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Qualitative analysis on a cubic predator-prey system with diffusion

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Abstract. In this paper, we study a cubic predator-prey model with diffusion. We first establish the global stability of the trivial and nontrivial constant steady states for the reaction diffusion system, and then prove the existence and non-existence results concerning non-constant positive stationary solutions by using topological argument and the energy method, respectively.

Key words: cubic predator-prey model, diffusion, steady-state, existence and non-existence.

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1 Introduction

Huang etc. in [6] proposed a cubic differential system, which can be considered a generalization of the predator-prey models and the mathematical form of the system satisfies the following:

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = X(b_1 + b_2X - b_3X^2) - b_4XY, \\ \dfrac{\mathrm{d}Y}{\mathrm{d}t} = -cY + (\alpha X - \beta Y)Y, \end{cases} \tag{1.1}$$

where X and Y represent the densities of prey and predator species at time t respectively. $b_3, b_4, c, \alpha, \beta$ are positive constants, and b_1 is non-negative, and the sign of b_2 is undetermined. When $b_2 < 0$ and $b_3 = 0$, the system (1.1) becomes the standard predator-prey model. The more detailed biological implication for the model, one may further refer to [6] and the references therein.

In [6], the authors introduced the following scaling transformations,

$$X = \frac{cu}{\alpha}, \quad Y = \frac{cv}{\beta}, \quad t = \frac{\tau}{c}$$

and rewrite t as τ, then system (1.1) turns into

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = u(a_1 + a_2u - a_3u^2) - kuv, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = v(-1 + u - v), \end{cases} \tag{1.2}$$

where $a_1 = b_1/c,\ a_2 = b_2/\alpha,\ a_3 = b_3c/\alpha^2$ and $k = b_4/\beta$. a_1 is non-negative, and the sign of a_2 is undetermined, a_3 and k are positive constants. For system (1.2), in [6], the authors studied the properties of the equilibrium points, the existence of a uniqueness limit cycle, and the conditions for three limit cycles.

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In the case that the densities of the predator and prey are spatially inhomogeneous in a bounded domain with smooth boundary $\Omega \subset \mathbf{R}^n$, instead of the ordinary differential system (1.2), we are led to consider the following reaction-diffusion system:

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1 \Delta u = u(a_1 + a_2 u - a_3 u^2 - kv), & \text{in } \ \Omega \times (0, \infty), \\ \dfrac{\partial v}{\partial t} - d_2 \Delta v = v(-1 + u - v), & \text{in } \ \Omega \times (0, \infty), \\ \partial_\nu u = \partial_\nu v = 0, & \text{on } \ \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) \geq 0, \not\equiv 0, \quad v(x, 0) = v_0(x) \geq 0, \not\equiv 0, & \text{on } \ \overline{\Omega}, \end{cases} \tag{1.3}$$

where $d_i > 0\,(i = 1, 2)$ is the diffusion coefficient corresponding to u and v. Here, ν is the outward unit normal vector on $\partial\Omega$ and $\partial_\nu = \partial/\partial\nu$. The admissible initial data $u_0(x)$ and $v_0(x)$ are continuous functions on $\overline{\Omega}$. The homogeneous Neumann boundary condition means that (1.3) is self-contained and has no population flux across the boundary $\partial\Omega$. The study of predator-prey models has a long history, we refer to [2, 10] for background on ODE models and to [1, 3, 4, 7, 12–15, 17] for diffusive models.

First of all, we note that (1.3) has two trivial non-negative constant steady states, namely, $E_0 = (0, 0)$, $E_1 = (u^{**}, 0)$, where $u^{**} = (a_2 + \sqrt{a_2^2 + 4a_1 a_3})/(2a_3)$. Simple analysis shows that model (1.3) has the only positive constant steady-state solution if and only if $a_1 + a_2 > a_3$. We denote this steady state by (u^*, v^*), where

$$u^* = \frac{(a_2 - k) + \sqrt{(a_2 - k)^2 + 4(a_1 + k)a_3}}{2a_3} \quad \text{and} \quad v^* = u^* - 1.$$

Another aspect of our goal is to investigate the corresponding steady-state problem of the reaction-diffusion system (1.3), which may display the dynamical behavior of solutions to (1.3) as time goes to infinity. This steady-state problem satisfies

$$\begin{cases} -d_1 \Delta u = u(a_1 + a_2 u - a_3 u^2 - kv), & \text{in } \ \Omega, \\ -d_2 \Delta v = v(-1 + u - v), & \text{in } \ \Omega, \\ \partial_\nu u = \partial_\nu v = 0, & \text{on } \ \partial\Omega. \end{cases} \tag{1.4}$$

It is clear that only non-negative solutions of (1.4) are of realistic interest.

The remaining content in our paper is organized as follows. In section 2, we mainly analyze the global stability of constant steady states to (1.3). Then, in section 3, we give a priori estimates of upper and lower bounds for positive solutions of (1.4), and finally in section 4 we derive some non-existence and existence results of positive non-constant solutions of (1.4).

2 Some properties of solutions to (1.3) and stability of (u^*, v^*)

In this section, we are mainly concerned with some properties of solutions to (1.3) and the global stability of (u^*, v^*) for system (1.3). Throughout this section, let $(u(x, t), v(x, t))$ be the unique solution of (1.3). It is easily seen that $(u(x, t), v(x, t))$ exists globally and is positive, namely, $u(x, t)$, $v(x, t) > 0$ for all $x \in \overline{\Omega}$ and $t > 0$.

2.1 Some properties of the solutions to (1.3)

The following assertions characterize the global stability of each of the trivial non-negative constant steady states, and the boundedness of the positive solutions to (1.3).

Theorem 2.1 *Let* $(u(x,t), v(x,t))$ *be the solution to* (1.3).

