-\theta].
\end{aligned}$$

Proof. Let the operator $T : P \to P$ be defined by (3.6). From the proof of Theorem 3.1, we know that T is completely continuous. Now, we will verify that all the conditions of Lemma 2.4 are satisfied. The proof is based on the following steps.

*Step*1. We will show that (H1) implies $T : \overline{P}(\alpha, r_2; \beta, L_2) \to \overline{P}(\alpha, r_2; \beta, L_2)$.

In fact, for $u \in \overline{P}(\alpha, r_2; \beta, L_2)$, $||u||_\infty \leq r_2$, $||u'||_\infty \leq L_2$. In view of (H1), we have

$$||Tu||_\infty = \max_{t \in [0,1]} |\Upsilon(t) + \sum_{i=1}^{n-1} \frac{a_i}{(i-1)!} \int_0^t (t-s)^{i-1} u(s) ds + \int_0^1 G(t, s) f(s, u(s), u'(s)) ds|$$

$$\leq \omega + r_2 \sum_{i=1}^{n-1} \frac{a_i}{i!} + \sigma_1 \max_{(t,u,v)\in[0,1]\times[0,r_2]\times[-L_2,L_2]} f(t,u,v) \leq r_2,$$

$$\begin{aligned}
||(Tu)'||_\infty &= \max_{t\in[0,1]} |\Upsilon'(t) + a_1 u(t) + \sum_{i=2}^{n-1} \frac{a_i}{(i-2)!} \int_0^t (t-s)^{i-2} u(s)ds \\
&\quad + \int_0^1 \frac{\partial G(t,s)}{\partial t} f(s,u(s),u'(s))ds| \\
&\leq \omega^* + r_2 \sum_{i=1}^{n-1} \frac{a_i}{(i-1)!} + \sigma_3 \max_{(t,u,v)\in[0,1]\times[0,r_2]\times[-L_2,L_2]} f(t,u,v) \\
&\leq L_2.
\end{aligned}$$

Thus, $Tu \in \overline{P}(\alpha, r_2; \beta, L_2)$.

*Step*2. To check the condition (C1) in Lemma 2.4, we choose $x^*(t) \equiv \frac{b}{\theta}$ on $[0,1]$. It is easy to see that $x^* \in \{x \in \overline{P}(\alpha, \frac{b}{\theta}; \beta, L_2; \psi, b) : \psi(x) > b\}$. For $u \in \overline{P}(\alpha, \frac{b}{\theta}; \beta, L_2; \psi, b)$, from (H2), one gets

$$\begin{aligned}
\min_{t\in[\theta,1-\theta]} |Tu(t)| &= \min_{t\in[\theta,1-\theta]} |\Upsilon(t) + \sum_{i=1}^{n-1} \frac{a_i}{(i-1)!} \int_0^t (t-s)^{i-1} u(s)ds + \int_0^1 G(t,s) f(s,u(s),u'(s))ds| \\
&\geq \omega^{**} + \sigma_2 \min_{(t,u,v)\in[0,1]\times[b,\frac{b}{\theta}]\times[-L,L]} f(t,u,v) > b,
\end{aligned}$$

then we can obtain $\psi(Tu) > b$.

*Step*3. It is similar to *Step*1 that we can prove $T : \overline{P}(\alpha, r_1; \beta, L_1) \to P(\alpha, r_1; \beta, L_1)$ by condition (H3), that is, (C2) in Lemma 2.4 holds.

*Step*4. We verify that (C3) in Lemma 2.4 is satisfied. For $u \in \overline{P}(\alpha, r_2; \beta, L_2; \psi, b)$ with $\alpha(Tu) > \frac{b}{\theta}$, from *Step*1, we know that $\alpha(Tu) \leq r_2$, Then, from (H4), we can obtain

$$\begin{aligned}
\min_{t\in[\theta,1-\theta]} |Tu(t)| &\geq \min_{t\in[\theta,1-\theta]} |\Upsilon(t) + \sum_{i=1}^{n-1} \frac{a_i}{(i-1)!} \int_0^t (t-s)^{i-1} u(s)ds + \int_\theta^{1-\theta} G(t,s) f(s,u(s),u'(s))ds| \\
&\geq \omega^{**} + \sigma_4 \cdot \min_{(t,u,v)\in[\theta,1-\theta]\times[b,r_2]\times[-L,L]} f(t,u,v) \\
&\geq \theta r_2 \geq \theta\alpha(Tu) > b.
\end{aligned}$$

