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Property A of differential equations with positive and negative term

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Abstract. In the paper, we elaborate new technique for the investigation of the asymptotic properties for third order differential equations with positive and negative term

$$\Big(b(t)\,\big(a(t)x'(t)\big)'\Big)' + p(t)f(x(\tau(t))) - q(t)h(x(\sigma(t))) = 0.$$

We offer new easily verifiable criteria for property A. We support our results with illustrative examples.

Keywords: third order differential equations, delay argument, property A.

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1 Introduction

We consider the third order differential equation with positive and negative term

$$\Big(b(t)\,\big(a(t)x'(t)\big)'\Big)' + p(t)f(x(\tau(t))) - q(t)h(x(\sigma(t))) = 0, \tag{E}$$

where

(H_1) $a(t), b(t), p(t), q(t), \tau(t), \sigma(t) \in C([t_0, \infty))$ are positive;

(H_2) $f(u), h(u) \in C(\mathbb{R})$, $uf(u) > 0$, $uh(u) > 0$ for $u \neq 0$, h is bounded, f is nondecreasing;

(H_3) $-f(-uv) \geq f(uv) \geq f(u)f(v)$ for $uv > 0$;

(H_4) $\tau(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \infty$, $\lim_{t\to\infty} \sigma(t) = \infty$.

We consider the canonical case of (E), that is

(H_5) $\displaystyle\int_{t_0}^{\infty} \frac{1}{b(s)}\,\mathrm{d}s = \int_{t_0}^{\infty} \frac{1}{a(s)}\,\mathrm{d}s = \infty.$

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By a solution of (E) we understand a function $x(t)$ with derivatives $a(t)x'(t)$, $b(t)(a(t)x'(t))'$ continuous on $[T_x,\infty))$, $T_x \geq t_0$, which satisfies Eq. (E) on $[T_x,\infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. A solution of (E) is said to be oscillatory if it has arbitrarily large zeros and otherwise, it is called nonoscillatory. Equation (E) is said to be oscillatory if all its solutions are oscillatory.

The investigation of the higher order differential equations (see [1–7]) essentially makes use of some generalization of a lemma of Kiguradze [5, 6]. In the lemma, from the fixed sign of the highest derivative, we deduce the structure of possible nonoscillatory solutions. Since the positive and negative term are included in (E), we are not able to fix the sign of the third order quasi-derivative for an eventually positive solution. So the authors mainly study properties of (E) in the partial case when either $p(t) \equiv 0$ or $q(t) \equiv 0$.

In what follows we shall assume that

$$(H_6)\quad \int_{t_0}^{\infty} \frac{1}{a(t)} \int_t^{\infty} \frac{1}{b(s)} \int_s^{\infty} q(u)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t < \infty.$$

It will be shown that this condition reduces the influence of the negative term and this permit us to study property A. By property A we mean the situation when every nonoscillatory solution of (E) tends to zero at infinity.

2 Main results

In this paper we provide easily verifiable conditions for property A of studied equation. To simplify our notation, we denote

$$B(t) = \int_{t_1}^{t} \frac{1}{b(s)}\,\mathrm{d}s\,\mathrm{d}s,$$

and

$$J(t) = \int_{t_1}^{t} \frac{1}{a(s)} \int_{t_1}^{s} \frac{1}{b(u)}\,\mathrm{d}u\,\mathrm{d}s,$$

where $t \geq t_1 \geq t_0$ and t_1 is large enough.

Theorem 2.1. *Let for all t_1 large enough*

$$\int_{t_1}^{\infty} \frac{1}{a(v)} \int_v^{\infty} \frac{1}{b(s)} \int_s^{\infty} p(u)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}v = \infty \tag{2.1}$$

and

$$\int_{t_1}^{\infty} p(s) f\left(J(\tau(s))\right) \mathrm{d}s = \infty. \tag{2.2}$$

Assume that

$$\limsup_{t\to\infty} \left\{ \frac{1}{B(\tau(t))} \int_{t_1}^{\tau(t)} p(s) f\left(J(\tau(s))\right) B(s)\,\mathrm{d}s + \int_{\tau(t)}^{t} p(s) f\left(J(\tau(s))\right) \mathrm{d}s \right.$$
$$\left. + f\left(B(\tau(t))\right) \int_t^{\infty} p(s) f\left(\frac{J(\tau(s))}{B(\tau(s))}\right) \mathrm{d}s \right\} > \limsup_{u\to 0} \frac{u}{f(u)}. \tag{2.3}$$

Then (E) has property A.