(i) *Assume that* $a_1 = 0,\ a_2 \le 0$, *then*

$$(u(x,t), v(x,t)) \to (0,0), \quad \textit{uniformly on } \overline{\Omega} \ \textit{ as } \ t \to \infty. \tag{2.1}$$

(ii) *Assume that* $a_1 = 0$ *and* $0 < a_2 \le a_3$, *or* $a_1 > 0$ *and* $a_1 + a_2 \le a_3$, *then*

$$(u(x,t), v(x,t)) \to (u^{**}, 0), \quad \textit{uniformly on } \overline{\Omega} \ \textit{ as } \ t \to \infty. \tag{2.2}$$

(iii) *Assume that* $a_1 \ge 0,\ a_1 + a_2 > a_3$, *then, for* $0 < \varepsilon \ll 1$, *there exists* $t_0 \gg 1$ *such that*

$$u(x,t) \le \frac{a_2 + \sqrt{a_2^2 + 4a_1a_3}}{2a_3} + \varepsilon, \quad v(x,t) \le \frac{a_2 + \sqrt{a_2^2 + 4a_1a_3}}{2a_3} - 1 + \varepsilon, \tag{2.3}$$

for all $x \in \overline{\Omega}$ *and* $t \ge t_0$.

Before proving the above conclusions, we need to introduce the following lemma, which can be proved using the comparison principle (see also [17]).

Lemma 2.1 *Assume that* $f(s)$ *is a positive* C^1 *function for* $s \ge 0$, *constants* $d > 0,\ \beta \ge 0$. *Let* $T \in [0,\infty)$ *and* $\omega \in C^{2,1}(\Omega \times (0,\infty)) \cap C^{1,0}(\overline{\Omega} \times [0,\infty))$ *be a positive function.*

(i) *If* ω *satisfies*

$$\begin{cases} \dfrac{\partial \omega}{\partial t} - d\Delta\omega \le (\ge) \omega^{1+\beta} f(\omega)(\alpha - \omega), & (x,t) \in \ \Omega \times (T,\infty), \\ \partial_\nu \omega = 0, & (x,t) \in \ \partial\Omega \times [T,\infty), \end{cases}$$

and the constant $\alpha > 0$. *Then*

$$\limsup_{t\to\infty} \max_{\overline{\Omega}} \omega(\cdot,t) \le \alpha \ (\liminf_{t\to\infty} \min_{\overline{\Omega}} \omega(\cdot,t) \ge \alpha).$$

(ii) *If* ω *satisfies*

$$\begin{cases} \dfrac{\partial \omega}{\partial t} - d\Delta\omega \le \omega^{1+\beta} f(\omega)(\alpha - \omega) & (x,t) \in \ \Omega \times (T,\infty), \\ \partial_\nu \omega = 0 & (x,t) \in \ \partial\Omega \times [T,\infty), \end{cases}$$

and the constant $\alpha \le 0$. *Then*

$$\limsup_{t\to\infty} \max_{\overline{\Omega}} \omega(\cdot,t) \le 0.$$

In the following, we give the proof of Theorem 2.1.

Proof of Theorem 2.1. (i) From the first equation of (1.3) we see that

$$\frac{\partial u}{\partial t} - d_1\Delta u \le u^2(a_2 - a_3u). \tag{2.4}$$

Since $a_2 \leq 0$, by Lemma 2.1, we have

$$\limsup_{t\to\infty} \max_{\overline{\Omega}} u(\cdot,t) \leq 0. \tag{2.5}$$

In view of u is positive, we obtain

$$\lim_{t\to\infty} u(\cdot,t) = 0$$

uniformly on $\overline{\Omega}$. For any given $\varepsilon > 0$ small enough, there is a $T_1 \gg 1$, such that

$$u(x,t) \leq \varepsilon, \quad \forall x \in \overline{\Omega},\ t \geq T_1.$$

From the second equation of (1.3) we have, for $x \in \Omega$ and $t > T_1$,

$$\frac{\partial v}{\partial t} - d_2 \Delta v \leq v(-1 + \varepsilon - v).$$

Thanks to Lemma 2.1 and the arbitrariness of $\varepsilon > 0$, it follows that

$$\limsup_{t\to\infty} \max_{\overline{\Omega}} v(\cdot,t) \leq 0.$$

Since v is also positive, we arrive at

$$\lim_{t\to\infty} v(\cdot,t) = 0$$

uniformly on $\overline{\Omega}$.

Before proving (ii), we firstly prove (iii). From the first equation of (1.3) we see that

$$\frac{\partial u}{\partial t} - d_1 \Delta u \leq u(a_1 + a_2 u - a_3 u^2) = a_3 u\Big(u + \frac{-a_2 + \sqrt{a_2^2 + 4a_1 a_3}}{2a_3}\Big)\Big(\frac{a_2 + \sqrt{a_2^2 + 4a_1 a_3}}{2a_3} - u\Big).$$

By Lemma 2.1, one gets

$$\limsup_{t\to\infty} \max_{\overline{\Omega}} u(\cdot,t) \leq \frac{a_2 + \sqrt{a_2^2 + 4a_1 a_3}}{2a_3}. \tag{2.6}$$

For any given $\varepsilon > 0$, there exists $T_2 \gg 1$, such that

$$u(x,t) \leq \frac{a_2 + \sqrt{a_2^2 + 4a_1 a_3}}{2a_3} + \varepsilon, \quad \forall x \in \overline{\Omega},\ t \geq T_2. \tag{2.7}$$

By the second equation of (1.3) we have, for $x \in \Omega$ and $t > T_2$,

$$\frac{\partial v}{\partial t} - d_2 \Delta v \leq v(-1 + \frac{a_2 + \sqrt{a_2^2 + 4a_1 a_3}}{2a_3} + \varepsilon - v). \tag{2.8}$$

Since $a_1 + a_2 > a_3$, then $-1 + (a_2 + \sqrt{a_2^2 + 4a_1 a_3})/(2a_3) > 0$. Thanks to Lemma 2.1 again,

$$\limsup_{t\to\infty} \max_{\overline{\Omega}} v(\cdot,t) \leq \frac{a_2 + \sqrt{a_2^2 + 4a_1 a_3}}{2a_3} - 1 + \varepsilon,$$

which asserts our result (iii).

