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Asymptotic behaviour of neutral differential equations of third-order with negative term

Dedicated to Professor Tibor Krisztin on the occasion of his 60th birthday

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Abstract. We derive new comparison theorems and oscillation criteria for neutral differential equations of third order with negative term. We show that one can deduce oscillation criteria for the equation with negative term from those for the equation with positive term. We give some examples and show applications to equation with symmetric operator.

Keywords: oscillation of solutions, neutral equation, functional equation.

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1 Introduction

Functional and neutral differential equations play an important role in many applications and have a long and rich history with a substantial contribution of Hungarian mathematicians, among them Tibor Krisztin who focused among others on asymptotic properties of delay and neutral functional differential equations of first order and applications, see e.g. [11] and [12].

Recently, much attention has been devoted to the oscillation of the neutral differential equation with a positive term

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}\left[x(t)+a(t)x(\gamma(t))\right]'\right)'\right)' + q(t)f\left(x(\delta(t))\right) = 0, \tag{E+}$$

see e.g. [1, 2, 7–9, 15] and references given there.

The aim of this paper is to consider the third-order neutral differential equation with negative term

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}\left[z(t)+a(t)z(\gamma(t))\right]'\right)'\right)' - q(t)f\left(z(\delta(t))\right) = 0 \tag{E−}$$

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where $t \geq t_0$. Moreover, we will consider the linear version of this equation

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}\left[z(t)+a(t)z\big(\gamma(t)\big)\right]'\right)'\right)' - q(t)z\big(\delta(t)\big) = 0. \tag{L$-$}$$

Observe that differential operators in both equations, i.e. in equations (E+) and (E−), are mutually adjoint and therefore functions in operators are interchanged.

We will always assume that

(i) $p, r, q, a, \gamma, \delta \in C[t_0, \infty)$, $p(t)$, $r(t)$, $q(t)$, $\gamma(t)$, $\delta(t)$ are positive for $t \geq t_0$,

(ii) $\int_{t_0}^{\infty} p(t)\,\mathrm{d}t = \int_{t_0}^{\infty} r(t)\,\mathrm{d}t = \infty$,

(iii) $\gamma(t) \leq t$, $\lim_{t\to\infty} \gamma(t) = \infty$,

(iv) $\lim_{t\to\infty} \delta(t) = \infty$,

(v) $0 \leq a(t) \leq a_0 < 1$ for $t \geq t_0$,

(vi) $f \in C(\mathbb{R}, \mathbb{R})$, f is odd, $f(v)v > 0$ for $v \neq 0$.

It will be convenient to set for each solution $z(t)$ of (E−)

$$v(t) = z(t) + a(t)z\big(\gamma(t)\big). \tag{1.1}$$

If v is a function defined by (1.1), then functions

$$v^{[0]} = v, \qquad v^{[1]} = \frac{1}{p(t)}v', \qquad v^{[2]} = \frac{1}{r(t)}\left(\frac{1}{p(t)}v'\right)' = \frac{1}{r(t)}\big(v^{[1]}\big)'.$$

are called quasiderivatives of v.

A solution z of (E−) is said to be *proper* if it exists on the interval $[t_0, \infty)$ and satisfies the condition

$$\sup\{|z(s)| : t \leq s < \infty\} > 0 \quad \text{for any } t \geq t_0.$$

A proper solution is called *oscillatory* or *nonoscillatory* according to whether it does or does not have arbitrarily large zeros.

Definition 1.1. Equation (E−) is said to have *property B* if any proper solution z of (E−) is either oscillatory or satisfies

$$\lim_{t\to\infty} \frac{|z(t)|}{\int_{t_0}^{t} p(s)\int_{t_0}^{s} r(u)\,\mathrm{d}u\,\mathrm{d}s} = \infty. \tag{1.2}$$

Later we show that (1.2) is equivalent to $\lim_{t\to\infty}\left|v^{[2]}(t)\right| = \infty$ (see Lemma 3.6). Hence, in case $a(t) \equiv 0$ this yields the original definition of property B introduced by I. Kiguradze for ordinary differential equations (see [10]).

To simplify notation, we set

$$L_3^A(\cdot) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{r(t)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{p(t)}\frac{\mathrm{d}}{\mathrm{d}t}(\cdot), \qquad L_3(\cdot) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{p(t)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{r(t)}\frac{\mathrm{d}}{\mathrm{d}t}(\cdot).$$

2 Preliminaries: linear ODE and FDE

First, consider the special case of (L−), where $a(t) \equiv 0$ and $\delta(t) = t$, i.e. the third-order linear differential equation

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}z'(t)\right)'\right)' - q(t)z(t) = 0. \tag{2.1}$$

For completeness, we summarize basic results concerning the oscillatory behaviour of (2.1), which we will need in our later consideration.

It is a well-known fact (see for instance [10]) that all nonoscillatory solutions x of (2.1) can be divided into the two classes:

$$\mathcal{M}_1 = \left\{z \text{ solution of } (2.1),\ \exists\, T_z : z(t)z^{[1]}(t) > 0,\ z(t)z^{[2]}(t) < 0 \text{ for } t \geq T_z\right\}$$
$$\mathcal{M}_3 = \left\{z \text{ solution of } (2.1),\ \exists\, T_z : z(t)z^{[1]}(t) > 0,\ z(t)z^{[2]}(t) > 0 \text{ for } t \geq T_z\right\}.$$

Definition 2.1. Equation (2.1) is said to have *property B* if any proper solution z of (2.1) is either oscillatory or satisfies

$$\left|z^{[i]}(t)\right| \uparrow \infty \quad \text{as } t \to \infty,\ i = 0, 1, 2.$$

Equation (2.1) is said to have *weak property B* if any proper solution x of (2.1) is either oscillatory or belongs to $\mathcal{M}_3$.

Equation (2.1) is said to be *oscillatory* if it has at least one oscillatory solution, otherwise it is said to be *nonoscillatory*.