Thus, $\psi(Tu) > b$, (C3) in Lemma 2.3 is satisfied.
Therefore, the operator T has three fixed points $x_1, x_2, x_3 \in \overline{P}(\alpha, r_2; \beta, L_2)$ with

$$\begin{aligned}
&x_1 \in P(\alpha, r_1; \beta, L_1), \quad x_2 \in P(\alpha, r_2; \beta, L_2; \psi, b), \\
&x_3 \in \overline{P}(\alpha, r_2; \beta, L_2) \setminus (P(\alpha, r_2; \beta, L_2; \psi, b) \cup P(\alpha, r_1; \beta, L_1)).
\end{aligned}$$

Then the BVP (1.1)-(1.2) has at least three positive solutions x_1, x_2 and x_3 such that

$$\begin{aligned}
&0 \le x_i(t) \le r_i, \quad ||x_i'||_\infty \le L_i, \quad i = 1, 2,\\
&r_1 \le x_3(t) \le r_2, \quad -L_1 \le x_3'(t) \le L_2, \quad t \in [0, 1],\\
&x_2(t) > b, \quad x_3(t) \le b, \quad t \in [\theta, 1-\theta].
\end{aligned}$$

□

4. Examples

In this section, we give two applications to illustrate our main results.

Example 4.1. Consider the problem

$$\begin{cases} D_{0+}^{\frac{5}{4}} x(t) - \frac{1}{e} D_{0+}^{\frac{1}{4}} x(t) = f(t, x(t), x'(t)), \quad 0 < t < 1,\\ x(1) - x(0) = \frac{1}{e}\int_0^1 x(s)ds, \ x'(0) = 1, \end{cases} \tag{4.1}$$

where

$$f(t, u, v) = t + \frac{2}{e^2}(t - t^2)u + \frac{t}{e}\left(\frac{1}{2} - \frac{1}{e}\right)v.$$

Corresponding to the BVP (1.1)-(1.2), $\tau = \frac{5}{4}$, $n = 2$, $a_1 = \frac{1}{e}$ and $b_1 = 1$. In order to apply Theorem 3.1, choose

$$g(t) = 1 + t, \quad h(t) = \frac{2}{e^2}(t - t^2), \quad l(t) = \frac{t}{e}\left(\frac{1}{2} - \frac{1}{e}\right).$$

It is easy to see that $||h||_\infty = \frac{1}{2e^2}$, $||l||_\infty = \frac{1}{2e} - \frac{1}{e^2}$. Notice that

$$G(t, s) = \frac{1}{\Gamma(\frac{5}{4})}\begin{cases} (e - t)(1 - s)^{\frac{1}{4}} + (t - s)^{\frac{1}{4}}, & 0 \le s \le t \le 1,\\ (e - t)(1 - s)^{\frac{1}{4}}, & 0 \le t \le s \le 1. \end{cases}$$

We can calculate that

$$\sigma_1 = \max_{t\in[0,1]}\int_0^1 G(t, s)ds = \frac{e}{\Gamma(\frac{9}{4})}, \quad \sigma_3 = \max_{t\in[0,1]}\int_0^1 \left|\frac{\partial G(t, s)}{\partial t}\right|ds = \frac{9}{4\Gamma(\frac{5}{4})}.$$

Obviously, the inequality in (3.5) is satisfied. Thus, by Theorem 3.1, the BVP (4.1) has at least one positive solution. □

Example 4.2. Consider

$$\begin{cases} D_{0+}^{\frac{9}{4}} x(t) - \frac{1}{8} D_{0+}^{\frac{5}{4}} x(t) - \frac{1}{4} D_{0+}^{\frac{1}{4}} x(t) = f(t, x(t), x'(t)), \quad 0 < t < 1,\\ x(1) - x(0) = \frac{1}{8}\int_0^1 x(s)ds + \frac{1}{4}\int_0^1 (1 - s)x(s)ds,\\ x'(0) = 1, \ x''(0) = 1, \end{cases} \tag{4.2}$$

where

$$f(t,u,v)=\begin{cases}(\frac{1}{10})^{t+1}+\frac{u^2}{10^3}+\frac{|v|}{10^3}, & u\in[0,10],\\ (\frac{1}{10})^{t+1}+\frac{11}{125}u^2+\frac{|v|}{10^3}-\frac{87}{10}, & u\in[10,15],\\ (\frac{1}{10})^{t+1}+\frac{|v|}{10^3}+\frac{111}{10}, & u\in[15,45],\\ (\frac{1}{10})^{t+1}+\frac{111}{450}u+\frac{|v|}{10^3}, & u\in[45,+\infty].\end{cases}$$