Now, we begin to verify (ii). In order to obtain the result, we need to consider two different cases.

Case 1. $a_1 = 0$, $0 < a_2 \le a_3$. By (2.4) and Lemma 2.1 we have

$$\limsup_{t\to\infty} \max_{\overline{\Omega}} u(\cdot,t) \le \frac{a_2}{a_3}. \tag{2.9}$$

For any given $\varepsilon > 0$, there exists $T_3 \gg 1$, such that

$$u(x,t) \le \frac{a_2}{a_3} + \varepsilon, \quad \forall x \in \overline{\Omega},\ t \ge T_3.$$

By the second equation of (1.3) we have, for $x \in \Omega$ and $t > T_3$,

$$\frac{\partial v}{\partial t} - d_2 \Delta v \le v(-1 + \frac{a_2}{a_3} + \varepsilon - v).$$

Thanks to Lemma 2.1, we obtain

$$\limsup_{t\to\infty} \max_{\overline{\Omega}} v(\cdot,t) \le -1 + \frac{a_2}{a_3} + \varepsilon.$$

By the arbitrariness of $\varepsilon > 0$, it follows that

$$\limsup_{t\to\infty} \max_{\overline{\Omega}} v(\cdot,t) \le -1 + \frac{a_2}{a_3} \le 0.$$

Since v is positive, we have

$$\lim_{t\to\infty} v(\cdot,t) = 0$$

uniformly on $\overline{\Omega}$.

Case 2. $a_1 > 0$, $a_1 + a_2 \le a_3$. In this case, the inequalities (2.6)-(2.8) also hold. In view of $a_1 + a_2 \le a_3$, then $-1 + (a_2 + \sqrt{a_2^2 + 4a_1a_3})/(2a_3) \le 0$. By the arbitrariness of $\varepsilon > 0$, it follows that

$$\limsup_{t\to\infty} \max_{\overline{\Omega}} v(\cdot,t) \le -1 + \frac{a_2 + \sqrt{a_2^2 + 4a_1a_3}}{2a_3} \le 0.$$

Consequently,

$$\lim_{t\to\infty} v(\cdot,t) = 0$$

uniformly on $\overline{\Omega}$ as above. For any given $\varepsilon > 0$ small enough, there is a $T_4 \gg 1$, such that

$$v(x,t) \le \varepsilon, \quad \forall x \in \overline{\Omega},\ t \ge T_4.$$

From the first equation of (1.3) we have, for $x \in \Omega$ and $t > T_4$,

$$\begin{aligned} \frac{\partial u}{\partial t} - d_1 \Delta u &\ge u(a_1 + a_2 u - a_3 u^2 - k\varepsilon) \\ &= a_3 u\Big(u + \frac{-a_2 + \sqrt{a_2^2 + 4(a_1 - k\varepsilon)a_3}}{2a_3}\Big)\Big(\frac{a_2 + \sqrt{a_2^2 + 4(a_1 - k\varepsilon)a_3}}{2a_3} - u\Big). \end{aligned}$$

Also by Lemma 2.1, we have

$$\limsup_{t\to\infty} \min_{\overline{\Omega}} u(\cdot,t) \ge \frac{a_2 + \sqrt{a_2^2 + 4(a_1 - k\varepsilon)a_3}}{2a_3}.$$

Hence, it follows that

$$\limsup_{t\to\infty}\min_{\overline{\Omega}} u(\cdot,t) \geq \frac{a_2+\sqrt{a_2^2+4a_1a_3}}{2a_3}$$

since ε is arbitrary small. This combined with (2.6) yields

$$\lim_{t\to\infty} u(\cdot,t) = \frac{a_2+\sqrt{a_2^2+4a_1a_3}}{2a_3}$$

uniformly on $\overline{\Omega}$. Thus, the proof is complete. □

2.2 Local stability of (u^*, v^*) to system (1.3)

By Theorem 2.1, from now on, without special statement, we always assume that $a_1 + a_2 > a_3$, which guarantees the existence of (u^*, v^*). In this subsection, we will analyze the local stability of (u^*, v^*) to (1.3). To this end, we first introduce some notations.

Let $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$ be the eigenvalues of the operator $-\Delta$ on Ω with the homogeneous Neumann boundary condition. Set $\mathbf{X}_j$ is the eigenspace corresponding to μ_j. Let

$$\mathbf{X} = \{(u,v) \in [C^1(\overline{\Omega})]^2 \mid \partial_\nu u = \partial_\nu v = 0 \quad \text{on } \partial\Omega\},$$

$\{\phi_{jl}; l = 1,\ldots,m(\mu_j)\}$ be an orthonormal basis of $\mathbf{X}_j$, and $\mathbf{X}_{jl} = \{\mathbf{c}\phi_{jl} |\ \mathbf{c} \in \mathbf{R}^2\}$. Here $m(\mu_j)$ is the multiplicity of μ_j. Then

$$\mathbf{X} = \bigoplus_{j=0}^{\infty} \mathbf{X}_j \quad \text{and} \quad \mathbf{X}_j = \bigoplus_{l=1}^{m(\mu_j)} \mathbf{X}_{jl}. \tag{2.10}$$

Theorem 2.2 *The positive constant solution* (u^*, v^*) *to system* (1.3) *is uniformly asymptotically stable provided that* $a_1 + a_2 > a_3$ *and* $4a_1a_3 + 2k(2a_3 - a_2) + a_2^2 > 0$ (*in the sense of* [5]).