Theorem A ([6, Theorem 6])**.** *If*

$$\int_{t_0}^{\infty} q(t) \int_{t_0}^{t} r(s) \int_{t_0}^{s} p(v)\, \mathrm{d}v\, \mathrm{d}s\, \mathrm{d}t < \infty, \tag{2.2}$$

then (2.1) *is nonoscillatory.*

Theorem B ([3, Lemma 2.2])**.** *Equation* (2.1) *has weak property B if and only if it is oscillatory.*

Theorem C ([3, Theorem 2.2])**.** *Equation* (2.1) *has property B if and only if it is oscillatory and*

$$\int_{t_0}^{\infty} q(t) \int_{t_0}^{t} p(s) \int_{t_0}^{s} r(v)\, \mathrm{d}v\, \mathrm{d}s\, \mathrm{d}t = \infty. \tag{2.3}$$

Proposition 2.2. *The class $\mathcal{M}_3 \neq \emptyset$ for* (2.1)*. If* (2.2) *holds, then $\mathcal{M}_1 \neq \emptyset$ for* (2.1)*.*

Proof. The first part follows from [14, Lemma 2]. The second part follows from Theorems A and B. □

Proposition 2.3. *Every solution z of* (2.1) *from class $\mathcal{M}_3$ satisfies*

$$|z^{[2]}(t)| < \infty$$

if and only if

$$\int_{t_0}^{\infty} q(t) \int_{t_0}^{t} p(s) \int_{t_0}^{s} r(v)\, \mathrm{d}v\, \mathrm{d}s\, \mathrm{d}t < \infty.$$

Proof. It follows from the proof of Theorem 1 in [4]. □

Proposition 2.4 ([5, Theorem 7])**.** *Consider equation* (2.1)*, where* $p(t) = r(t)$ *for large* t*. Then* (2.1) *has property B if and only if it has weak property B.*

Now consider the linear functional differential equation

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}z'(t)\right)'\right)' - q(t)z\big(\delta(t)\big) = 0. \tag{2.4}$$

The classification of nonoscillatory solutions of (2.4) and definitions of property B and weak property B are the same as for equation (2.1).

One of our main tools will be the comparison method for third-order linear functional differential equations

$$L_3^{\mathcal{A}} y(t) - q_1(t) y\big(\delta_1(t)\big) = 0 \tag{2.5}$$

and

$$L_3^{\mathcal{A}} z(t) - q_2(t) z\big(\delta_2(t)\big) = 0 \tag{2.6}$$

where $q_1(t) \geq q_2(t) > 0$ and $\lim_{t\to\infty} \delta_1(t) = \lim_{t\to\infty} \delta_2(t) = \infty$.

Proposition 2.5. *Assume*

$$\delta_1(t) \geq \delta_2(t) \quad \textit{and} \quad q_1(t) \geq q_2(t) \quad \textit{for } t \geq t_1.$$

a) If there exists a solution $y \in \mathcal{M}_1$ *of* (2.5)*, then there exists a solution* $z \in \mathcal{M}_1$ *of* (2.6)*.*

b) If there exists a solution $y \in \mathcal{M}_3$ *of* (2.5) *such that* $|y^{[2]}(t)| < \infty$*, then there exists a solution* $z \in \mathcal{M}_3$ *of* (2.6) *such that* $|z^{[2]}(t)| < \infty$*.*

Proof. It follows from [13, Theorem 2-ii)] and its proof. □

Proposition 2.6. *If* $\delta(t) \leq t$ *and*

$$\int_{t_0}^{\infty} q(t) \int_{t_0}^{t} p(s) \int_{t_0}^{s} r(v)\, \mathrm{d}v\, \mathrm{d}s\, \mathrm{d}t < \infty, \tag{2.7}$$

then equation (2.4) *has a solution* $x \in \mathcal{M}_3$ *such that* $\lim_{t\to\infty} |z^{[2]}(t)| < \infty$*.*

Proof. By Propositions 2.2 and 2.3, equation (2.1) has a solution z in the class $\mathcal{M}_3$ such that $\lim_{t\to\infty} |z^{[2]}(t)| < \infty$. Now the conclusion follows from Proposition 2.5 b). □

The following theorem extends Proposition 2.4 to functional differential equations. This result is interesting in the light of results from the book [17], where various criteria for weak property B are given.

Theorem 2.7. *Assume that* $p(t) = r(t)$*,* $\delta(t) \leq t$ *and*

$$\liminf_{t\to\infty} \frac{\int_{t_0}^{\delta(t)} p(s) \int_{t_0}^{s} p(u)\, \mathrm{d}u\, \mathrm{d}s}{\int_{t_0}^{t} p(s) \int_{t_0}^{s} p(u)\, \mathrm{d}u\, \mathrm{d}s} > 0. \tag{2.8}$$

Then (2.4) *has property B if and only if it has weak property B.*

Proof. "$\Longrightarrow$": It is immediate.

"$\Longleftarrow$": Assume by contradiction that there exists a solution $z \in \mathcal{M}_3$ of (2.4) such that $\lim_{t\to\infty} z^{[2]}(t) = c > 0$, i.e. there exists $\varepsilon > 0$ such that

$$c - \varepsilon \le z^{[2]}(t) \le c.$$

Integrating this inequality twice from t_0 to t we obtain

$$(c-\varepsilon)\int_{t_0}^{t} p(s)\int_{t_0}^{s} p(u)\,\mathrm{d}u\,\mathrm{d}s \le z(t) \le c\int_{t_0}^{t} p(s)\int_{t_0}^{s} p(u)\,\mathrm{d}u\,\mathrm{d}s.$$

Therefore using (2.8) we get

$$1 \ge \frac{z(\delta(t))}{z(t)} \ge \frac{(c-\varepsilon)\int_{t_0}^{\delta(t)} p(s)\int_{t_0}^{s} p(u)\,\mathrm{d}u\,\mathrm{d}s}{c\int_{t_0}^{t} p(s)\int_{t_0}^{s} p(u)\,\mathrm{d}u\,\mathrm{d}s} > 0. \tag{2.9}$$

Consider the linear equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{p(t)}y'(t)\right)'\right)' - q(t)\frac{z(\delta(t))}{z(t)}y(t) = 0. \tag{2.10}$$

Then $y = z$ is a solution of (2.10). By Theorem C, we get

$$\int_{t_0}^{\infty} q(t)\frac{z(\delta(t))}{z(t)}\int_{t_0}^{t} p(s)\int_{t_0}^{s} p(v)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t < \infty.$$

In view of (2.9), we get

$$\int_{t_0}^{\infty} q(t)\int_{t_0}^{t} p(s)\int_{t_0}^{s} p(v)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t < \infty.$$

By Theorem A, the linear equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{p(t)}x'(t)\right)'\right)' - q(t)x(t) = 0$$

is nonoscillatory and it has a solution $x \in \mathcal{M}_1$ by Proposition 2.2.

Consider the delayed equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{p(t)}z'(t)\right)'\right)' - q(t)\frac{x(t)}{x(\delta(t))}z(\delta(t)) = 0. \tag{2.11}$$

Then $z = x$ is a solution of (2.11). Since x is increasing and $\delta(t) \le t$, we have

$$\frac{x(t)}{x(\delta(t))} \ge 1 \quad \text{for large } t.$$

By the comparison theorem for the functional differential equations (Proposition 2.5), equation (2.4) has a solution $x \in \mathcal{M}_1$, a contradiction. □

Example 2.8. Consider the equation

$$z''' - q(t)z(kt) = 0 \tag{2.12}$$

where $0 < k < 1$. A quick computation shows that condition (2.8) holds and therefore (2.12) has property B if and only if it has weak property B.