Corresponding to the BVP (1.1)-(1.2), we have $\tau=\frac{9}{4}$, $n=3$, $a_1=\frac{1}{8}$, $a_2=\frac{1}{4}$, $b_1=b_2=1$. Thus, we can obtain $B(t)=4-\frac{1}{2}(t+t^2)$, $\Upsilon(t)=\frac{23}{4}+\frac{9}{32}(t-t^2)$, and

$$G(t,s)=\frac{1}{\Gamma(\frac{9}{4})}\begin{cases}(4-\frac{1}{2}(t+t^2))(1-s)^{\frac{5}{4}}+(t-s)^{\frac{5}{4}}, & 0\le s\le t\le 1,\\ (4-\frac{1}{2}(t+t^2))(1-s)^{\frac{5}{4}}, & 0\le t\le s\le 1.\end{cases}$$

By choosing $\theta=\frac{1}{3}$, one gets

$$\begin{aligned}&\omega=\frac{745}{128},\ \omega^*=\frac{9}{32},\ \omega^{**}=\frac{91}{16},\\ &\sigma_1\le\frac{5}{\Gamma(\frac{13}{4})},\ \sigma_2\ge\frac{31+3^{-\frac{1}{4}}}{9\Gamma(\frac{13}{4})},\ \sigma_3=\frac{5}{3\Gamma(\frac{9}{4})},\ \sigma_4=\frac{31(4\cdot 2^{\frac{1}{4}}-1)}{81\cdot 3^{\frac{1}{4}}\Gamma(\frac{13}{4})}.\end{aligned}$$

Taking $r_1=10$, $r_2=60$, $b=15$, $L_1=5$ and $L_2=40$, using $\Gamma(\frac{1}{4})\approx 3.62$, we have

$$\begin{aligned}&(1)\qquad f(t,u,v)\le\min\{\frac{\frac{3}{4}r_2-\omega}{\frac{5}{\Gamma(\frac{13}{4})}},\ \frac{L_2-\omega^*-\frac{1}{4}r_2}{\frac{5}{3\Gamma(\frac{9}{4})}}\}\approx\min\{19.7,\ 16.8\}=16.8,\\ &\qquad\qquad\qquad\qquad\text{for } (t,u,v)\in[0,1]\times[0,60]\times[-40,40],\\ &(2)\qquad f(t,u,v)>\frac{b-\omega^*}{\frac{31+3^{-\frac{1}{4}}}{9\Gamma(\frac{13}{4})}}\approx 6.7,\quad\text{for } (t,u,v)\in[0,1]\times[15,45]\times[-40,40],\\ &(3)\qquad f(t,u,v)<\min\{\frac{\frac{3}{4}r_1-\omega}{\frac{5}{\Gamma(\frac{13}{4})}},\ \frac{L_1-\omega^*-\frac{1}{4}r_1}{\frac{5}{3\Gamma(\frac{9}{4})}}\}\approx\min\{0.8,\ 0.5\}=0.5,\\ &\qquad\qquad\qquad\qquad\text{for } (t,u,v)\in[0,1]\times[0,10]\times[-5,5],\\ &(4)\qquad f(t,u,v)\ge\frac{\theta r_2-\omega^{**}}{\frac{31(4\cdot 2^{\frac{1}{4}}-1)}{81\cdot 3^{\frac{1}{4}}\Gamma(\frac{13}{4})}}\approx 10.24,\ \text{for } (t,u,v)\in[\frac{1}{3},\frac{2}{3}]\times[15,60]\times[-40,40],\end{aligned}$$

that is, f satisfies the conditions (H1)-(H4) of Theorem 3.3. Hence, by Theorem 3.3, the BVP (4.2) has at least three positive solutions x_1, x_2 and x_3 such that

$$\begin{aligned}&0\le x_1(t)\le 10,\ \ 0\le x_2(t)\le 60,\ \ ||x_1'||_\infty\le 5,\ \ ||x_2'||_\infty\le 40,\\ &10\le x_3(t)\le 60,\quad -5\le x_3'(t)\le 40,\quad t\in[0,1],\\ &x_2(t)>15,\ \ x_3(t)\le 15,\ \ t\in[\frac{1}{3},\frac{2}{3}].\end{aligned}$$

□

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