Proof. The linearization of (1.3) at (u^*, v^*) is

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{L}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(u-u^*, v-v^*) \\ f_2(u-u^*, v-v^*) \end{pmatrix},$$

where $f_i(z_1, z_2) = O(z_1^2 + z_2^2), i = 1, 2$, and

$$\mathcal{L} = \begin{pmatrix} d_1\Delta - u^*(2a_3u^* - a_2) & -ku^* \\ v^* & d_2\Delta - v^* \end{pmatrix}.$$

For each j, $j = 0, 1, 2, \cdots, X_j$ is invariant under the operator $\mathcal{L}$, and ξ is an eigenvalue of $\mathcal{L}$ on X_j if and only if ξ is an eigenvalue of the matrix

$$A_j = \begin{pmatrix} -d_1\mu_j - u^*(2a_3u^* - a_2) & -ku^* \\ v^* & -d_2\mu_j - v^* \end{pmatrix}$$

$$\begin{aligned}\det A_j &= d_1d_2\mu_j^2 + [d_1v^* + d_2u^*(2a_3u^* - a_2)]\mu_j + u^*v^*(k + 2a_3u^* - a_2),\\ \mathrm{Tr}A_j &= -(d_1 + d_2)\mu_j - u^*(2a_3u^* - a_2) - v^* \leq -u^*(2a_3u^* - a_2) - v^*,\end{aligned}$$

where $\det A_j$ and $\mathrm{Tr}A_j$ are respectively the determinant and trace of A_j. It is easy to check that $\mathrm{detA_j} > 0$ and $\mathrm{TrA_j} < 0$ if $u^* > a_2/(2a_3)$, i.e. $4a_1a_3 + 2k(2a_3 - a_2) + a_2^2 > 0$. The same analysis as in [16] gives that the spectrum of $\mathcal{L}$ lies in $\{\mathrm{Re}\xi < -\delta\}$ for some positive δ independent of $i \geq 0$. It is known that (u^*, v^*) is uniformly asymptotically stable and the proof is complete. □

2.3 Global stability of (u^*, v^*) to system (1.3)

In this subsection, we will be devoted to the global stability of (u^*, v^*) for system (1.3).

Theorem 2.3 *Assume that $a_1 + a_2 > a_3$ and $a_1a_3 + k(a_3 - a_2) > 0$, then (u^*, v^*) is globally asymptotically stable.*

Proof. In order to give the proof, we need to construct a Lyapunov function. First, we define

$$E(u)(t) = \int_\Omega \Big\{u(x,t) - u^* - u^* \ln \frac{u(x,t)}{u^*}\Big\}\mathrm{d}x,$$
$$E(v)(t) = \int_\Omega \Big\{v(x,t) - v^* - v^* \ln \frac{v(x,t)}{v^*}\Big\}\mathrm{d}x.$$

We note that $E(u)(t)$ and $E(v)(t)$ are non-negative, $E(u)(t) = 0$ and $E(v)(t) = 0$ if and only if $(u(x,t), v(x,t)) = (u^*, v^*)$. Furthermore, easy computations yield that

$$\begin{aligned}\frac{dE(u)}{dt} &= \int_\Omega \Big\{(1 - \frac{u^*}{u})u_t\Big\}\mathrm{d}x = \int_\Omega \Big\{ - d_1\frac{u^*|\nabla u|^2}{u^2} + (u-u^*)(a_1 + a_2u - a_3u^2 - kv)\Big\}\mathrm{d}x \\ &= \int_\Omega \Big\{ - d_1\frac{u^*|\nabla u|^2}{u^2} + (u-u^*)(-a_2u^* + a_3(u^*)^2 + kv^* + a_2u - a_3u^2 - kv)\Big\}\mathrm{d}x \\ &= \int_\Omega \Big\{ - d_1\frac{u^*|\nabla u|^2}{u^2} - [a_3(u+u^*) - a_2](u-u^*)^2 - k(u-u^*)(v-v^*)\Big\}\mathrm{d}x.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{dE(v)}{dt} &= \int_\Omega \Big\{(1 - \frac{v^*}{v})v_t\Big\}\mathrm{d}x = \int_\Omega \Big\{ - d_2\frac{v^*|\nabla v|^2}{v^2} + (v-v^*)(-1 + u - v)\Big\}\mathrm{d}x \\ &= \int_\Omega \Big\{ - d_2\frac{v^*|\nabla v|^2}{v^2} + (v-v^*)(-u^* + v^* + u - v)\Big\}\mathrm{d}x \\ &= \int_\Omega \Big\{ - d_2\frac{v^*|\nabla v|^2}{v^2} - (v-v^*)^2 + (u-u^*)(v-v^*)\Big\}\mathrm{d}x.\end{aligned}$$

Now define

$$E(t) = E(u)(t) + kE(v)(t).$$

Hence

$$\begin{aligned}\frac{dE(t)}{dt} &= \frac{dE(u)(t)}{dt} + k\frac{dE(v)(t)}{dt} \\ &= \int_\Omega \Big\{ - d_1\frac{u^*|\nabla u|^2}{u^2} - d_2k\frac{v^*|\nabla v|^2}{v^2} - [a_3(u+u^*) - a_2](u-u^*)^2 - k(v-v^*)^2\Big\}\mathrm{d}x \\ &\le \int_\Omega \Big\{ - [a_3(u+u^*) - a_2](u-u^*)^2 - k(v-v^*)^2\Big\}\mathrm{d}x.\end{aligned}$$

When $u^* > a_2/a_3$, i.e., $a_1a_3 + k(a_3 - a_2) > 0$ then $\mathrm{d}E(t)/\mathrm{d}t \le 0$, and the equality holds if and only if $(u, v) = (u^*, v^*)$. Hence, the standard arguments together with (iii) of Theorem 2.1 and Theorem 2.2 deduce that (u^*, v^*) attracts all solutions of (1.3). This finishes the proof. $\square$

3 A priori estimates for positive solutions to (1.4)

From now on, our aim is to investigate the steady-state problem (1.4). In this section, we will deduce a priori estimates of positive upper and lower bounds for positive solutions of (1.4). To this end, we first cite two known results.