We finish this part by recalling a result concerning equivalence between property B and property A. Consider the equation

$$L_3 x(t) + q(t)x(t) = 0. \tag{2.13}$$

The equation (2.13) is said to have property A if any proper solution x of (2.13) is either oscillatory or satisfies $\left|x^{[i]}(t)\right| \downarrow 0$ as $t \to \infty$, $i = 0, 1, 2$.

Theorem D ([4, Theorem 1]). *Equation (2.13) has property A if and only if equation (2.1) has property B.*

3 Basic properties of neutral equations

In this section we establish some auxiliary results concerning the properties of solutions of neutral equation (E−).

Lemma 3.1. *Let z be a nonoscillatory solution of (E−) and let v be defined by (1.1). Then v, $v^{[1]}$, $v^{[2]}$ are monotone for large t.*

Proof. The proof is similar to the proof of Lemma 1 in [8] and therefore it is omitted. □

Again, as in the case of ordinary (functional) differential equations, we can divide all solutions of (E−) into two classes.

Lemma 3.2. *Let z be a nonoscillatory solution of (E−) and let v be defined by (1.1). Then there are only two possible classes of solutions*

$$\mathcal{M}_1 = \left\{ z \text{ solution}, \ \exists\, T_z : v(t)v^{[1]}(t) > 0, \ v(t)v^{[2]}(t) < 0 \text{ for } t \geq T_z \right\},$$
$$\mathcal{M}_3 = \left\{ z \text{ solution}, \ \exists\, T_z : v(t)v^{[1]}(t) > 0, \ v(t)v^{[2]}(t) > 0 \text{ for } t \geq T_z \right\}.$$

Proof. Without loss of generality we may assume that there exists t_1 such that $z\big(\gamma(t)\big) > 0$, $z\big(\delta(t)\big) > 0$, $z(t) > 0$ for $t \geq t_1$. Then $v(t) \geq z(t) > 0$ and from (E−)

$$\big(v^{[2]}(t)\big)' = q(t) f\big(z(\delta(t))\big) > 0.$$

Therefore $v^{[2]}$ is increasing and there exists $t_2 \geq t_1$ such that there are two possibilities. Either $v^{[2]}(t) > 0$ or $v^{[2]}(t) < 0$ for $t \geq t_2$.

Assume that $v^{[2]}(t) > 0$ for $t \geq t_2$. Then there exists an $M > 0$ such that

$$v^{[2]}(t) \geq M > 0.$$

Integrating from t_2 to t we get

$$v^{[1]}(t) - v^{[1]}(t_2) \geq M \int_{t_2}^{t} r(s)\,\mathrm{d}s,$$
$$v^{[1]}(t) \geq v^{[1]}(t_2) + M \int_{t_2}^{t} r(s)\,\mathrm{d}s.$$

Letting $t \to \infty$ and using the fact that $\int_{t_0}^{\infty} r(t)\,\mathrm{d}t = \infty$, we get $v^{[1]}(t) \to \infty$, i.e. $v^{[1]}(t) > 0$ eventually, i.e. z is from the class $\mathcal{M}_3$.

Now assume that $v^{[2]}(t) < 0$ for $t \geq t_2$. Therefore $v^{[1]}$ is decreasing and there exists $t_3 \geq t_2$ such that there are two possibilities, either $v^{[1]}(t) > 0$ or $v^{[1]}(t) < 0$ for $t \geq t_3$. Assume that $v^{[1]}(t) < 0$. Then there exists a constant $N > 0$ such that

$$v^{[1]}(t) \leq -N < 0.$$

Integrating this inequality from t_3 to t we have

$$v(t) \leq v(t_2) - N \int_{t_3}^{t} p(s)\,\mathrm{d}s.$$

Letting $t \to \infty$ and using the fact that $\int_{t_0}^{\infty} p(t)\,\mathrm{d}t = \infty$ we get $v(t) \to -\infty$, i.e. $v(t) < 0$ eventually, a contradiction. Hence $v^{[1]}(t) > 0$ and z is from the class $\mathcal{M}_1$. □

Lemma 3.3. *Every solution z of (E−) satisfies*

$$(1 - a_0)|v(t)| \leq |z(t)| \leq |v(t)| \tag{3.1}$$

for $t \geq T$, where v is defined by (1.1).

Proof. The proof is similar to the proof of Lemma 2 in [8] and therefore it is omitted. □

First, we prove a lemma which helps with characterizing solutions from the class $\mathcal{M}_1$.

Lemma 3.4. *Assume that z is a solution of (E−) from the class $\mathcal{M}_1$. Then*

$$\lim_{t\to\infty} v^{[2]}(t) = 0.$$

Proof. Let $z \in \mathcal{M}_1$. Without loss of generality we may assume that z is eventually positive, i.e. there exists $T \geq t_0$ such that $v(t) > 0$, $v^{[1]}(t) > 0$, $v^{[2]}(t) < 0$ for $t \geq T$.

Assume by contradiction that

$$\lim_{t\to\infty} \big(-v^{[2]}(t) \big) = l > 0.$$

Then we have

$$\big(v^{[1]}(t)\big)' \leq -l r(t)$$

and by integrating from T to t we get

$$v^{[1]}(t) \leq v^{[1]}(T) - l \int_{T}^{t} r(s)\,\mathrm{d}s.$$

Letting $t \to \infty$ we get a contradiction. □

The following lemmas summarize results concerning solutions from the class $\mathcal{M}_3$.

Lemma 3.5. *Let z be an eventually positive solution of (E−) from the class $\mathcal{M}_3$, then*

$$\lim_{t\to\infty} \big|v^{[i]}(t)\big| = \infty, \qquad i = 0, 1.$$

Moreover, if f is increasing, $f(uv) \geq f(u)f(v)$ for all positive u, v and

$$\int_{t_0}^{\infty} q(s) f\left(\int_{t_0}^{\delta(s)} p(u) \int_{t_0}^{u} r(w)\,\mathrm{d}w\,\mathrm{d}u \right) \mathrm{d}s = \infty \tag{3.2}$$

holds, then

$$\lim_{t\to\infty} \big|v^{[2]}(t)\big| = \infty.$$

Proof. Set $y = v^{[1]}$ and $x = v^{[2]}$. Then z is a solution of (E−) if and only if (v, y, x) is a solution of the system

$$\begin{aligned} v'(t) &= p(t)y(t) \\ y'(t) &= r(t)x(t) \\ x'(t) &= q(t)f\left(z(\delta(t))\right). \end{aligned} \tag{3.3}$$

Let $z \in \mathcal{M}_3$. Then the vector $(v,\, y,\, x)$ is a solution of system (3.3) such that

$$\operatorname{sgn} z(t) = \operatorname{sgn} v(t) = \operatorname{sgn} y(t) = \operatorname{sgn} x(t) \quad \text{for large } t.$$

We show that

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} x(t) = \infty.$$

There exists $T \ge t_0$ such that $v(t) > 0$, $y(t) > 0$, $x(t) > 0$ for $t \ge T$. As y is eventually increasing, there exists $T_1 \ge T$ and $K > 0$ such that

$$v'(t) = p(t)y(t) \ge p(t)K \quad \text{for } t \ge T_1.$$

Integrating in $[T_1, t]$ we get

$$v(t) \ge K \int_{T_1}^{t} p(s)\,\mathrm{d}s.$$

Using the assumption $\int_{t_0}^{\infty} p(t)\,\mathrm{d}t = \infty$ we get $\lim_{t\to\infty} v(t) = \infty$.

