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# SECOND-ORDER DIFFERENTIAL INCLUSIONS WITH ALMOST CONVEX RIGHT-HAND SIDES

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Abstract. We study the existence of solutions of a boundary second order differential inclusion under conditions that are strictly weaker than the usual assumption of convexity on the values of the right-hand side.

## 1. Introduction

The existence of solutions for second order differential inclusions of the form $\ddot{u}(t) \in F(t, u(t), \dot{u}(t))(t \in [0,1])$ with boundary conditions, where $F : [0,1] \times E \times E \rightrightarrows E$ is a convex compact multifunction, Lebesgue-measurable on $[0,1]$, upper semicontinuous on $E \times E$ and integrably compact in finite and infinite dimensional spaces has been studied by many authors see for example [1],[7]. Our aim in this article is to provide an existence result for the differential inclusion with two-point boundary conditions in a finite dimensional space $E$ of the form

$$(P_F)\begin{cases} \ddot{u}(t) \in F(u(t), \dot{u}(t)), \quad a.e. \ t \in [a,b], \ (0 \leq a < b < +\infty) \\ u(a) = u(b) = v_0, \end{cases}$$

where $F : E \times E \rightrightarrows E$ is an upper semicontinuous multifunction with almost convex values, i.e., the convexity is replaced by a strictly weaker condition.

For the first order differential inclusions with almost convex values we refer the reader to [5].

After some preliminaries, we present a result which is the existence of $\mathbf{W}_E^{2,1}([a,b])$-solutions of $(P_F)$ where $F$ is a convex valued multifunction. Using this convexified problem we show that the differential inclusion $(P_F)$ has solutions if the values of $F$ are almost convex. As an example of the almost convexity of the values of the right-hand side, notice that, if $F(t,x,y)$ is a convex set not containing the origin then the boundary of $F(x,y)$, $\partial F(x,y)$, is almost convex.

## 2. Notation and preliminaries

Throughout, $(E, \|.\|)$ is a real separable Banach space and $E'$ is its topological dual, $\overline{\mathbf{B}}_E$ is the closed unit ball of $E$ and $\sigma(E,E')$ the weak topology on $E$. We denote by $\mathbf{L}_E^1([a,b])$ the space of all Lebesgue-Bochner integrable $E$ valued mappings defined on $[a,b]$.

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Let $\mathbf{C}_E([a,b])$ be the Banach space of all continuous mappings $u : [a,b] \to E$ endowed with the sup-norm, and $\mathbf{C}^1_E([a,b])$ be the Banach space of all continuous mappings $u : [a,b] \to E$ with continuous derivative, equipped with the norm

$$\|u\|_{\mathbf{C}^1} = \max\{\max_{t\in[a,b]} \|u(t)\|, b \max_{t\in[a,b]} \|\dot{u}(t)\|\}.$$

Recall that a mapping $v : [a,b] \to E$ is said to be scalarly derivable when there exists some mapping $\dot{v} : [a,b] \to E$ (called the weak derivative of $v$) such that, for every $x' \in E'$, the scalar function $\langle x', v(\cdot)\rangle$ is derivable and its derivative is equal to $\langle x', \dot{v}(\cdot)\rangle$. The weak derivative $\ddot{v}$ of $\dot{v}$ when it exists is the weak second derivative.

By $\mathbf{W}^{2,1}_E([a,b])$ we denote the space of all continuous mappings in $\mathbf{C}_E([a,b])$ such that their first derivatives are continuous and their second weak derivatives belong to $\mathbf{L}^1_E([a,b])$.

For a subset $A \subset E$, $co(A)$ denotes its convex hull and $\overline{co}(A)$ its closed convex hull.

Let $X$ be a vector space, a set $K \subset X$ is called almost convex if for every $\xi \in co(K)$ there exist $\lambda_1$ and $\lambda_2$, $0 \le \lambda_1 \le 1 \le \lambda_2$, such that $\lambda_1\xi \in K$, $\lambda_2\xi \in K$.

Note that every convex set is almost convex.

## 3. The Main result

We begin with a lemma which summarizes some properties of some Green type function. It will after be used in the study of our boundary value problems (see [1], [7] and [3]).

**Lemma 3.1.** *Let $E$ be a separable Banach space, $v_0 \in E$ and $G : [a,b] \times [a,b] \to \mathbb{R}$ $(0 \le a < b < \infty)$ be the function defined by*

$$G(t,s) = \begin{cases} -\dfrac{1}{b}(b-t)(s-a) & \text{if} \quad a \le s \le t \le b, \\ -\dfrac{1}{b}(t-a)(b-s) & \text{if} \quad a \le t \le s \le b. \end{cases}$$

*Then the following assertions hold.*

*(1) If $u \in \mathbf{W}^{2,1}_E([a,b])$ with $u(a) = u(b) = v_0$, then*

$$u(t) = v_0 + \frac{b}{b-a}\int_a^b G(t,s)\ddot{u}(s)ds, \quad \forall t \in [a,b].$$

*(2) $G(.,s)$ is derivable on $[a,b[$ for every $s \in [a,b]$, except on the diagonal, and its derivative is given by*

$$\frac{\partial G}{\partial t}(t,s) = \begin{cases} \dfrac{1}{b}(s-a) & \text{if} \quad a \le s < t \le b \\ -\dfrac{1}{b}(b-s) & \text{if} \quad a \le t < s \le b. \end{cases}$$

*(3)* $G(.,.)$ *and* $\dfrac{\partial G}{\partial t}(.,.)$ *satisfy*

$$\sup_{t,s\in[a,b]}|G(t,s)|\leq b,\qquad \sup_{t,s\in[a,b],t\neq s}|\frac{\partial G}{\partial t}(t,s)|\leq 1. \tag{3.1}$$

*(4) For* $f\in \mathbf{L}^1_E([a,b])$ *and for the mapping* $u_f:[a,b]\to E$ *defined by*

$$u_f(t)=v_0+\frac{b}{b-a}\int_a^b G(t,s)f(s)ds,\quad \forall t\in[a,b] \tag{3.2}$$