Lemma 3.1 (Maximum principle [8]) *Suppose that* $g \in C(\overline{\Omega} \times \mathbf{R})$.

(i) *Assume that* $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *and satisfies*

$$\Delta w(x) + g(x, w(x)) \geq 0 \ \text{ in } \Omega, \quad \partial_\nu w \leq 0 \ \text{ on } \partial\Omega.$$

If $w(x_0) = \max_{\overline{\Omega}} w$, *then* $g(x_0, w(x_0)) \geq 0$.

(ii) *Assume that* $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *and satisfies*

$$\Delta w(x) + g(x, w(x)) \leq 0 \ \text{ in } \Omega, \quad \partial_\nu w \geq 0 \ \text{ on } \partial\Omega.$$

If $w(x_0) = \min_{\overline{\Omega}} w$, *then* $g(x_0, w(x_0)) \leq 0$.

Lemma 3.2 (Harnack inequality [9]) *Let* $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *be a positive solution to* $\Delta w(x) + c(x)w(x) = 0$ *in* Ω *subject to the homogeneous Neumann boundary condition where* $c(x) \in C(\overline{\Omega})$. *Then there exists a positive constant* $C^* = C^*(\|c\|_\infty, \Omega)$ *such that*

$$\max_{\overline{\Omega}} w \leq C^* \min_{\overline{\Omega}} w.$$

Theorem 3.1 *Assume that* $a_1 + a_2 > a_3$, *then the positive solution* (u, v) *of* (1.4) *satisfies*

$$\max_{\overline{\Omega}} u(x) < \frac{a_2 + \sqrt{a_2^2 + 4a_1a_3}}{2a_3}, \qquad \max_{\overline{\Omega}} v(x) < \frac{a_2 + \sqrt{a_2^2 + 4a_1a_3}}{2a_3} - 1.$$

Proof. Assume that (u, v) is a positive solution of (1.4). We set

$$u(x_1) = \max_{\overline{\Omega}} u, \quad v(x_2) = \max_{\overline{\Omega}} v.$$

Applying Lemma 3.1 to (1.4), we obtain that

$$a_1 + a_2u(x_1) - a_3u^2(x_1) - kv(x_1) \geq 0, \tag{3.1}$$

$$-1 + u(x_2) - v(x_2) \geq 0. \tag{3.2}$$

From (3.1), it follows that

$$a_3u^2(x_1) - a_2u(x_1) - a_1 \leq -kv(x_1) < 0 \Rightarrow u(x_1) < \frac{a_2 + \sqrt{a_2^2 + 4a_1a_3}}{2a_3}.$$

If $a_1 + a_2 > a_3$, then in view of (3.2), it is easy to see that

$$v(x_2) \leq u(x_2) - 1 < \frac{a_2 + \sqrt{a_2^2 + 4a_1a_3}}{2a_3} - 1.$$

The proof is complete. □

Theorem 3.2 *Assume that $a_1 + a_2 > a_3$, let d be an arbitrary fixed positive number, then, there exists a positive constant $\underset{\sim}{C}$ only depending on a_1, a_2, a_3, k, d and Ω such that if $d_1, d_2 \geq d$, any positive solution (u, v) of (1.4) satisfies*

$$\min_{\overline{\Omega}} u(x) > \underset{\sim}{C} \qquad \min_{\overline{\Omega}} v(x) > \underset{\sim}{C}.$$

Proof. Since $\int_\Omega v(-1+u-v)\mathrm{d}x = 0$, there exists $x_0 \in \overline{\Omega}$ such that $v(x_0)(-1+u(x_0)-v(x_0)) = 0$, that is

$$u(x_0) = 1 + v(x_0).$$

It follows that $\max_{\overline{\Omega}} u(x) \geq 1$. Let $c_1(x) = d_1^{-1}[a_1 + a_2 u - a_3 u^2 - kv]$, by Theorem 3.1 and Lemma 3.2, there exists a positive constant C_1, such that

$$\min_{\overline{\Omega}} u(x) \geq \frac{\max_{\overline{\Omega}} u(x)}{C_1} \geq \frac{1}{C_1}. \tag{3.3}$$

Now, it suffices to verify the lower bounds of $v(x)$. We shall prove by contradiction.

Suppose that Theorem 3.2 is not true, then there exists a sequence $\{d_{2,i}\}_{i=1}^{\infty}$ with $d_{2,i} \geq d$ and the positive solution (u_i, v_i) of (1.4) corresponding to $d_2 = d_{2,i}$, such that

$$\min_{\overline{\Omega}} v_i(x) \to 0 \quad \text{as} \quad i \to \infty.$$

By the Harnack inequality, we know that there is a positive constant C_2 independent of i such that $\max_{\overline{\Omega}} v_i(x) \leq C_2 \min_{\overline{\Omega}} v_i(x)$. Consequently,

$$v_i(x) \to 0 \qquad \text{uniformly on } \overline{\Omega}, \quad as \quad i \to \infty.$$

Let $w_i = v_i / \|v_i\|_\infty$ and (u_i, w_i) satisfies the following elliptic model

$$\begin{cases} -d_1 \Delta u_i = u_i(a_1 + a_2 u_i - a_3 u_i^2 - k v_i) & \text{in } \Omega, \\ -d_{2,i} \Delta w_i = w_i(-1 + u_i - v_i) & \text{in } \Omega, \\ \partial_\nu u_i = \partial_\nu w_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Moreover, integrating over Ω by parts, we have

$$\int_\Omega u_i(a_1 + a_2 u_i - a_3 u_i^2 - k v_i)\mathrm{d}x = 0, \qquad \int_\Omega w_i(-1 + u_i - v_i)\mathrm{d}x = 0. \tag{3.5}$$