Proof. Assume that (E) possesses an eventually positive solution $x(t)$ on (T_x, ∞), $T_x \geq t_0$. We introduce the auxiliary function $z(t)$ associated to $x(t)$ by

$$z(t) = x(t) + \int_t^\infty \frac{1}{a(v)} \int_v^\infty \frac{1}{b(s)} \int_s^\infty q(u)h(x(\sigma(u)))\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}v. \tag{2.4}$$

It follows from (H6) and the boundedness of $h(u)$ that definition of function $z(t)$ is correct and $z(t)$ exists for all $t \geq T_x$. It is useful to notice that $z(t) > x(t) > 0$, $z'(t) < x'(t)$ and

$$\Big(b(t)\left(a(t)z'(t)\right)'\Big)' = -p(t)f(x(\tau(t))) < 0. \tag{2.5}$$

Therefore, Kiguradze's lemma implies that either

$$z(t) \in \mathcal{N}_0 \iff a(t)z'(t) < 0, \qquad b(t)\left(a(t)z'(t)\right)' > 0,$$

or

$$z(t) \in \mathcal{N}_2 \iff a(t)z'(t) > 0, \qquad b(t)\left(a(t)z'(t)\right)' > 0,$$

eventually, let us say for $t \geq t_1 \geq T_x$.

At first, assume that $z(t) \in \mathcal{N}_2$. Using the fact that $b(t)\big(a(t)z'(t)\big)'$ is decreasing, we have

$$\begin{aligned} a(t)z'(t) &\geq \int_{t_1}^t b(s)\left(a(s)z'(s)\right)' \frac{1}{b(s)}\,\mathrm{d}s \geq b(t)\left(a(t)z'(t)\right)' \int_{t_1}^t \frac{1}{b(s)}\,\mathrm{d}s \\ &= b(t)\left(a(t)z'(t)\right)' B(t). \end{aligned} \tag{2.6}$$

In view of (2.6), we see that $\left(\frac{a(t)z'(t)}{B(t)}\right)' \leq 0$, consequently $a(t)z'(t)/B(t)$ is decreasing. Then

$$\begin{aligned} x(t) &\geq \int_{t_1}^t x'(s)\,\mathrm{d}s \geq \int_{t_1}^t \frac{a(s)z'(s)}{B(s)} \frac{B(s)}{a(s)}\,\mathrm{d}s \geq \frac{a(t)z'(t)}{B(t)} \int_{t_1}^t \frac{1}{a(s)} B(s)\,\mathrm{d}s \\ &= \frac{a(t)z'(t)}{B(t)} J(t). \end{aligned}$$

Setting the last estimate into (2.5), we see that that $y(t) = a(t)z'(t)$ is a positive solution of the differential inequality

$$\left(b(t)y'(t)\right)' + p(t)f\left(\frac{J(\tau(t))}{B(\tau(t))} y(\tau(t))\right) \leq 0 \tag{2.7}$$

and, what is more, $y(t)/B(t)$ is decreasing and $b(t)y'(t) > 0$.