*one has* $u_f(a)=u_f(b)=v_0$.
*Furthermore, the mapping* $u_f$ *is derivable, and its derivative* $\dot u_f$ *satisfies*

$$\lim_{h\to 0}\frac{u_f(t+h)-u_f(t)}{h}=\dot u_f(t)=\frac{b}{b-a}\int_a^b\frac{\partial G}{\partial t}(t,s)f(s)ds, \tag{3.3}$$

*for all* $t\in[a,b]$. *Consequently,* $\dot u_f$ *is a continuous mapping from* $[a,b]$ *into the space* $E$.
*(5) The mapping* $\dot u_f$ *is scalarly derivable, that is, there exists a mapping* $\ddot u_f:[a,b]\to E$ *such that, for every* $x'\in E'$, *the scalar function* $\langle x',\dot u_f(.)\rangle$ *is derivable, with* $\dfrac{d}{dt}\langle x',\dot u_f(t)\rangle=\langle x',\ddot u_f(t)\rangle$, *furthermore*

$$\ddot u_f=f\ \text{ a.e. on } [a,b]. \tag{3.4}$$

Let us mention a useful consequence of Lemma 3.1.

**Proposition 3.2.** *Let* $E$ *be a separable Banach space and let* $f:[a,b]\to E$ *be a continuous mapping (respectively a mapping in* $\mathbf{L}^1_E([a,b])$*). Then the mapping*

$$u_f(t)=v_0+\frac{b}{b-a}\int_a^b G(t,s)f(s)ds,\ \forall t\in[a,b]$$

*is the unique* $\mathbf{C}^2_E([a,b])$*-solution (respectively* $\mathbf{W}^{2,1}_E([a,b])$*-solution) to the differential equation*

$$\begin{cases}\ddot u(t)=f(t),\quad \forall t\in[a,b],\\ u(a)=u(b)=v_0.\end{cases}$$

The following is an existence result for a second order differential inclusion with boundary conditions and a convex valued right hand side. It will be used in the proof of our main theorem.

**Proposition 3.3.** *Let* $E$ *be a finite dimensional space,* $F:E\times E\rightrightarrows E$ *be a convex compact valued multifunction, upper semicontinuous on* $E\times E$. *Suppose that there is a nonnegative function* $m\in\mathbf{L}^1_{\mathbb{R}}([a,b])$ *such that* $F(x,y)\subset m(t)\overline{\mathbf{B}}_E$ *for all* $x,y\in[a,b]$. *Let* $v_0\in E$. *Then the* $\mathbf{W}^{2,1}_E([a,b])$*-solutions set of the problem*

$$(P_F)\begin{cases}\ddot u(t)\in F(u(t),\dot u(t)),\ \text{ a.e. } t\in[a,b],\\ u(a)=u(b)=v_0,\end{cases}$$

*is nonempty and compact in* $\mathbf{C}^1_E([a,b])$.

*Proof. Step 1.* Let

$$\mathbf{S} = \{f \in \mathbf{L}^1_E([a,b]) : \ \|f(t)\| \le m(t), a.e.\, t \in [a,b]\}$$

and

$$\mathbf{X} = \{u_f : [a,b] \to E : \ \ u_f(t) = v_0 + \frac{b}{b-a}\int_a^b G(t,s)f(s)ds, \forall t \in [a,b], f \in \mathbf{S}\}.$$

Obviously $\mathbf{S}$ and $\mathbf{X}$ are convex. Let us prove that $\mathbf{S}$ is a $\sigma(\mathbf{L}^1_E([a,b]), \mathbf{L}^\infty_E([a,b]))$-compact subset of $\mathbf{L}^1_E([a,b])$. Indeed, let $(f_n)$ be a sequence of $\mathbf{S}$. It is clear that $(f_n)$ is bounded in $\mathbf{L}^\infty_E([a,b])$, taking a subsequence if necessary, we may conclude that $(f_n)$ weakly* or $\sigma(\mathbf{L}^\infty_E([a,b]), \mathbf{L}^1_E([a,b]))$-converges to some mapping $f \in \mathbf{L}^\infty_E([a,b]) \subset \mathbf{L}^1_E([a,b])$. Consequently, for all $y(\cdot) \in \mathbf{L}^1_E([a,b])$ we have

$$\lim_{n\to\infty} \langle f_n(\cdot), y(\cdot)\rangle = \langle f(\cdot), y(\cdot)\rangle.$$

Let $z(\cdot) \in \mathbf{L}^\infty_E([a,b]) \subset \mathbf{L}^1_E([a,b])$, then

$$\lim_{n\to\infty} \langle f_n(\cdot), z(\cdot)\rangle = \langle f(\cdot), z(\cdot)\rangle.$$

This shows that $(f_n)$ weakly or $\sigma(\mathbf{L}^1_E([a,b]), \mathbf{L}^\infty_E([a,b]))$-converges to $f(\cdot)$ and that $\|f(t)\| \le m(t)$ a.e on $[a,b]$ since $\mathbf{S}$ is convex and strongly closed in $\mathbf{L}^1_E([a,b])$ and hence it is weakly closed in $\mathbf{L}^1_E([a,b])$.
Now, let us prove that $\mathbf{X}$ is compact in $\mathbf{C}^1_E([a,b])$ equipped with the norm $\|\cdot\|_{\mathbf{C}^1}$. For any $u_f \in \mathbf{X}$ and all $t, \tau \in [a,b]$ we have

$$\begin{aligned}\|u_f(t) - u_f(\tau)\| &\le \frac{b}{b-a}\int_a^b |G(t,s) - G(\tau,s)|\|f(s)\|ds \\ &\le \frac{b}{b-a}\int_a^b |G(t,s) - G(\tau,s)|m(s)ds\end{aligned}$$

and by the relation (3.3) in Lemma 3.1

$$\begin{aligned}\|\dot{u}_f(t) - \dot{u}_f(\tau)\| &\le \frac{b}{b-a}\int_a^b |\frac{\partial G}{\partial t}(t,s) - \frac{\partial G}{\partial t}(\tau,s)|\|f(s)\|ds \\ &\le \frac{b}{b-a}\int_a^b |\frac{\partial G}{\partial t}(t,s) - \frac{\partial G}{\partial t}(\tau,s)|m(s)ds.\end{aligned}$$