The embedding theory and the standard regularity theory of elliptic equations guarantee that there is a subsequence of (u_i, w_i) also denoted by itself, and two non-negative functions $u, w \in C^2(\overline{\Omega})$, such that $(u_i, w_i) \to (u, w)$ in $[C^2(\overline{\Omega})]^2$ as $i \to \infty$. Since $\|w_i\|_\infty = 1$, we have $\|w\|_\infty = 1$. Since (u_i, w_i) satisfy (3.5), so do (u, w), i.e.

$$\int_\Omega u(a_1 + a_2 u - a_3 u^2)\mathrm{d}x = 0, \qquad \int_\Omega w(-1 + u)\mathrm{d}x = 0. \tag{3.6}$$

By (3.3) and Theorem 3.1, we have $0 < u \leq (a_2 + \sqrt{a_2^2 + 4a_1 a_3})/(2a_3)$, and when u lies in this interval, $a_1 + a_2 u - a_3 u^2 \geq 0$. As a result, by the first integral identity of (3.6) we obtain $u = (a_2 + \sqrt{a_2^2 + 4a_1 a_3})/(2a_3)$. In view of $a_1 + a_2 > a_3$, so $u = (a_2 + \sqrt{a_2^2 + 4a_1 a_3})/(2a_3) > 1$, and the second integral identity of (3.6) yields $\int_\Omega w\mathrm{d}x = 0$, which implies a contradiction. This completes the proof. □

4 Non-existence and existence for non-constant solutions to (1.4)

4.1 Non-existence of positive non-constant solutions

In this subsection, based on the priori estimates in Section 3 for positive solutions to (1.4), we present some results for non-existence of positive non-constant solutions of (1.3) as the diffusion coefficient d_1 or d_2 is sufficiently large.

Note that μ_1 be the smallest positive eigenvalue of the operator $-\Delta$ in Ω subject to the homogeneous Neumann boundary condition. Now, using the energy estimates, we can claim

Theorem 4.1 (i) *There exists a positive constant $\tilde{d_1} = \tilde{d_1}(a_1, a_2, a_3, k, \Omega)$ such that* (1.4) *has no non-constant positive solutions provided that $\mu_1 d_1 > \tilde{d_1}$ and $\mu_1 d_2 > (a_2 + \sqrt{a_2^2 + 4a_1a_3})/(2a_3) - 1$;*

(ii) *There exists a positive constant $\tilde{d_2} = \tilde{d_2}(a_1, a_2, a_3, k, \Omega)$ such that* (1.4) *has no non-constant positive solutions provided that $\mu_1 d_2 > \tilde{d_2}$ and $\mu_1 d_1 > a_1 + \mid a_2 \mid (a_2 + \sqrt{a_2^2 + 4a_1a_3})/a_3$.*

Proof. Let (u, v) be any positive solution of (1.4) and denote $\bar{g} = (1/|\Omega|)\int_\Omega g\,\mathrm{d}x$. Then, multiplying the corresponding equation in (1.4) by $u - \bar{u}$ and $v - \bar{v}$ respectively, integrating over Ω, we obtain

$$
\begin{aligned}
& d_1 \int_\Omega |\nabla(u-\bar{u})|^2 \mathrm{d}x = \int_\Omega (a_1 u + a_2 u^2 - a_3 u^3 - kuv)(u-\bar{u})\,\mathrm{d}x \\
&= \int_\Omega \Big[a_1(u-\bar{u}) + a_2(u^2-\bar{u}^2) - a_3(u^3-\bar{u}^3) - k(uv-\bar{u}\bar{v})\Big](u-\bar{u})\,\mathrm{d}x \\
&= \int_\Omega \Big[a_1 + a_2(u+\bar{u}) - a_3(u^2+u\bar{u}+\bar{u}^2) - kv\Big](u-\bar{u})^2\,\mathrm{d}x - k\int_\Omega \bar{u}(u-\bar{u})(v-\bar{v})\mathrm{d}x \\
&\le \Big[a_1 + \frac{\mid a_2 \mid (a_2 + \sqrt{a_2^2+4a_1a_3})}{a_3} + C(\varepsilon, a_1, a_2, a_3, k, \Omega)\Big] \int_\Omega (u-\bar{u})^2 \mathrm{d}x + \varepsilon \int_\Omega (v-\bar{v})^2 \mathrm{d}x.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& d_2 \int_\Omega |\nabla(v-\bar{v})|^2 \mathrm{d}x = \int_\Omega (-v + uv - v^2)(v-\bar{v})\,\mathrm{d}x \\
&= \int_\Omega [-1 + u - (v+\bar{v})](v-\bar{v})^2 \mathrm{d}x + \int_\Omega \bar{v}(u-\bar{u})(v-\bar{v})\,\mathrm{d}x \\
&\le (\frac{a_2 + \sqrt{a_2^2+4a_1a_3}}{2a_3} - 1 + \varepsilon) \int_\Omega (v-\bar{v})^2 \mathrm{d}x + C(\varepsilon, a_1, a_2, a_3, k, \Omega) \int_\Omega (u-\bar{u})^2 \mathrm{d}x.
\end{aligned}
$$

Consequently, there exists $0 < \varepsilon \ll 1$ which depends only on a_1, a_2, a_3, k, Ω , such that

$$
\begin{aligned}
& \int_\Omega \Big\{ d_1 |\nabla(u-\bar{u})|^2 + d_2 |\nabla(v-\bar{v})|^2 \Big\}\,\mathrm{d}x \\
& \quad \le \Big[a_1 + \frac{\mid a_2 \mid (a_2 + \sqrt{a_2^2+4a_1a_3})}{a_3} + C(\varepsilon, a_1, a_2, a_3, k, \Omega)\Big] \int_\Omega (u-\bar{u})^2\,\mathrm{d}x \\
& \quad + \Big(\frac{a_2 + \sqrt{a_2^2+4a_1a_3}}{2a_3} - 1 + \varepsilon\Big) \int_\Omega (v-\bar{v})^2 \mathrm{d}x.
\end{aligned} \tag{4.1}
$$