Since $x(t)$ is eventually increasing, there exists $T_2 \ge T_1$ and $L > 0$ such that

$$y'(t) = r(t)x(t) \ge r(t)L \quad \text{for } t \ge T_2$$

and integrating in $[T_2, t]$ we obtain

$$y(t) \ge L \int_{T_2}^{t} r(s)\,\mathrm{d}s. \tag{3.4}$$

Using the assumption $\int_{t_0}^{\infty} r(t)\,\mathrm{d}t = \infty$ we get $\lim_{t\to\infty} y(t) = \infty$, which completes the proof of the first part of the assertion.

Now integrating the first equation of (3.3) and using (3.4) we obtain

$$v\big(\delta(t)\big) \ge \int_{T_1}^{\delta(t)} p(s)y(s)\,\mathrm{d}s \ge L \int_{T_1}^{\delta(t)} p(s) \int_{T_1}^{s} r(u)\,\mathrm{d}u\,\mathrm{d}s.$$

From here and (3.1)

$$z\big(\delta(t)\big) \ge \frac{L}{1-a_0} \int_{T_1}^{\delta(t)} p(s) \int_{T_1}^{s} r(u)\,\mathrm{d}u\,\mathrm{d}s. \tag{3.5}$$

Using the third equation of (3.3) and (3.5), there exists $T_2 \ge T_1$ such that

$$x'(t) = q(t)f\left(z(\delta(t))\right) \ge q(t)f\left(\frac{L}{1-a_0} \int_{T_1}^{\delta(t)} p(s) \int_{T_1}^{s} r(u)\,\mathrm{d}u\,\mathrm{d}s\right).$$

Using the fact that $f(uv) \ge f(u)f(v)$ and integrating the last inequality from T_2 to t we have

$$x(t) \ge f\left(\frac{L}{1-a_0}\right) \int_{T_2}^{t} q(s) f\left(\int_{T_2}^{\delta(s)} p(u) \int_{T_1}^{u} r(w)\,\mathrm{d}w\,\mathrm{d}u\right)\mathrm{d}s$$

and taking into account that (3.2) holds, we get $\lim_{t\to\infty} x(t) = \infty$, which completes the proof. □

Lemma 3.6. *Let z be a nonoscillatory solution of* (E−) *from the class $\mathcal{M}_3$. Then*

$$\lim_{t\to\infty} \left|v^{[2]}(t)\right| = \infty$$

if and only if

$$\lim_{t\to\infty} \frac{|z(t)|}{\int_{t_0}^t p(s) \int_{t_0}^s r(u)\,\mathrm{d}u\,\mathrm{d}s} = \infty. \tag{3.6}$$

Proof. Without loss of generality, we may assume that there exists $t_1 \geq t_0$ such that $z(t) > 0$, $v^{[1]}(t) > 0$ and $v^{[2]}(t) > 0$ for $t \geq t_1$.

"$\Rightarrow$": Observe that since $\int_{t_0}^t r(s)\,\mathrm{d}s$ is increasing function, there exists $t_2 \geq t_1$ such that for $T \geq t_2$ we have

$$\int_{t_2}^T p(t) \int_{t_2}^t r(s)\,\mathrm{d}s\,\mathrm{d}t \geq \int_{t_2}^T p(t) \int_{t_2}^{t_2+1} r(s)\,\mathrm{d}s\,\mathrm{d}t \geq K \int_{t_2}^T p(t)\,\mathrm{d}t,$$

i.e. the fact that $\int_{t_0}^\infty p(t)\,\mathrm{d}t = \infty$ implies $\int_{t_0}^\infty p(t) \int_{t_0}^t r(s)\,\mathrm{d}s\,\mathrm{d}t = \infty$. This justifies the following computations.

Using (3.1), Lemma 3.5 and the L'Hospital rule we get

$$\begin{aligned}
\lim_{t\to\infty} \frac{z(t)}{\int_{t_0}^t p(s) \int_{t_0}^s r(u)\,\mathrm{d}u\,\mathrm{d}s} &\geq (1-a_0) \lim_{t\to\infty} \frac{v(t)}{\int_{t_0}^t p(s) \int_{t_0}^s r(u)\,\mathrm{d}u\,\mathrm{d}s} \\
&= (1-a_0) \lim_{t\to\infty} \frac{v'(t)}{p(t) \int_{t_0}^t r(s)\,\mathrm{d}s} = (1-a_0) \lim_{t\to\infty} \frac{v^{[1]}(t)}{\int_{t_0}^t r(s)\,\mathrm{d}s} \\
&= (1-a_0) \lim_{t\to\infty} \frac{\left(v^{[1]}(t)\right)'}{r} = (1-a_0) \lim_{t\to\infty} v^{[2]}(t) = \infty,
\end{aligned}$$

which proves the first part of the assertion.

"$\Leftarrow$": Suppose there exists a positive constant L such that $v^{[2]}(\infty) < L$, i.e. as $v^{[2]}$ is increasing $v^{[2]}(t) < L$ for $t_0 \leq t \leq \infty$.

Integrating this inequality twice from t_1 to t we obtain

$$v(t) < v(t_1) + v^{[1]}(t_1) \int_{t_1}^t p(s)\,\mathrm{d}s + L \int_{t_1}^t p(s) \int_{t_1}^s r(u)\,\mathrm{d}u\,\mathrm{d}s,$$

which implies

$$\frac{v(t)}{\int_{t_1}^t p(s) \int_{t_1}^s r(u)\,\mathrm{d}u\,\mathrm{d}s} < \frac{v(t_1)}{\int_{t_1}^t p(s) \int_{t_1}^s r(u)\,\mathrm{d}u\,\mathrm{d}s} + \frac{v^{[1]}(t_1) \int_{t_1}^t p(s)\,\mathrm{d}s}{\int_{t_1}^t p(s) \int_{t_1}^s r(u)\,\mathrm{d}u\,\mathrm{d}s} + L.$$

Obviously, using L'Hospital's rule we get

$$\lim_{t\to\infty} \frac{\int_{t_1}^t p(s)\,\mathrm{d}s}{\int_{t_1}^t p(s) \int_{t_1}^s r(u)\,\mathrm{d}u\,\mathrm{d}s} = \frac{1}{\int_{t_1}^t r(s)\,\mathrm{d}s} = 0$$

and so

$$\lim_{t\to\infty} \frac{v(t)}{\int_{t_1}^t p(s) \int_{t_1}^s r(u)\,\mathrm{d}u\,\mathrm{d}s} < \infty.$$

Since $v(t) \geq z(t)$, we get a contradiction with (3.6). $\square$

For equation (L−) we have the following characterization of solutions from the class $\mathcal{M}_3$.