On the other hand, an integration of (2.7) from t to ∞ and then from t_1 to t yields

$$\begin{aligned} y(t) &\geq \int_{t_1}^t \frac{1}{b(u)} \int_u^\infty p(s)f\left(\frac{J(\tau(s))}{B(\tau(s))} y(\tau(s))\right) \mathrm{d}s\,\mathrm{d}u \\ &= \int_{t_1}^t \frac{1}{b(u)} \int_u^t p(s)f\left(\frac{J(\tau(s))}{B(\tau(s))} y(\tau(s))\right) \mathrm{d}s\,\mathrm{d}u \\ &\quad + \int_{t_1}^t \frac{1}{b(u)} \int_t^\infty p(s)f\left(\frac{J(\tau(s))}{B(\tau(s))} y(\tau(s))\right) \mathrm{d}s\,\mathrm{d}u \\ &= \int_{t_1}^t p(s)f\left(\frac{J(\tau(s))}{B(\tau(s))} y(\tau(s))\right) B(s)\,\mathrm{d}s + B(t) \int_t^\infty p(s)f\left(\frac{J(\tau(s))}{B(\tau(s))} y(\tau(s))\right) \mathrm{d}s. \end{aligned}$$

Thus,

$$y(\tau(t)) \geq \int_{t_1}^{\tau(t)} p(s) f\left(\frac{J(\tau(s))}{B(\tau(s))} y(\tau(s))\right) B(s)\, \mathrm{d}s + B(\tau(t)) \int_{\tau(t)}^{t} p(s) f\left(\frac{J(\tau(s))}{B(\tau(s))} y(\tau(s))\right) \mathrm{d}s + B(\tau(t)) \int_{t}^{\infty} p(s) f\left(\frac{J(\tau(s))}{B(\tau(s))} y(\tau(s))\right) \mathrm{d}s.$$

Employing (H_3) and the facts that $y(t)$ is increasing and $y(t)/B(t)$ is decreasing, we have

$$y(\tau(t)) \geq f\left(\frac{y(\tau(t))}{B(\tau(t))}\right) \int_{t_1}^{\tau(t)} p(s) f\left(J(\tau(s))\right) B(s)\, \mathrm{d}s + B(\tau(t)) f\left(\frac{y(\tau(t))}{B(\tau(t))}\right) \int_{\tau(t)}^{t} p(s) f\left(J(\tau(s))\right) \mathrm{d}s + B(\tau(t)) f\left(y(\tau(s))\right) \int_{t}^{\infty} p(s) f\left(\frac{J(\tau(s))}{B(\tau(s))}\right) \mathrm{d}s. \tag{2.8}$$

Therefore, setting $u = y(\tau(t))/B(\tau(t))$, we get

$$\frac{u}{f(u)} \geq \frac{1}{B(\tau(t))} \int_{t_1}^{\tau(t)} p(s) f\left(J(\tau(s))\right) B(s)\, \mathrm{d}s + \int_{\tau(t)}^{t} p(s) f\left(J(\tau(s))\right) \mathrm{d}s + f\left(B(\tau(t))\right) \int_{t}^{\infty} p(s) f\left(\frac{J(\tau(s))}{B(\tau(s))}\right) \mathrm{d}s. \tag{2.9}$$

Condition (2.2) implies that $y(t)/B(t) \to 0$ as $t \to \infty$. Indeed, if we admit $y(t)/B(t) \to \ell > 0$, then $y(t)/B(t) \geq \ell$ and setting the last inequality into (2.7), we obtain

$$0 \geq \left(b(t) y'(t)\right)' + p(t) f\left(\ell J(\tau(t))\right).$$

An integration from t_1 to ∞ yields

$$b(t_1) y'(t_1) \geq f(\ell) \int_{t_1}^{\infty} p(s) f\left(J(\tau(s))\right) \mathrm{d}s,$$

which contradicts with (2.2). Now, taking lim sup on the both sides of (2.9), one gets a contradiction with (2.3).

Now, we assume that $z(t) \in \mathcal{N}_0$. Since $z(t)$ is positive and decreasing, there exists $\lim_{t\to\infty} z(t) = 2\ell \geq 0$. It follows from (2.4) that $\lim_{t\to\infty} x(t) = 2\ell$. If we admit that $\ell > 0$, then $x(\tau(t)) \geq \ell > 0$, eventually. An integration of (2.5) yields

$$b(t) \left(a(t) z'(t)\right)' \geq \int_{t}^{\infty} p(s) f\big(x(\tau(s))\big)\, \mathrm{d}s \geq f(\ell) \int_{t}^{\infty} p(s)\, \mathrm{d}s.$$

Integrating from t to ∞ and consequently from t_1 to ∞ one gets

$$z(t_1) \geq f(\ell) \int_{t_1}^{\infty} \frac{1}{a(v)} \int_{v}^{\infty} \frac{1}{b(s)} \int_{s}^{\infty} p(u)\, \mathrm{d}u\, \mathrm{d}s\, \mathrm{d}v$$

which contradicts with (2.1) and the proof is complete. □

For partial case of (*E*) we have the following easily verifiable criterion.