Since $m \in \mathbf{L}^1_{\mathbb{R}}([a,b])$ and the function $G$ is uniformly continuous we get the equicontinuity of the sets $\mathbf{X}$ and $\{\dot{u}_f : \ u_f \in \mathbf{X}\}$. On the other hand, for any $u_f \in \mathbf{X}$ and for all $t \in [a,b]$ we have by the relations (3.1), (3.2) and (3.3)

$$\|u_f(t)\| \le \|v_0\| + \frac{b^2}{b-a}\|m\|_{\mathbf{L}^1} \text{ and } \|\dot{u}_f(t)\| \le \frac{b}{b-a}\|m\|_{\mathbf{L}^1},$$

that is, the sets $\mathbf{X}(t)$ and $\{\dot{u}_f(t) : \ u_f \in \mathbf{X}\}$ are relatively compact in the finite dimensional space $E$. Hence, we conclude that $\mathbf{X}$ is relatively compact in $(\mathbf{C}^1_E([a,b]), \|\cdot\|_{\mathbf{C}^1})$. We claim that $\mathbf{X}$ is closed in $(\mathbf{C}^1_E([a,b]), \|\cdot\|_{\mathbf{C}^1})$. Fix any sequence $(u_{f_n})$ of $\mathbf{X}$ converging to $u \in \mathbf{C}^1_E([a,b])$. Then, for each $n \in \mathbb{N}$

$$u_{f_n}(t) = v_0 + \frac{b}{b-a}\int_a^b G(t,s)f_n(s)ds, \ \forall t \in [a,b]$$

and $f_n \in \mathbf{S}$. Since $\mathbf{S}$ is $\sigma(\mathbf{L}^1_E([a,b]), \mathbf{L}^\infty_E([a,b]))$-compact, by extracting a subsequence if necessary we may conclude that $(f_n)$ $\sigma(\mathbf{L}^1_E([a,b]), \mathbf{L}^\infty_E([a,b]))$-converges to $f \in \mathbf{S}$. Putting for all $t \in [a,b]$

$$u_f(t) = v_0 + \frac{b}{b-a}\int_a^b G(t,s)f(s)ds,$$

we obtain for all $z(\cdot) \in \mathbf{L}^\infty_E([a,b])$ and for all $t \in [a,b]$

$$\lim_{n\to\infty}\langle f_n(\cdot), G(t,\cdot)z(\cdot)\rangle = \langle f(\cdot), G(t,\cdot)z(\cdot)\rangle.$$

Hence

$$\begin{aligned}\lim_{n\to\infty}\int_a^b \langle G(t,s)f_n(s), z(s)\rangle ds &= \lim_{n\to\infty}\int_a^b \langle f_n(s), G(t,s)z(s)\rangle ds \\ &= \int_a^b \langle f(s), G(t,s)z(s)\rangle ds \\ &= \int_a^b \langle G(t,s)f(s), z(s)\rangle ds.\end{aligned}$$

In particular, for $z(\cdot) = \chi_{[a,b]}(\cdot)e_j$, where $\chi_{[a,b]}(\cdot)$ stands for the characteristic function of $[a,b]$ and $(e_j)$ a basis of $E$, we obtain

$$\lim_{n\to\infty}\int_a^b \langle G(t,s)f_n(s), \chi_{[a,b]}(s)e_j\rangle ds = \int_a^b \langle G(t,s)f(s), \chi_{[a,b]}(s)e_j\rangle ds,$$

or equivalently

$$\langle \lim_{n\to\infty}\int_a^b G(t,s)f_n(s)ds, e_j\rangle = \langle \int_a^b G(t,s)f(s)ds, e_j\rangle,$$

which entails

$$\lim_{n\to\infty}(v_0 + \frac{b}{b-a}\int_a^b G(t,s)f_n(s)ds) = v_0 + \frac{b}{b-a}\int_a^b G(t,s)f(s)ds = u_f(t).$$

Consequently, the sequence $(u_{f_n})$ converges to $u_f$ in $\mathbf{C}_E([a,b])$. By the same arguments, we prove that the sequence $(\dot{u}_{f_n})$ with

$$\dot{u}_{f_n}(t) = \frac{b}{b-a}\int_a^b \frac{\partial G}{\partial t}(t,s)f_n(s)ds, \ \forall t \in [a,b]$$

converges to $\dot{u}_f$ in $\mathbf{C}_E([a,b])$. That is, $(u_{f_n})$ converges to $u_f$ in $\mathbf{C}^1_E([a,b])$. This shows that $\mathbf{X}$ is compact in $(\mathbf{C}^1_E([a,b]), \|\cdot\|_{\mathbf{C}^1})$.

*Step 2.* Observe that a mapping $u : [a, b] \to E$ is a $\mathbf{W}_E^{2,1}([a, b])$-solution of $(P_F)$ iff there exists $u_f \in \mathbf{X}$ and $f(t) \in F(u_f(t), \dot{u}_f(t))$ for a.e $t \in [a, b]$.