Lemma 3.7. *Let z be a nonoscillatory solution of (L−) from the class $\mathcal{M}_3$. Then the following assertions are equivalent:*

i)

$$\int_{t_0}^{\infty} q(s) \int_{t_0}^{\delta(s)} p(u) \int_{t_0}^{u} r(w)\,\mathrm{d}w\,\mathrm{d}u\,\mathrm{d}s = \infty,$$

ii)

$$\lim \left|v^{[i]}(t)\right| = \infty, \quad \textit{for } i = 0, 1, 2,$$

iii)

$$\lim_{t\to\infty} \frac{|z(t)|}{\int_{t_0}^{t} p(s) \int_{t_0}^{s} r(u)\,\mathrm{d}u\,\mathrm{d}s} = \infty.$$

Proof. "*i*) $\Longrightarrow$ *ii*)" It follows from Lemma 3.5.

"*ii*) $\Longrightarrow$ *i*)" Without loss of generality, we may assume that there exists $t_1 \geq t_0$ such that $z(\delta(t)) > 0$, $v^{[1]}(t) > 0$ and $v^{[2]}(t) > 0$ for $t \geq t_1$. Assume by contradiction that

$$\int_{t_0}^{\infty} q(s) \int_{t_0}^{\delta(s)} p(u) \int_{t_0}^{u} r(w)\,\mathrm{d}w\,\mathrm{d}u\,\mathrm{d}s < \infty. \tag{3.7}$$

We can choose $T \geq t_1$ such that

$$\int_{T}^{t} q(s) \int_{T}^{\delta(s)} p(u) \int_{T}^{u} r(w)\,\mathrm{d}w\,\mathrm{d}u\,\mathrm{d}s < 1$$

for every $t \geq T$. Integrating equation (L−) from T to t and using Lemma 3.3 we get

$$v^{[2]}(t) = v^{[2]}(T) + \int_{T}^{t} q(s) z\big(\delta(s)\big)\,\mathrm{d}s \leq v^{[2]}(T) + \int_{T}^{t} q(s) v\big(\delta(s)\big)\,\mathrm{d}s. \tag{3.8}$$

We can express v and $v^{[1]}$ as follows:

$$v(t) = v(T) + \int_{T}^{t} v'(s)\,\mathrm{d}s = v(T) + \int_{T}^{t} p(s) v^{[1]}(s)\,\mathrm{d}s, \tag{3.9}$$

$$v^{[1]}(t) = v^{[1]}(T) + \int_{T}^{t} r(s) v^{[2]}(s)\,\mathrm{d}s. \tag{3.10}$$

Using (3.10) in (3.9) and setting $t = \delta(t)$ we obtain

$$v\big(\delta(t)\big) = v(T) + v^{[1]}(T) \int_{T}^{\delta(t)} p(s)\,\mathrm{d}s + \int_{T}^{\delta(t)} p(s) \int_{T}^{s} r(u) v^{[2]} u\,\mathrm{d}u\,\mathrm{d}s. \tag{3.11}$$

Substituting (3.11) in (3.8) gives

$$\begin{aligned} v^{[2]}(t) \leq v^{[2]}(T) + v(T) \int_{T}^{t} q(s)\,\mathrm{d}s + v^{[1]}(T) \int_{T}^{t} q(s) \int_{T}^{\delta(s)} p(u)\,\mathrm{d}u\,\mathrm{d}s \\ + \int_{T}^{t} q(s) \int_{T}^{\delta(s)} p(u) \int_{T}^{u} r(w) v^{[2]}(w)\,\mathrm{d}w\,\mathrm{d}u\,\mathrm{d}s. \end{aligned}$$

Since $v^{[2]}$ is increasing, it follows that

$$v^{[2]}(t) \leq \frac{v^{[2]}(T) + v(T) \int_T^t q(s)\,\mathrm{d}s + v^{[1]}(T) \int_T^t q(s) \int_T^{\delta(s)} p(u)\,\mathrm{d}u\,\mathrm{d}s}{1 - \int_T^t q(s) \int_T^{\delta(s)} p(u) \int_T^u r(w)\,\mathrm{d}w\,\mathrm{d}u\,\mathrm{d}s}.$$

Moreover, (3.7) implies that $\int_T^{\infty} q(s)\,\mathrm{d}s < \infty$ and $\int_T^{\infty} q(s) \int_T^{\delta(s)} p(u)\,\mathrm{d}u\,\mathrm{d}t < \infty$, thus $v^{[2]}(t) < \infty$.

"*ii*) $\Longleftrightarrow$ *iii*)" It follows from Lemma 3.6.

□

4 Comparison theorems for the superlinear case

We state separately comparison theorems for neutral linear equation (L−) with the advanced argument $\delta(t) \ge t$ and with delay argument $\delta(t) \le t$. Similarly, we state comparison theorems for neutral nonlinear equation (E−). In this section we assume that

$$\limsup_{|u|\to\infty} \frac{u}{f(u)} < \infty. \tag{4.1}$$

In particular, if $f(u) = u^\lambda \operatorname{sgn} u$, then (4.1) is satisfied for $\lambda \ge 1$.

Theorem 4.1. *Assume that $\delta(t) \ge t$. If the linear ordinary differential equation*

$$L_3^A y(t) - (1 - a_0)q(t)\, y(t) = 0 \tag{4.2}$$

has property B, then equation (L−) has also property B.

Proof. Let (4.2) have property B and without loss of generality let z be a solution of (L−) such that $z(t) > 0$ for $t \ge t_1$, $t_1 \ge t_0$ and $v(t)$ be defined by (1.1). Then v is nondecreasing and so $v(t) \le v\big(\delta(t)\big)$. Using Lemma 3.3 we get the following estimate

$$1 - a_0 \le \frac{z\big(\delta(t)\big)}{v\big(\delta(t)\big)} \le \frac{z\big(\delta(t)\big)}{v(t)}. \tag{4.3}$$

Assume by contradiction that $z \in \mathcal{M}_1$ and consider the equation

$$L_3^A y(t) - q(t)\frac{z\big(\delta(t)\big)}{v(t)} y(t) = 0. \tag{4.4}$$

This equation has a solution $y = v$ satisfying $y(t) > 0$, $y^{[1]}(t) > 0$, $y^{[2]}(t) < 0$ for large t, i.e. y is a solution of (4.4) from the class $\mathcal{M}_1$. Since (4.3) holds, equation (4.4) is a majorant of (4.2) and by Proposition 2.5a), $\mathcal{M}_1 \ne \emptyset$ for (4.2), a contradiction.