Thanks to the well-known Poincaré Inequality

$$
\mu_1 \int_\Omega (g-\bar{g})^2\,\mathrm{d}x \le \int_\Omega |\nabla(g-\bar{g})|^2\,\mathrm{d}x,
$$

we yield from (4.1) that

$$\begin{aligned}\mu_1 &\int_\Omega \Big\{ d_1(u-\bar{u})^2 + d_2(v-\bar{v})^2 \Big\}\, \mathrm{d}x \\ &\leq \Big[a_1 + \frac{\mid a_2 \mid (a_2 + \sqrt{a_2^2 + 4a_1a_3})}{a_3} + C(\varepsilon, a_1, a_2, a_3, k, \Omega) \Big] \int_\Omega (u-\bar{u})^2\, \mathrm{d}x \\ &+ \Big(\frac{a_2 + \sqrt{a_2^2+4a_1a_3}}{2a_3} - 1 + \varepsilon \Big) \int_\Omega (v-\bar{v})^2 \mathrm{d}x. \end{aligned} \tag{4.2}$$

It is clear that there exists $\tilde{d}_1$ depending only on a_1, a_2, a_3, k, Ω, such that when $\mu_1 d_1 > \tilde{d}_1$ and $\mu_1 d_2 > (a_2 + \sqrt{a_2^2 + 4a_1a_3})/(2a_3) - 1$, $u \equiv \bar{u}$ =const., in turn, $v \equiv \bar{v}$ =const., which asserts our result (i).

As above, we have

$$\begin{aligned}\mu_1 \int_\Omega \Big\{ d_1(u-\bar{u})^2 + d_2(v-\bar{v})^2 \Big\}\, \mathrm{d}x \leq \Big[a_1 + \frac{\mid a_2 \mid (a_2 + \sqrt{a_2^2 + 4a_1a_3})}{a_3} + \varepsilon \Big] \\ \times \textstyle\int_\Omega (u-\bar{u})^2\, \mathrm{d}x + C(\varepsilon, a_1, a_2, a_3, k, \Omega) \int_\Omega (v-\bar{v})^2 \mathrm{d}x. \end{aligned} \tag{4.3}$$

The remaining arguments are rather similar as above. The proof is complete. □

4.2 Existence of positive non-constant solutions

This subsection is concerned with the existence of non-constant positive solutions to (1.4). The main tool to be used is the topological degree theory. To set up a suitable framework where the topological degree theory can apply, let us first introduce some necessary notations.

Let $\mathbf{X}$ be as in section 2. For simplicity, we write

$$\mathbf{u} = (u, v), \quad \mathbf{u}^* = (u^*, v^*).$$

We also denote the following sets

$$\mathbf{D} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad \mathbf{G}(\mathbf{u}) = \begin{pmatrix} a_1u + a_2u^2 - a_3u^3 - kuv \\ -v + uv - v^2 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \theta & -ku^* \\ v^* & -v^* \end{pmatrix},$$

where $\theta = u^*(a_2 - 2a_3u^*)$. Then $D_{\mathbf{u}}\mathbf{G}(\mathbf{u}^*) = \mathbf{A}$. Moreover, (1.4) can be written as

$$\begin{cases} -\Delta \mathbf{u} = \mathbf{D}^{-1}\mathbf{G}(\mathbf{u}), & x \in \Omega, \\ \partial_\nu \mathbf{u} = 0, & x \in \partial\Omega. \end{cases} \tag{4.4}$$

Furthermore, $\mathbf{u}$ solves (4.4) if and only if it satisfies

$$f(d_1, d_2; \mathbf{u}) := \mathbf{u} - (\mathbf{I} - \Delta)^{-1}\{\mathbf{D}^{-1}\mathbf{G}(\mathbf{u}) + \mathbf{u}\} = 0 \quad \text{on } \mathbf{X}, \tag{4.5}$$

where $(\mathbf{I} - \Delta)^{-1}$ is the inverse of $\mathbf{I} - \Delta$ with the homogeneous Neumann boundary condition. Direct computation gives

$$D_{\mathbf{u}} f(d_1, d_2; \mathbf{u}^*) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\{\mathbf{D}^{-1}\mathbf{A} + \mathbf{I}\}. \tag{4.6}$$

In order to apply the degree theory to obtain the existence of positive non-constant solutions, our first aim is to compute the index of $f(d_1, d_2; \mathbf{u})$ at $\mathbf{u}^*$. By the Leray-Schauder Theorem (see [11]), we have that if 0 is not the eigenvalue of (4.6), then

$$\mathrm{index}(f(d_1, d_2; \cdot), \mathbf{u}^*) = (-1)^r,$$

where r is the number of negative eigenvalues of (4.6).

It is easy to see that, for each integer $j \geq 0$, $\mathbf{X}_j$ is invariant under $D_{\mathbf{u}} f(d_1, d_2; \mathbf{u}^*)$, and ξ is an eigenvalue of $D_{\mathbf{u}} f(d_1, d_2; \mathbf{u}^*)$ on $\mathbf{X}_j$ if and only if $\xi(1 + \mu_j)$ is an eigenvalue of the matrix

$$\mathbf{M}(\mu_j) := \mu_j \mathbf{I} - \mathbf{D}^{-1}\mathbf{A} = \begin{pmatrix} \mu_j - \theta d_1^{-1} & k u^* d_1^{-1} \\ -v^* d_2^{-1} & \mu_j + v^* d_2^{-1} \end{pmatrix}.$$

Thus, $D_{\mathbf{u}} f(d_1, d_2; \mathbf{u}^*)$ is invertible if and only if, for all $j \geq 0$, the matrix $\mu_j \mathbf{I} - \mathbf{D}^{-1}\mathbf{A}$ is nonsingular.