For any Lebesgue-measurable mappings $v, w : [a, b] \to E$, there is a Lesbegue-measurable selection $s \in \mathbf{S}$ such that $s(t) \in F(v(t), w(t))$ a.e. Indeed, there exist sequences $(v_n)$ and $(w_n)$ of simple $E$-valued functions such that $(v_n)$ converges pointwise to $v$ and $(w_n)$ converges pointwise to $w$ for $E$ endowed by the strong topology. Notice that the multifunctions $F(v_n(.), w_n(.))$ are Lebesgue-measurable. Let $s_n$ be a Lesbegue-measurable selection of $F(v_n(.), w_n(.))$. As $s_n(t) \in F(v_n(t), w_n(t)) \subset m(t)\overline{\mathbf{B}}_E$ for all $t \in [a, b]$ and $\mathbf{S}$ is $\sigma(\mathbf{L}_E^1([a, b]), \mathbf{L}_E^\infty([a, b]))$-compact in $\mathbf{L}_E^1([a, b])$, by Eberlein-Šmulian theorem, we may extract from $(s_n)$ a subsequence $(s'_n)$ which converges $\sigma(\mathbf{L}_E^1([a, b]), \mathbf{L}_E^\infty([a, b]))$ to some mapping $s \in \mathbf{S}$. Here we may invoke the fact that $\mathbf{S}$ is a weakly compact metrizable set in the separable Banach space $\mathbf{L}_E^1([a, b])$. Now, application of the Mazur's trick to $(s'_n)$ provides a sequence $(z_n)$ with $z_n \in co\{s'_m : \quad m \geq n\}$ such that $(z_n)$ converges almost every where to $s$. Then, for almost every $t \in [a, b]$

$$\begin{aligned} s(t) &\in \bigcap_{k \geq 0} \overline{\{z_n(t) : \ n \geq k\}} \\ &\subset \bigcap_{k \geq 0} \overline{co}\{s'_n(t) : \ n \geq k\}. \end{aligned}$$

As $s'_n(t) \in F(v_n(t), w_n(t))$, we obtain

$$\begin{aligned} s(t) &\in \bigcap_{k \geq 0} \overline{co}(\bigcup_{n \geq k} F(v_n(t), w_n(t))) \\ &= \overline{co}(\limsup_{n \to \infty} F(v_n(t), w_n(t))), \end{aligned}$$

using the pointwise convergence of $(v_n(\cdot))$ and $(w_n(\cdot))$ to $v(\cdot)$ and $(w(\cdot))$ respectively, the upper semicontinuity of $F$ and the compactness of its values we get

$$s(t) \in \overline{co}(F(v(t), w(t))) = F(v(t), w(t))$$

since $F(v(t), w(t))$ is a closed convex set.

*Step 3.* Let us consider the multifunction $\Phi : \mathbf{S} \rightrightarrows \mathbf{S}$ defined by

$$\Phi(f) = \{g \in \mathbf{S} : \ g(t) \in F(u_f(t), \dot{u}_f(t)) \ \ a.e.t \in [a, b]\}$$

where $u_f \in \mathbf{X}$. In view of Step 2, $\Phi(f)$ is a nonempty set. These considerations lead us to the application of the Kakutani-ky Fan fixed point theorem to the multifunction $\Phi(.)$. It is clear that $\Phi(f)$ is a convex weakly compact subset of $\mathbf{S}$. We need to check that $\Phi$ is upper semicontinuous on the convex weakly compact metrizable set $\mathbf{S}$. Equivalently, we need to prove that the graph of $\Phi$ is sequentially weakly compact in $\mathbf{S} \times \mathbf{S}$. Let $(f_n, g_n)$ be a sequence in the graph of $\Phi$. $(f_n) \subset \mathbf{S}$. By extracting a subsequence we may suppose that $(f_n)$ $\sigma(\mathbf{L}^1_E([a,b]),\mathbf{L}^\infty_E([a,b]))$ converges to $f \in \mathbf{S}$. It follows that the sequences $(u_{f_n})$ and $(\dot{u}_{f_n})$ converge pointwise to $u_f$ and $\dot{u}_f$ respectively. On the other hand, $g_n \in \Phi(f_n) \subset \mathbf{S}$. We may suppose that $(g_n)$ converges weakly to some element $g \in \mathbf{S}$. As $g_n(t) \in F(u_{f_n}(t), \dot{u}_{f_n}(t))$ a.e., by repeating the arguments given in Step 2, we obtain that $g(t) \in F(u_f(t), \dot{u}_f(t))$ a.e. This shows that the graph of $\Phi$ is weakly compact in the weakly compact set $\mathbf{S}\times\mathbf{S}$. Hence $\Phi$ admits a fixed point, that is, there exists $f \in \mathbf{S}$ such that $f \in \Phi(f)$ and so $f(t) \in F(u_f(t), \dot{u}_f(t))$ for almost every $t \in [a,b]$. Equivalently (see Lemma 3.1) $\ddot{u}_f(t) \in F(u_f(t), \dot{u}_f(t))$ for almost evert $t \in [a,b]$ with $u_f(a) = \dot{u}_f(b) = v_0$, what in turn, means that the mapping $u_f$ is a $\mathbf{W}^{2,1}_E([a,b])$-solution of the problem $(P_F)$. Compactness of the solutions set follows easily from the compactness in $\mathbf{C}^1_E([a,b])$ of $\mathbf{X}$ given in Step 1, and the preceding arguments. ■

Now, we present an existence result of solutions to the problem $(P_F)$ if we suppose on $F$ a linear growth condition.

**Theoreme 3.4.** *Let $E$ be a finite dimensional space and $F : E \times E \rightrightarrows E$ be a convex compact valued multifunction, upper semicontinuous on $E \times E$. Suppose that there is two nonnegative functions $p$ and $q$ in $\mathbf{L}^1_{\mathbb{R}}([a,b])$ with $\|p+q\|_{\mathbf{L}^1_{\mathbb{R}}} < \dfrac{b-a}{b^2}$ such that $F(x,y) \subset (p(t)\|x\|+bq(t)\|y\|)\overline{\mathbf{B}}_E$ for all $t \in [a,b]$ and for all $(x,y) \in E \times E$. Let $v_0 \in E$. Then the $\mathbf{W}^{2,1}_E([a,b])$-solutions set of the problem $(P_F)$ is nonempty and compact in $\mathbf{C}^1_E([a,b])$.*

For the proof of our Theorem we need the following Lemma.