Now assume that $z \in \mathcal{M}_3$ and assume by contradiction that $\lim_{t\to\infty} v^{[2]}(t) < \infty$. Consider the equation

$$L_3^A y(t) - q(t)\frac{z\big(\delta(t)\big)}{v(t)} y(t) = 0. \tag{4.5}$$

This equation has a solution $y = v$ satisfying $y(t) > 0$, $y^{[1]}(t) > 0$, $y^{[2]}(t) > 0$ for large t, i.e. y is a solution of (4.5) from the class $\mathcal{M}_3$ such that $\lim_{t\to\infty} y^{[2]}(t) < \infty$. Since (4.3) holds, equation (4.5) is a majorant of (4.2) and by Proposition 2.5b) there exists a solution $y \in \mathcal{M}_3$ of (4.2) such that $z^{[2]}(t) < \infty$, a contradiction. $\square$

We extend the previous theorem for nonlinear equation (E−).

Theorem 4.2. *Assume that (4.1) holds and $\delta(t) \ge t$. If for every $K > 0$ the linear ordinary differential equation*

$$L_3^A y(t) - Kq(t)y(t) = 0 \tag{4.6}$$

has property B, then equation (E−) has also property B.

Proof. Let (4.6) have property B for every $K > 0$ and let $v(t)$ be defined by (1.1). Without loss of generality, we may assume that there exists $t_1 \geq t_0$ such that z is a solution of (E−) and $z\big(\delta(t)\big) > 0$ for $t \geq t_1$.

Observe that if $0 < z(t) < \infty$, then f being continuous, we can assume that there exists $c > 0$ such that

$$\frac{f\big(z(\delta(t))\big)}{z\big(\delta(t)\big)} \geq c$$

for large t and if $z(t) \to \infty$ then (4.1) gives

$$\liminf_{t\to\infty} \frac{f\big(z(\delta(t))\big)}{z\big(\delta(t)\big)} > 0.$$

From here and (4.3)

$$\frac{f\big(z(\delta(t))\big)}{v(t)} = \frac{f\big(z(\delta(t))\big)}{z\big(\delta(t)\big)} \frac{z\big(\delta(t)\big)}{v(t)} \geq c_1(1-a_0).$$

Now we proceed similarly to the proof of the previous theorem. Consider the linear equation

$$L_3^A y(t) - q(t)\frac{f\big(z(\delta(t))\big)}{v(t)} y(t) = 0. \tag{4.7}$$

Taking $K \geq c_1(1-a_0)$, we get that equation (4.7) is a majorant of (4.6) for this choice.

Now assume by contradiction, that (E−) has a solution $z \in \mathcal{M}_1$. Therefore, equation (4.7) has a solution $y = v$ from the class $\mathcal{M}_1$. Using Proposition 2.5 a) we get that there exists a solution $z \in \mathcal{M}_1$ of (4.6), a contradiction.

Now assume by contradiction that equation (E−) has a solution z from the class $\mathcal{M}_3$ such that $\lim_{t\to\infty} v^{[2]}(t) < \infty$. Then equation (4.7) has a solution $y = v$ from the class $\mathcal{M}_3$ such that $\lim_{t\to\infty} y^{[2]}(t) < \infty$. Using Proposition 2.5 b) we get a contradiction. □

Now we prove similar theorems for equations with delay, the main difference is in the fact that now we compare equations (L−) and (E−) with delay differential equations.

Theorem 4.3. *Assume that $\delta(t) \leq t$. If the linear delay equation*

$$L_3^A y(t) - (1-a_0)q(t)\, y\big(\delta(t)\big) = 0 \tag{4.8}$$

has property B, then equation (L−) has also property B.

Proof. Let (4.8) have property B and without loss of generality let z be a solution of (L−) such that $z\big(\delta(t)\big) > 0$ for $t \geq t_1$, $t_1 \geq t_0$ and $v(t)$ be defined by (1.1). Using Lemma 3.3 we get the following estimate

$$\frac{z\big(\delta(t)\big)}{v\big(\delta(t)\big)} \geq 1 - a_0. \tag{4.9}$$

Assume by contradiction that $z \in \mathcal{M}_1$ and consider the delay equation

$$L_3^A y(t) - q(t)\frac{z\big(\delta(t)\big)}{v\big(\delta(t)\big)} y\big(\delta(t)\big) = 0. \tag{4.10}$$

This equation has a solution $y = v$ satisfying $y(t) > 0$, $y^{[1]}(t) > 0$, $y^{[2]}(t) < 0$ for large t, i.e. y is a solution of (4.10) from the class $\mathcal{M}_1$. Since (4.9) holds, equation (4.10) is a majorant of (4.8) and by Proposition 2.5 a), $\mathcal{M}_1 \neq \emptyset$ for (4.8), a contradiction.

Now assume that $z \in \mathcal{M}_3$ and assume by contradiction that $\lim_{t\to\infty} v^{[2]}(t) < \infty$. Consider the equation

$$L_3^A y(t) - q(t)\frac{z(\delta(t))}{v(t)}y(t) = 0. \tag{4.11}$$

This equation has a solution $y = v$ satisfying $y(t) > 0$, $y^{[1]}(t) > 0$, $y^{[2]}(t) > 0$ for large t, i.e. y is a solution of (4.11) from the class $\mathcal{M}_3$ and moreover $\lim_{t\to\infty} y^{[2]}(t) < \infty$. Since (4.9) holds, equation (4.11) is a majorant of (4.8) and by Proposition 2.5 b) there exists a solution $y \in \mathcal{M}_3$ of (4.8) such that $z^{[2]}(t) < \infty$, a contradiction. □

Theorem 4.4. *Assume that* (4.1) *holds and* $\delta(t) \le t$. *If for every* $K > 0$ *the linear delay equation*

$$L_3^A y(t) - Kq(t)y(\delta(t)) = 0 \tag{4.12}$$

has property B, then equation (E−) *has also property B.*

Proof. Let (4.12) have property B for every $K > 0$ and let $v(t)$ be defined by (1.1). Without loss of generality we may assume that there exists $t_1 \ge t_0$ such that z is a solution of (E−) such that $z(\delta(t)) > 0$ for $t \ge t_1$..