Corollary 2.2. *Let (2.1) hold and for all t_1 large enough*

$$\int_{t_1}^{\infty} p(s)J(\tau(s))\,\mathrm{d}s = \infty. \tag{2.10}$$

Assume that

$$\limsup_{t\to\infty}\left\{\frac{1}{B(\tau(t))}\int_{t_1}^{\tau(t)} p(s)J(\tau(s))B(s)\,\mathrm{d}s + \int_{\tau(t)}^{t} p(s)J(\tau(s))\,\mathrm{d}s \right.$$
$$\left. + B(\tau(t))\int_{t}^{\infty} p(s)\frac{J(\tau(s))}{B(\tau(s))}\,\mathrm{d}s\right\} > 1. \tag{2.11}$$

Then the trinomial differential equation

$$\left(b(t)\left(a(t)x'(t)\right)'\right)' + p(t)x(\tau(t)) - q(t)h(x(\sigma(t))) = 0 \tag{E_L}$$

has property A.

Theorem 2.3. *Let (2.1) hold and for all t_1 large enough*

$$\int_{t_1}^{\infty}\frac{1}{b(u)}\int_{u}^{\infty} p(s)f\left(\frac{J(\tau(s))}{B(\tau(s))}\right)\mathrm{d}s\,\mathrm{d}u = \infty. \tag{2.12}$$

Assume that

$$\begin{aligned}\limsup_{t\to\infty}\Bigg\{ & f\left(\frac{1}{B(\tau(t))}\right)\int_{t_1}^{\tau(t)} p(s)f\left(J(\tau(s))\right)B(s)\,\mathrm{d}s \\ & B(\tau(t))f\left(\frac{1}{B(\tau(t))}\right)\int_{\tau(t)}^{t} p(s)f\left(J(\tau(s))\right)\mathrm{d}s \\ & + f\left(B(\tau(t))\right)\int_{t}^{\infty} p(s)f\left(\frac{J(\tau(s))}{B(\tau(s))}\right)\mathrm{d}s\Bigg\} > \limsup_{v\to\infty}\frac{v}{f(v)}.\end{aligned} \tag{2.13}$$

Then (E) has property A.

Proof. Assume that $x(t)$ is a positive solution of (E). Proceeding exactly as in the proof of Theorem 2.1 we verify that the associated function $z(t)$ belongs to the class $\mathcal{N}_0$ or $\mathcal{N}_2$.

If $z(t) \in \mathcal{N}_2$, then $y(t) = a(t)z'(t)$ satisfies (2.8). We claim that condition (2.12) implies that $y(t) \to \infty$ as $t \to \infty$. Really, if not, then $y(t) \to K$ as $t \to \infty$. An integration of (2.7) yields

$$b(t)y'(t) \geq \int_{t}^{\infty} p(s)f\left(\frac{J(\tau(s)}{B(\tau(s))}y(\tau(s))\right)\mathrm{d}s.$$

Integrating once more, we get

$$\begin{aligned}K &\geq \int_{t_1}^{\infty}\frac{1}{b(u)}\int_{u}^{\infty} p(s)f\left(\frac{J(\tau(s)}{B(\tau(s))}y(\tau(s))\right)\mathrm{d}s\,\mathrm{d}u \\ &\geq f\left(y(\tau(t_1))\right)\int_{t_1}^{\infty}\frac{1}{b(u)}\int_{u}^{\infty} p(s)f\left(\frac{J(\tau(s)}{B(\tau(s))}\right)\mathrm{d}s\,\mathrm{d}u,\end{aligned}$$

which contradicts with (2.12) and we conclude that $y(t) \to \infty$ as $t \to \infty$.