**Lemma 3.5.** *Let $E$ be a finite dimensional space. Suppose that the hypotheses of Theorem 3.4 are satisfied. If $u$ is a solution in $\mathbf{W}^{2,1}_E([a,b])$ of the problem $(P_F)$, then for all $t \in [a,b]$ we have*

$$\|u(t)\| \leq \alpha, \quad \|\dot{u}(t)\| \leq \frac{\alpha}{b}$$

*where*

$$\alpha = \frac{\|v_0\|}{1 - \dfrac{b^2}{b-a}\|p+q\|_{\mathbf{L}^1_{\mathbb{R}}}}.$$

*Proof.* Suppose that $u : [a,b] \to E$ is a $\mathbf{W}^{2,1}_E([a,b])$-solution of $(P_F)$. Then, there exists a measurable mapping $f : [a,b] \to E$ such that $f(t) \in F(u_f(t), \dot{u}_f(t))$ for almost every $t \in [a,b]$ and

$$u(t) = u_f(t) = v_0 + \frac{b}{b-a}\int_a^b G(t,s)f(s)ds \ \forall t \in [a,b].$$

Consequently, for all $t \in [a,b]$

$$\begin{aligned}
\|u(t)\| &= \|v_0 + \frac{b}{b-a}\int_a^b G(t,s)f(s)ds\| \\
&\le \|v_0\| + \frac{b}{b-a}\int_a^b |G(t,s)|\|f(s)\|ds \\
&\le \|v_0\| + \frac{b}{b-a}\int_a^b b(p(s)\|u(s)\| + bq(s)\|\dot{u}(s)\|)ds \\
&\le \|v_0\| + \frac{b}{b-a}\int_a^b b(p(s)\|u\|_{\mathbf{C}^1_E} + q(s)\|u\|_{\mathbf{C}^1_E})ds \\
&\le \|v_0\| + \frac{b^2}{b-a}\|u\|_{\mathbf{C}^1_E}\int_a^b (p(s)+q(s))ds,
\end{aligned}$$

and hence,

$$\|u(t)\| \le \|v_0\| + \frac{b^2}{b-a}\|p+q\|_{\mathbf{L}^1_{\mathbb{R}}}\|u\|_{\mathbf{C}^1_E}.$$

In the same way we have

$$\begin{aligned}
\|\dot{u}(t)\| &= \|\frac{b}{b-a}\int_a^b \frac{\partial G}{\partial t}(t,s)f(s)ds\| \le \frac{b}{b-a}\int_a^b |\frac{\partial G}{\partial t}(t,s)|\|f(s)\|ds \\
&\le \frac{b}{b-a}\int_a^b (p(s)\|u(s)\| + bq(s)\|\dot{u}(s)\|)ds \le \frac{b}{b-a}\|p+q\|_{\mathbf{L}^1_{\mathbb{R}}}\|u\|_{\mathbf{C}^1_E},
\end{aligned}$$

and hence

$$b\|\dot{u}(t)\| \le \frac{b^2}{b-a}\|p+q\|_{\mathbf{L}^1_{\mathbb{R}}}\|u\|_{\mathbf{C}^1_E} \le \|v_0\| + \frac{b^2}{b-a}\|p+q\|_{\mathbf{L}^1_{\mathbb{R}}}\|u\|_{\mathbf{C}^1_E}.$$

These last inequalities show that

$$\|u\|_{\mathbf{C}^1_E} \le \|v_0\| + \frac{b^2}{b-a}\|p+q\|_{\mathbf{L}^1_{\mathbb{R}}}\|u\|_{\mathbf{C}^1_E},$$

or

$$(1 - \frac{b^2}{b-a}\|p+q\|_{\mathbf{L}^1_{\mathbb{R}}})\|u\|_{\mathbf{C}^1_E} \le \|v_0\|,$$

equivalently

$$\|u\|_{\mathbf{C}^1_E} \le \frac{\|v_0\|}{1 - \dfrac{b^2}{b-a}\|p+q\|_{\mathbf{L}^1_{\mathbb{R}}}} = \alpha.$$

By the definition of $\|u\|_{\mathbf{C}^1_E}$ we conclude that for all $t \in [a,b]$

$$\|u(t)\| \le \alpha \quad and \quad \|\dot{u}(t)\| \le \frac{\alpha}{b}.$$

*Proof of Theorem 3.4.* Let us consider the mapping $\varphi_\kappa : E \to E$ defined by

$$\varphi_\kappa(x) \quad = \quad \begin{cases} \|x\| \; if \; \|x\| \le \kappa \\ \frac{\kappa x}{\|x\|} \; if \; \|x\| > \kappa, \end{cases}$$

and consider the multifunction $F_0 : E \times E \rightrightarrows E$ defined by

$$F_0(x, y) = F(\varphi_\alpha(x), \varphi_{\frac{\alpha}{b}}(y)).$$

Then $F_0$ inherits the hypotheses on $F$, and furthermore, for all $(x, y) \in E \times E$

$$\begin{aligned} F_0(x, y) = \quad & F(\varphi_\alpha(x), \varphi_{\frac{\alpha}{b}}(y)) \\ & \subset (p(t)\|\varphi_\alpha(x)\| + bq(t)\|\varphi_{\frac{\alpha}{b}}(y)\|)\overline{\mathbf{B}}_E \\ & \subset (p(t)\alpha + b\frac{1}{b}q(t)\alpha)\overline{\mathbf{B}}_E = \alpha(p(t) + q(t))\overline{\mathbf{B}}_E = \beta(t)\overline{\mathbf{B}}_E. \end{aligned}$$