We proceed similarly to proof of Theorem 4.2. If $0 < z(t) < \infty$, then f being continuous, we can assume that there exists $c > 0$ such that

$$\frac{f(z(\delta(t)))}{z(\delta(t))} \ge c$$

for large t. If $z(t) \to \infty$, then (4.1) gives

$$\liminf_{t\to\infty} \frac{f(z(\delta(t)))}{z(\delta(t))} > 0.$$

From here and (4.9) we obtain

$$\frac{f(z(\delta(t)))}{v(\delta(t))} = \frac{f(z(\delta(t)))}{z(\delta(t))}\frac{z(\delta(t))}{v(\delta(t))} \ge c_1(1 - a_0).$$

Now we proceed similarly to the proof of the previous theorem. Consider the linear delay equation

$$L_3^A y(t) - q(t)\frac{f(z(\delta(t)))}{v(\delta(t))}y(\delta(t)) = 0. \tag{4.13}$$

Taking $K \ge c_1(1 - a_0)$, we get that equation (4.13) is a majorant of (4.12) for this choice.

Now assume by contradiction, that (E−) has a solution $z \in \mathcal{M}_1$. Therefore, equation (4.13) has a solution $y = v$ from the class $\mathcal{M}_1$. Using Proposition 2.5 a) we get that there exists a solution $z \in \mathcal{M}_1$ of (4.12), a contradiction.

Now assume by contradiction that equation (E−) has a solution z from the class $\mathcal{M}_3$ such that $\lim_{t\to\infty} v^{[2]}(t) < \infty$. Then equation (4.13) has a solution $y = v$ from the class $\mathcal{M}_3$ such that $\lim_{t\to\infty} y^{[2]}(t) < \infty$. Using Proposition 2.5 b) we get a contradiction. □

Remark 4.5. There exists various criteria for equation (2.13) to have property A. Using Theorem D and our comparison theorems for neutral equations we can derive new oscillation criteria for equations (E−) and (L−), moreover we can derive new criteria even in the case where $g(t) = t$. To ilustrate this see Examples 6.1 and 6.2 below.

Corollary 4.6. *Assume that $\delta(t) < t$ and there exists function $\tau(t)$ such that (5.3) holds. Moreover, assume that*

$$\limsup_{t\to\infty} \int_t^{\tau(t)} q(s) \int_{t_0}^{\delta(s)} r(u) \int_{t_0}^{u} p(w)\,\mathrm{d}w\,\mathrm{d}u\,\mathrm{d}s > \frac{1}{(1-a_0)}. \tag{4.14}$$

Then $\mathcal{M}_1 = \emptyset$ for (L−).

If in addition (3.2) holds, then (L−) has property B.

Proof. Applying [16, Theorem 3.5] to equation (4.8) and using Theorem 4.3 we get the conclusion. □

5 Oscillation criteria for the sublinear case

In this section we additionally assume that f is increasing, $f(uv) \geq f(u)f(v)$ and

$$\lim_{u\to 0} \frac{u}{f(u)} = 0. \tag{5.1}$$

In particular, if $f(u) = u^\lambda \operatorname{sgn} u$, then (5.1) is satisfied for $0 < \lambda < 1$. Therefore, we refer to this case as to the "sublinear" case.

Theorem 5.1. *Assume that (5.1) holds, $\delta(t)$ is nondecreasing, $\delta(t) < t$ and there exists function $\tau(t)$ such that*

$$\tau(t) \in C\big([t_0,\infty),\mathbb{R}\big), \qquad \tau(t) > t, \qquad \delta\big(\tau(t)\big) \leq t. \tag{5.2}$$

Moreover, assume that

$$\limsup_{t\to\infty} \int_t^{\tau(t)} q(s) f\left(\int_{t_0}^{\delta(s)} r(u) \int_{t_0}^{u} p(w)\,\mathrm{d}w\,\mathrm{d}u\right)\mathrm{d}s > 0. \tag{5.3}$$

Then $\mathcal{M}_1 = \emptyset$ for (E−).

If in addition (3.2) holds, then (E−) has property B.

Proof. Let $z \in \mathcal{M}_1$. Without loss of generality we can assume that z is an eventually positive solution, i.e. there exists $t_1 \geq t_0$ such that $z(t) > 0$, $v^{[1]}(t) > 0$ and $v^{[2]}(t) < 0$ for $t \geq t_1$. Let t_2 be such that $\delta(t) \geq t_1$ for $t \geq t_2$. Because

$$\big(v^{[2]}(t)\big)' = q(t) f\left(z(\delta(t))\right) > 0$$

for $t \geq t_2$, $v^{[2]}$ is a negative increasing function. Therefore we have

$$0 \leq -v^{[2]}(t) < \infty.$$

Integrating equation (E−) from t to ∞ we get

$$v^{[2]}(\infty) - v^{[2]}(t) = \int_t^\infty q(s) f\left(z(\delta(s))\right)\mathrm{d}s.$$

Using the fact that $0 \leq -v^{[2]}(t) < \infty$ and Lemma 3.3 we obtain the inequality

$$\begin{aligned}
-v^{[2]}(t) &= \int_t^\infty q(s) f\left(z(\delta(s))\right)\mathrm{d}s \geq f(1-a_0)\int_t^\infty q(s) f\left(v(\delta(s))\right)\mathrm{d}s \geq \\
&\geq f(1-a_0)\int_t^{\tau(t)} q(s) f\left(v(\delta(s))\right)\mathrm{d}s. \qquad (5.4)
\end{aligned}$$

Integrating the identity $-v^{[2]} = -v^{[2]}$ twice, for the first time from t to ∞ and for the second time from t_1 to t, we obtain

$$v(t) \geq \int_{t_1}^{t} p(s) \int_{s}^{\infty} r(u)\big(-v^{[2]}(u)\big)\,\mathrm{d}u\,\mathrm{d}s.$$

By changing the order of integration we get

$$v(t) \geq \int_{t_1}^{t} r(s)\big(-v^{[2]}(s)\big) \int_{t_1}^{s} p(u)\,\mathrm{d}u\,\mathrm{d}s$$

for $t \geq t_1$ and therefore for $t \geq t_2$ we have

$$v\big(\delta(t)\big) \geq \int_{t_1}^{\delta(t)} r(s)\big(-v^{[2]}(s)\big) \int_{t_1}^{s} p(u)\,\mathrm{d}u\,\mathrm{d}s.$$

Substituting the last inequality into (5.4) we get

$$-v^{[2]}(t) \geq f(1-a_0) \int_{t}^{\tau(t)} q(s) f\left(\int_{t_1}^{\delta(t)} r(u)\big(-v^{[2]}(u)\big) \int_{t_1}^{u} p(w)\,\mathrm{d}w\,\mathrm{d}u\right)\mathrm{d}s.$$