Therefore, setting $v = y(\tau(t))$, we get

$$\begin{aligned}\frac{v}{f(v)} \geq f\left(\frac{1}{B(\tau(t))}\right) \int_{t_1}^{\tau(t)} p(s) f\left(J(\tau(s))\right) B(s)\, \mathrm{d}s \\ + B(\tau(t)) f\left(\frac{1}{B(\tau(t))}\right) \int_{\tau(t)}^{t} p(s) f\left(J(\tau(s))\right) \mathrm{d}s \\ + B(\tau(t)) \int_t^\infty p(s) f\left(\frac{J(\tau(s))}{B(\tau(s))}\right) \mathrm{d}s.\end{aligned}$$

Taking lim sup on the both sides, we are led to contradiction with (2.13).

If $z(t) \in \mathcal{N}_0$, then proceeding as in the proof of Theorem 2.1, we verify that $x(t) \to 0$ as $t \to \infty$. The proof is complete. □

Remark 2.4. Theorems 2.1 and 2.3 are applicable for

$$f(u) = |u|^\beta \operatorname{sgn} u$$

with $0 < \beta \leq 1$ and $\beta \geq 1$, respectively.

Remark 2.5. The integral criteria (2.3) and (2.13) of Theorems 2.1 and 2.3 contain three terms and naturally they provide the better results then one term integral criteria that are in generally used.

To support Remark 2.5, we recall the well known result of Kiguradze and Chanturia [6]. The condition

$$\limsup_{t\to\infty} t \int_t^\infty s p(s)\, \mathrm{d}s > 2$$

guaranties property A of

$$y'''(t) + p(t) y(t) = 0. \tag{E_1}$$

On the other hand it follows from Theorem 2.1 that (E_1) has property A provided that

$$\limsup_{t\to\infty} \left\{ t \int_t^\infty s p(s)\, \mathrm{d}s + \frac{1}{t} \int_{t_1}^{t} s^3 p(s)\, \mathrm{d}s \right\} > 2.$$

We support the results obtained with a couple of illustrative examples.

Example 2.6. Consider the third order trinomial differential equation

$$\left(t^{1/3} \left(t^{1/2} x'(t) \right)' \right)' + \frac{p}{t^{13/6}} y(\lambda t) - \frac{q}{t^3} \arctan\left(y(\sigma(t)) \right) = 0, \tag{E_{x1}}$$

with $p > 0$, $q > 0$, $\lambda \in (0,1)$. Now $h(u) = \arctan(u)$ is bounded and conditions (2.1) and (2.10) holds true. Simple computation shows that

$$B(t) \sim \frac{3}{2} t^{2/3}, \qquad J(t) \sim \frac{9}{7} t^{7/6}$$

and (2.11) takes the form

$$p \lambda^{7/6} \left(\frac{27}{7} + \frac{9}{7} \ln\left(\frac{1}{\lambda}\right) \right) > 1, \tag{2.14}$$

which implies that every nonoscillatory solution of (E_{x1}) tends to zero as $t \to \infty$. For e.g. $\lambda = 1/2$ condition (2.14) reduces to $p > 0.4728$.

Our results are applicable also for the case when $\tau(t) \equiv t$.

Example 2.7. Consider the third order differential equation

$$\left(t^{1/3}\left(t^{1/2}x'(t)\right)'\right)' + \frac{p}{t^{13/6}}y(t) - \frac{q}{t^3}\arctan\left(y(\sigma(t))\right) = 0, \qquad (E_{x2})$$

with $p > 0$, $q > 0$. Property A of (E_{x2}) is guaranteed by (2.11), which reduces to

$$p > \frac{7}{27}.$$

3 Summary

The results obtained provide new technique for studying oscillation and asymptotic properties of third order differential equation with positive and negative terms. The criteria obtained are easily verifiable and applicable to wide class of differential equations.

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