Consequently, $F_0$ satisfies all the hypotheses of Proposition 3.3. Hence, we conclude the existence of a $\mathbf{W}_E^{2,1}([a, b])$-solution of the problem $(P_{F_0})$.
Now, let us prove that $u$ is a solution of $(P_{F_0})$ if and only if $u$ is a solution of $(P_F)$.
If $u$ is a solution of $(P_{F_0})$, there exists a measurable mapping $f_0$ such that $u = u_{f_0}$ and $f_0(t) \in F_0(u(t), \dot{u}(t))$, a.e., with for almost every $t \in [a, b]$

$$\|f_0(t)\| \le \beta(t) = \alpha(p(t) + q(t)).$$

Using this inequality and the fact that for all $t \in [a, b]$

$$u(t) = v_0 + \frac{b}{b-a}\int_a^b G(t, s)f_0(s)ds, \text{ and } \dot{u}(t) = \frac{b}{b-a}\int_a^b \frac{\partial G}{\partial t}(t, s)f_0(s)ds,$$

we obtain

$$\begin{aligned} \|u(t)\| \le \|v_0\| + \frac{b^2}{b-a}\|\beta\|_{\mathbf{L}^1_{\mathbb{R}}} = \|v_0\| + \frac{b^2}{b-a}\alpha\|p+q\|_{\mathbf{L}^1_{\mathbb{R}}} \\ = \|v_0\| + (\frac{b^2}{b-a})\frac{\|v_0\|}{1 - \frac{b^2}{b-a}\|p+q\|_{\mathbf{L}^1_{\mathbb{R}}}}\|p+q\|_{\mathbf{L}^1_{\mathbb{R}}} = \frac{\|v_0\|}{1 - \frac{b^2}{b-a}\|p+q\|_{\mathbf{L}^1_{\mathbb{R}}}} = \alpha, \end{aligned}$$

and

$$\begin{aligned} \|\dot{u}(t)\| \le \frac{b}{b-a}\|\beta\|_{\mathbf{L}^1_{\mathbb{R}}} = \frac{b}{b-a}\alpha\|p+q\|_{\mathbf{L}^1_{\mathbb{R}}} = (\frac{b}{b-a})\frac{\|v_0\|}{1 - \frac{b^2}{b-a}\|p+q\|_{\mathbf{L}^1_{\mathbb{R}}}}\|p+q\|_{\mathbf{L}^1_{\mathbb{R}}} \\ < (\frac{b}{b-a})(\frac{\|v_0\|}{1 - \frac{b^2}{b-a}\|p+q\|_{\mathbf{L}^1_{\mathbb{R}}}})(\frac{b-a}{b^2}) = \frac{\alpha}{b}. \end{aligned}$$

These last relations show that $\varphi_\alpha(u(t)) = u(t)$ and $\varphi_{\frac{\alpha}{b}}(\dot{u}(t)) = \dot{u}(t)$, or equivalently $F_0(u(t), \dot{u}(t)) = F(u(t), \dot{u}(t))$. Consequently, $u$ is a solution of $(P_F)$.

Suppose now that $u$ is a solution of $(P_F)$. By Lemma 3.5, we have for all $t \in [a,b]$

$$\|u(t)\| \leq \alpha \ \ and \ \ \|\dot{u}(t)\| \leq \frac{\alpha}{b}.$$

Then, $F(u(t),\dot{u}(t)) = F_0(u(t),\dot{u}(t))$, that is, $u$ is a solution of $(P_{F_0})$. ■

Now we are able to give our main result.

**Theoreme 3.6.** *Let $E$ be a finite dimensional space and $F : E \times E \rightrightarrows E$ be an almost convex compact valued multifunction, upper semicontinuous on $E \times E$ and satisfying the following assumptions:*
***(1)*** *there is two nonnegative functions $p, q \in \mathbf{L}^1_{\mathbb{R}}([a,b])$, satisfying $\|p+q\|_{\mathbf{L}^1_{\mathbb{R}}} < \dfrac{b-a}{b^2}$, such that $F(x,y) \subset (p(t)\|x\| + bq(t)\|y\|)\overline{\mathbf{B}}_E$ for all $(x,y) \in E \times E$,*
***(2)*** *$F(x,\xi y) \subseteq \xi F(x,y)$ for all $(x,y) \in E \times E$ and for every $\xi > 0$.*
*Let $v_0 \in E$. Then there is at least a $\mathbf{W}^{2,1}_E([a,b])$-solution of the problem $(P_F)$.*

For the proof we need the following result.

**Theoreme 3.7.** *Let $F : E \times E \rightrightarrows E$ be a multifunction upper semicontinuous on $E \times E$. Suppose that the assumption **(2)** in Theorem 3.6 is also satisfied. Let $v_0 \in E$ and let $x : [a,b] \to E$, be a solution of the problem*

$$(P_{co(F)}) \begin{cases} \ddot{u}(t) \in co(F(u(t),\dot{u}(t))), \ \ a.e. \ \ t \in [a,b], \\ u(a) = u(b) = v_0, \end{cases}$$

*and assume that there are two constants $\lambda_1$ and $\lambda_2$, satisfying $0 \leq \lambda_1 \leq 1 \leq \lambda_2$, such that for almost every $t \in [a,b]$, we have*

$$\lambda_1 \ddot{x}(t) \in F(x(t),\dot{x}(t)) \ \ and \ \ \lambda_2 \ddot{x}(t) \in F(x(t),\dot{x}(t)).$$

*Then there exists $t = t(\tau)$, a nondecreasing absolutely continuous map of the interval $[a,b]$ onto itself, such that the map $\tilde{x}(\tau) = x(t(\tau))$ is a solution of the problem $(P_F)$. Moreover $\tilde{x}(a) = \tilde{x}(b) = v_0$.*

*Proof. Step 1.* Let $[\alpha,\beta]$ $(0 \leq \alpha < \beta < +\infty)$ be an interval, and assume that there exist two constants $\lambda_1, \lambda_2$, with the properties stated above. Assume that $\lambda_1 > 0$. We claim that there exist two measurable subsets of $[\alpha,\beta]$, having characteristic functions $\mathcal{X}_1$ and $\mathcal{X}_2$ such that $\mathcal{X}_1 + \mathcal{X}_2 = \mathcal{X}_{[\alpha,\beta]}$, and an absolutely continuous function $s = s(\tau)$ on $[\alpha,\beta]$, satisfying $s(\alpha) - s(\beta) = \alpha - \beta$, such that

$$\dot{s}(\tau) = \frac{1}{\lambda_1}\mathcal{X}_1(\tau) + \frac{1}{\lambda_2}\mathcal{X}_2(\tau).$$