Considering the facts that $-v^{[2]}(t)$ is decreasing and $-v^{[2]}\big(\delta(t)\big)$ is nonincreasing and using the assumption (vii) we get

$$-v^{[2]}(t) \geq f(1-a_0) f\left(-v^{[2]}\left(\delta\big(\tau(t)\big)\right)\right) \int_{t}^{\tau(t)} q(s) f\left(\int_{t_1}^{\delta(t)} r(u) \int_{t_1}^{u} p(w)\,\mathrm{d}w\,\mathrm{d}u\right)\mathrm{d}s.$$

Since $-v^{[2]}(t)$ is positive, decreasing and $\delta(t) < t$ we have

$$\begin{aligned} 1 &\geq \frac{-v^{[2]}(t)}{-v^{[2]}\left(\delta\big(\tau(t)\big)\right)} \\ &\geq \frac{f(1-a_0) f\left(-v^{[2]}\left(\delta\big(\tau(t)\big)\right)\right) \int_{t}^{\tau(t)} q(s) f\left(\int_{t_1}^{\delta(t)} r(u) \int_{t_1}^{u} p(w)\,\mathrm{d}w\,\mathrm{d}u\right)\mathrm{d}s}{-v^{[2]}\left(\delta\big(\tau(t)\big)\right)}. \end{aligned}$$

By Lemma 3.4, we have $\lim_{t\to\infty} v^{[2]}(t) = 0$, and using (5.1) and (5.3) we get a contradiction, i.e. $\mathcal{M}_1 = \emptyset$. The rest of the assertion now follows from Lemma 3.5. □

6 Applications and examples

The following examples illustrate our comparison theorems.

Example 6.1. Consider the linear neutral equation

$$\big(z(t) + a(t)z\big(\gamma(t)\big)\big)''' - \frac{k}{t^3} z\big(\delta(t)\big) = 0$$

where $\delta(t) \geq t$. We show that this equation has property B for

$$k > \frac{2}{3(1-a_0)\sqrt{3}}.$$

Indeed, consider the corresponding linear ordinary differential equation

$$y'''(t) - (1-a_0)\frac{k}{t^3} y(t) = 0. \tag{6.1}$$

Using the result of [8, Example 6.1] we get that equation

$$y'''(t) + (1 - a_0)\frac{k}{t^3}y(t) = 0$$

has property A if

$$(1 - a_0)k > \frac{2}{3\sqrt{3}} \tag{6.2}$$

and using Theorem D we get that (6.1) has property B if (6.2) is satisfied. Applying Theorem 4.1 we obtain the assertion.

Example 6.2. Consider the neutral equation

$$\left(t\left(z(t) + a_0 z\left(\frac{t}{2}\right)\right)''\right)' - \frac{k}{t^2}z(t) = 0$$

where $a_0 \in [0, 1)$. We show that this equation has property B for every $k > 0$.

Consider the corresponding linear equation

$$(ty''(t))' - \frac{k}{(1 - a_0)t^3}y(t) = 0. \tag{6.3}$$

Applying [8, Corollary 6.3] to equation

$$(tx'(t))'' + \frac{k}{(1 - a_0)t^3}x = 0 \tag{6.4}$$

we obtain that (6.4) has property A for every $k > 0$ and using Theorem D we get that (6.3) has property A for every $k > 0$. Now Theorem 4.1 gives the assertion.

Example 6.3. Consider the equation

$$\big(z(t) + a(t)z(\gamma(t))\big)''' - \frac{1}{t^2}\left|z(\delta(t))\right|^{\lambda}\operatorname{sgn}\big(z(\delta(t))\big) = 0, \qquad t \geq 1$$

where $\lambda \geq 1$ and $\delta(t) \leq t$. We show that this equation has property B.

Consider the corresponding delay differential equation

$$y'''(t) - \frac{K}{t^2}y(\delta(t)) = 0, \tag{6.5}$$

where $K > 0$. Obviously

$$\int_1^{\infty} \frac{K}{t^2}\,\mathrm{d}t < \infty \qquad \text{and} \qquad \int_1^{\infty} \frac{K}{t}\,\mathrm{d}t = \infty,$$

thus using [18, Theorem 3.3] we have that (6.5) has property B for every $K > 0$. Now Theorem 4.4 yields the assertion.

We close with the following application to neutral equations with symmetric operator

$$\left(\frac{1}{p(t)}\left(\frac{1}{p(t)}\left[x(t) + a(t)z(\gamma(t))\right]'\right)'\right)' + q(t)x(\delta(t)) = 0 \tag{S, δ, a}$$

and

$$\left(\frac{1}{p(t)}\left(\frac{1}{p(t)}\left[z(t) + a(t)z(\gamma(t))\right]'\right)'\right)' - q(t)z(\delta(t)) = 0. \tag{S$^{\mathcal{A}}$, δ, a}$$

Following result extends [16, Corollary 4.2] to neutral equations with the delay argument.

Corollary 6.4. *Assume that $\delta(t)$ is nondecreasing, $\delta(t) < t$ and there exists function $\tau(t)$ such that* (5.3) *holds.*

Moreover, assume that

$$\limsup_{t\to\infty} \int_t^{\tau(t)} q(s) \int_a^{\delta(s)} p(u) \int_a^u p(v)\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}s > \frac{1}{1-a_0}. \tag{6.6}$$

Then equation (S, δ, a) *has property A and equation* ($S^{\mathcal{A}}$, δ, a) *has property B.*

Proof. Using [16, Lemma 3.4] and its proof we get that (6.6) implies that (3.2) hold and

$$\int_{t_0}^{\infty} q(t) \int_{t_0}^{t} p(s) \int_{t_0}^{s} r(v)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t = \infty.$$

Now Corollary 4.6 and [7, Theorem 1] give the assertion. □

Moreover, we have the following corollary for equations with symmetric operator and advanced argument.

Corollary 6.5. *Assume that $\delta(t) \geq t$ and*

$$\int_{t_0}^{\infty} q(t) \int_{t_0}^{t} p(s)\,\mathrm{d}s\,\mathrm{d}t = \infty. \tag{6.7}$$

Then equation (S, δ, a) *has property A and equation* ($S^{\mathcal{A}}$, δ, a) *has property B.*

Proof. According [6, Theorem 8 and 10] the linear equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{p(t)}x'\right)'\right)' + (1-a_0)q(t)x(t) = 0 \tag{6.8}$$

corresponding to (S, δ, a) has an oscillatory solution. By [3, Lemma 2.2] any nonoscillatory solution satisfies $x(t)x'(t) < 0$ for large t. This property is called weak property A (see e.g. [3]). From here and the proof of [4, Theorem 1] we get that linear equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{p(t)}z'\right)'\right)' - (1-a_0)q(t)z(t) = 0 \tag{6.9}$$

corresponding to ($S^{\mathcal{A}}$, δ, a) has weak property B. Using [5, Theorem 7] we get that equation (6.8) has property A and equation (6.9) has property B. Now the conclusion follows from Theorems [8, Theorem 5.1] and 4.1. □

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