Indeed, set

$$\gamma = \begin{cases} \dfrac{1}{2} & when \ \ \lambda_1 = \lambda_2 = 1 \\ \dfrac{\lambda_2 - 1}{\lambda_2 - \lambda_1} & otherwise. \end{cases}$$

With this definition we have that $0 \le \gamma \le 1$ and that both equalities

$$1 = \gamma + (1-\gamma) = \gamma\lambda_1 + (1-\gamma)\lambda_2.$$

In particular, we have

$$\int_\alpha^\beta 1dt = \int_\alpha^\beta [\frac{\gamma\lambda_1}{\lambda_1} + \frac{(1-\gamma)\lambda_2}{\lambda_2}]dt.$$

Applying Liapunov's theorem on the range of measures, to infer the existence of two subsets having characteristic functions $\mathcal{X}_1(.), \mathcal{X}_2(.)$ such that $\mathcal{X}_1 + \mathcal{X}_2 = \mathcal{X}_{[\alpha,\beta]}$ and with the property that

$$\int_\alpha^\beta 1dt = \int_\alpha^\beta [\frac{1}{\lambda_1}\mathcal{X}_1(t) + \frac{1}{\lambda_2}\mathcal{X}_2(t)]dt.$$

Define $\dot{s}(\tau) = \dfrac{1}{\lambda_1}\mathcal{X}_1(\tau) + \dfrac{1}{\lambda_2}\mathcal{X}_2(\tau)$. Then $\displaystyle\int_\alpha^\beta \dot{s}(\tau)d\tau = \beta - \alpha$.

*Step 2.* (a) Consider

$$C = \{\tau \in [a,b] : \ \ 0 \in F(x(\tau), \dot{x}(\tau))\}.$$

We have that $C$ is a closed set. Indeed, let $(\tau_n)$ be a sequence in $C$ converging to $\tau \in [a,b]$. Then, for each $n \in \mathbb{N}$,

$$0 \in F(x(\tau_n), \dot{x}(\tau_n)).$$

Since $F$ is upper semicontinuous with compact values we have that it's graph is closed, and since $x(\cdot)$ and $\dot{x}(\cdot)$ are continuous we get $0 \in F(x(\tau), \dot{x}(\tau))$, that is $C$ is closed.

(b) Consider the case in which $C$ is empty. In this case, it cannot be that $\lambda_1 = 0$, and the Step 1 can be applied to the interval $[a,b]$. Set $s(\tau) = a + \displaystyle\int_a^\tau \dot{s}(\omega)d\omega$, $s$ is increasing and we have $s(a) = a$ and $s(b) = a + \displaystyle\int_a^b \dot{s}(\omega)d\omega = a + b - a = b$, that is $s$ maps $[a,b]$ onto itself. Let $t : [a,b] \to [a,b]$ be its inverse, so $t(a) = a$; $t(b) = b$, and we have $\dfrac{d}{d\tau}s(t(\tau)) = \dot{s}(t(\tau))\dot{t}(\tau) = 1$. Then, $\dot{t}(\tau) = \frac{1}{\dot{s}(t(\tau))} = \lambda_1\mathcal{X}_1(t(\tau)) + \lambda_2\mathcal{X}_2(t(\tau))$, and $\ddot{t}(\tau) = 0$. Consider the map $\tilde{x}(\tau) = x(t(\tau))$. We have $\dfrac{d}{d\tau}\tilde{x}(\tau) = \dot{t}(\tau)\dot{x}(t(\tau))$, and $\dfrac{d^2}{d\tau^2}\tilde{x}(\tau) =$

$(\dot{t}(\tau))^2\ddot{x}(t(\tau))+\ddot{t}(\tau)\dot{x}(t(\tau))=\ddot{x}(t(\tau))(\dot{t}(\tau))^2$. Hence

$$\frac{1}{\dot{t}(\tau)}\frac{d^2}{d\tau^2}\tilde{x}(\tau)=\ddot{x}(t(\tau))(\dot{t}(\tau))=\ddot{x}(t(\tau))[\lambda_1\mathcal{X}_1(t(\tau))+\lambda_2\mathcal{X}_2(t(\tau))]$$

$$\in F(x(t(\tau)),\dot{x}(t(\tau)))=F(\tilde{x}(\tau),\frac{1}{\dot{t}(\tau)}\dot{\tilde{x}}(\tau)),$$

and by the assumption 2, we have

$$F(\tilde{x}(\tau),\frac{1}{\dot{t}(\tau)}\dot{\tilde{x}}(\tau))\subseteq\frac{1}{\dot{t}(\tau)}F(\tilde{x}(\tau),\dot{\tilde{x}}(\tau))$$

then we get

$$\frac{1}{\dot{t}(\tau)}\frac{d^2}{d\tau^2}\tilde{x}(\tau)\in\frac{1}{\dot{t}(\tau)}F(\tilde{x}(\tau),\dot{\tilde{x}}(\tau)).$$

Consequently

$$\frac{d^2}{d\tau^2}\tilde{x}(\tau)\in F(\tilde{x}(\tau),\dot{\tilde{x}}(\tau)).$$

(c) Now we shall assume that $C$ is nonempty. Let $c=\sup\{\tau;\ \tau\in C\}$, there is a sequence $(\tau_n)$ in $C$ such that $\underset{n\to\infty}{lim}\tau_n=c$. Since $C$ is closed we get $c\in C$. The complement of $C$ is open relative to $[a,b]$, it consists of at most countably many nonoverlapping open intervals $]a_i,b_i[$, with the possible exception of one of the form $[a_{i_i},b_{i_i}[$ with $a_{i_i}=a$ and one of the form $]a_{i_f},b_{i_f}]$ with $a_{i_f}=c$. For each $i$, apply Step 1 to the interval $]a_i,b_i[$ to infer the existence of $K_1^i$ and $K_2^i$, two subsets of $]a_i,b_i[$ with characteristic functions $\mathcal{X}_1^i(.)$, $\mathcal{X}_2^i(.)$ such that $\mathcal{X}_1^i+\mathcal{X}_2^i=\mathcal{X}_{]a_i,b_i[}$, setting

$$\dot{s}(\tau)=\frac{1}{\lambda_1}\mathcal{X}_1^i(\tau)+\frac{1}{\lambda_2}\mathcal{X}_2^i(\tau)$$

we obtain

$$\int_{a_i}^{b_i}\dot{s}(\omega)d\omega=b_i-a_i.$$

(d) On $[a,c]$ set

$$\dot{s}(\tau)=\frac{1}{\lambda_2}\mathcal{X}_C(\tau)+\sum_i(\frac{1}{\lambda_1}\mathcal{X}_1^i(\tau)+\frac{1}{\lambda_2}\mathcal{X}_2^i(\tau)),$$

where the sum is over all intervals contained in $[a,c]$, i.e., with the exception of $]c,b]$. We have that

$$\int_a^c\dot{s}(\omega)d\omega=\kappa\leq c-a$$

since $\lambda_2\geq 1$ and $\int_{a_i}^{b_i}\dot{s}(\omega)d\omega=b_i-a_i$. Setting $s(\tau)=a+\int_a^{\tau}\dot{s}(\omega)d\omega$, we obtain that $s$ is an invertible map from $[a,c]$ to $[a,\kappa+a]$.

(e) Define $t : [a, \kappa + a] \to [a, c]$ to be the inverse of $s(.)$. Extend $t(.)$ as an absolutely continuous map $\tilde{t}(.)$ on $[a, c]$, setting $\dot{\tilde{t}}(\tau) = 0$ for $\tau \in ]\kappa + a, c]$. We claim that the function $\tilde{x}(\tau) = x(\tilde{t}(\tau))$ is a solution to the problem $(P_F)$ on the interval $[a, c]$. Moreover, we claim that it satisfies $\tilde{x}(c) = x(c)$.
Observe that, as in (b), we have that for $\tau \in [a, \kappa + a]$, $\tilde{t}(\tau) = t(\tau)$ is invertible, such that $\dot{t}(\tau) = \lambda_2 \mathcal{X}_C(\tau) + \sum_i (\lambda_1 \mathcal{X}_1^i(\tau) + \lambda_2 \mathcal{X}_2^i(\tau))$. Since

$$\frac{d^2}{d\tau^2}\tilde{x}(\tau) = (\dot{t}(\tau))^2 \ddot{x}(t(\tau)) + \ddot{t}(\tau)\dot{x}(t(\tau)) = \ddot{x}(t(\tau))(\dot{t}(\tau))^2,$$

we get

$$\begin{aligned}\frac{1}{\dot{t}(\tau)}\frac{d^2\tilde{x}(\tau)}{d\tau^2} &= \ddot{x}(t(\tau))(\dot{t}(\tau)) = [\lambda_2 \mathcal{X}_C(t(\tau)) + \sum_i (\lambda_1 \mathcal{X}_1^i(t(\tau)) + \lambda_2 \mathcal{X}_2^i(t(\tau)))]\ddot{x}(t(\tau)) \\ &\in F(x(t(\tau)), \dot{x}(t(\tau))) = F(\tilde{x}(\tau), \frac{1}{\dot{t}(\tau)}\dot{\tilde{x}}(\tau)) \\ &\subseteq \frac{1}{\dot{t}(\tau)} F(\tilde{x}(\tau), \dot{\tilde{x}}(\tau)).\end{aligned}$$

Consequently

$$\frac{d^2}{d\tau^2}\tilde{x}(\tau) \in F(\tilde{x}(\tau), \dot{\tilde{x}}(\tau)).$$

In particular, from $t(\kappa + a) = c$ and $\dot{\tilde{t}}(\tau) = 0$ for all $\tau \in ]\kappa + a, c]$ we obtain

$$\tilde{t}(\tau) = \tilde{t}(\kappa + a) = t(\kappa + a), \ \forall \tau \in ]\kappa + a, c]$$

then

$$\tilde{x}(\kappa + a) = x(\tilde{t}(\kappa + a)) = x(\tilde{t}(\tau)) = \tilde{x}(\tau), \ \forall \tau \in ]\kappa + a, c]$$

so, on $]\kappa + a, c]$, $\tilde{x}$ is constant, and since $c \in C$ we have

$$\frac{d^2}{d\tau^2}\tilde{x}(\tau) = 0 \in F(x(c), \dot{x}(c)) = F(\tilde{x}(\kappa{+}a), \frac{1}{\dot{t}(\kappa + a)}\dot{\tilde{x}}(\kappa{+}a)) \subset F(\tilde{x}(\tau), \dot{\tilde{x}}(\tau)).$$

This proves the claim.
(f) It is left to define the solution on $[c, b]$. On it, $\lambda_1 > 0$ and the construction of Step 1 and (b) can be repeated to find a solution to problem $(P_F)$ on $[c, b]$. This completes the proof of the theorem. ■

*Proof of the Theorem 3.6.* In view of Theorem 3.4, and since $co(F) : E \times E \rightrightarrows E$ is a multifunction with compact values, upper semicontinuous on $E \times E$ and furthermore, for all $(x, y) \in E \times E$,

$$co(F(x, y)) \subset (p(t)\|x\| + bq(t)\|y\|)co(\overline{\mathbf{B}}_E) = (p(t)\|x\| + bq(t)\|y\|)\overline{\mathbf{B}}_E,$$

we conclude the existence of a $\mathbf{W}_E^{2,1}([a, b])$-solution $x$ of the problem $(P_{co(F)})$. By the almost convexity of the values of $F$, there exist two constants $\lambda_1$ and

$\lambda_2$, satisfying $0 \leq \lambda_1 \leq 1 \leq \lambda_2$, such that, for almost every $t \in [a, b]$, we have

$$\lambda_1 \ddot{x}(t) \in F(x(t), \dot{x}(t)) \text{ and } \lambda_2 \ddot{x}(t) \in F(x(t), \dot{x}(t)).$$

Using Theorem 3.7, we conclude the existence of a $\mathbf{W}_E^{2,1}([a, b])$-solution of the problem $(P_F)$.
This completes the proof of our main result. ∎

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