Denote

$$H(\mu; d_1, d_2) := d_1 d_2 \det \mathbf{M}(\mu) = d_1 d_2 \mu^2 + (v^* d_1 - \theta d_2)\mu + v^*(k u^* - \theta).$$

In addition, we also have that, if $H(\mu_j; d_1, d_2) \neq 0$, the number of negative eigenvalues of $D_{\mathbf{u}} f(d_1, d_2; \mathbf{u}^*)$ on $\mathbf{X}_j$ is odd if and only if $H(\mu_j; d_1, d_2) < 0$.

Let $m(\mu_j)$ be the algebraical multiplicity of μ_j. In conclusion, we can assert the following:

Proposition 4.1 *Suppose that, for all $j \geq 0$, the matrix $\mu_j \mathbf{I} - \mathbf{D}^{-1}\mathbf{A}$ is nonsingular. Then*

$$\mathrm{index}(f(d_1, d_2; \cdot), \mathbf{u}^*) = (-1)^r, \quad where \ \ r = \sum_{j \geq 0, H(\mu_j; d_1, d_2) < 0} m(u_j).$$

Now, we analyze the sign of $H(\mu; d_1, d_2)$. Simple computations give that if

$$(v^* d_1 - \theta d_2)^2 - 4 d_1 d_2 v^*(k u^* - \theta) > 0, \tag{4.7}$$

then $H(\mu; d_1, d_2) = 0$ has exactly two different roots $\mu_*(d_1, d_2)$ and $\mu^*(d_1, d_2)$:

$$\mu_*(d_1, d_2) = \frac{1}{2 d_1 d_2}\{\theta d_2 - v^* d_1 - \sqrt{(\theta d_2 - v^* d_1)^2 - 4 d_1 d_2 v^*(k u^* - \theta)}\},$$

$$\mu^*(d_1, d_2) = \frac{1}{2 d_1 d_2}\{\theta d_2 - v^* d_1 + \sqrt{(\theta d_2 - v^* d_1)^2 - 4 d_1 d_2 v^*(k u^* - \theta)}\}.$$

In fact, we observe that $\mu_*(d_1, d_2)$ and $\mu^*(d_1, d_2)$ are the two real roots of the matrix $\mathbf{M}(u)$. Moreover, $H(\mu; d_1, d_2) < 0$ if and only if $\mu \in (\mu_*(d_1, d_2), \mu^*(d_1, d_2))$.

We can claim the main result of this subsection as follows.

Theorem 4.2 *Assume that $a_1 + a_2 > a_3$ and $4 a_1 a_3 + 2k(2a_3 - a_2) + a_2^2 < 0$, or equivalently, $\theta > 0$, and satisfies $\theta / d_1 \in (\mu_s, \mu_{s+1})$ for some $s \geq 1$. If $\sum_{j=1}^{s} m(\mu_j)$ is odd, then there exists a positive constant $\hat{d}$ such that (1.4) has at least one non-constant positive solution for all $d_2 \geq \hat{d}$.*

Proof. First, it is clear that when d_2 is large enough then (4.7) holds, and a simple computation gives that the constant term $v^*(k u^* - \theta)$ of $H(\mu; d_1, d_2)$ is positive. Hence, we have $\mu^*(d_1, d_2) > \mu_*(d_1, d_2) > 0$. Moreover

$$\lim_{d_2 \to \infty} \mu^*(d_1, d_2) = \frac{\theta}{d_1}, \qquad \lim_{d_2 \to \infty} \mu_*(d_1, d_2) = 0.$$

As $\theta/d_1 \in (\mu_s, \mu_{s+1})$, it follows that there exists a $\hat{d}$ such that

$$\mu^*(d_1, d_2) \in (\mu_s, \mu_{s+1}), \quad \text{and} \quad 0 < \mu_*(d_1, d_2) < \mu_1 \quad \forall d_2 > \hat{d}.$$

On the other hand, by Theorem 4.1 we know that there exists $\tilde{d_1} > 0$ such that (1.4) has no non-constant positive solution if $d_1 > \tilde{d}$. Moreover, taking a larger $\hat{d}$ if necessary, we may assume that $\theta/d_1 < \mu_1$ for all $d_1 \geq \hat{d} \geq \tilde{d_1}$. Thus, we have

$$0 < \mu_*(d_1, d_2) < \mu^*(d_1, d_2) < \mu_1 \quad \text{for any fixed } d_1, d_2 \geq \hat{d}.$$

We are now in the position of proving (1.4) has at least one non-constant positive solution for any $d_2 \geq \hat{d}$ under the hypotheses of the theorem. On the contrary, suppose that this assertion is not true for some $d_2 \geq \hat{d}$. In the following, we will derive a contradiction by using a homotopy argument.

For such d_2 and $t \in [0, 1]$, we define

$$\mathbf{D}(t) = \begin{pmatrix} td_1 + (1-t)\hat{d} & 0 \\ 0 & td_2 + (1-t)\hat{d} \end{pmatrix},$$

and consider the problem

$$\begin{cases} -\Delta \mathbf{u} = \mathbf{D}^{-1}(t)\mathbf{G}(\mathbf{u}), & x \in \ \Omega, \\ \partial_\nu \mathbf{u} = 0, & x \in \ \partial\Omega. \end{cases} \tag{4.8}$$

It is clear that finding positive solutions of (1.4) becomes equivalent to finding positive solutions of (4.8) for $t = 1$. On the other hand, for $0 \leq t \leq 1$. $\mathbf{u}$ is a non-constant positive solution of (4.8) if and only if it is a solution of the problem

$$h(\mathbf{u}; t) = \mathbf{u} - (\mathbf{I} - \Delta)^{-1}\{\mathbf{D}^{-1}(t)\mathbf{G}(\mathbf{u}) + \mathbf{u}\} = 0 \quad \text{on } \mathbf{X}. \tag{4.9}$$

We note that

$$h(\mathbf{u}; 1) = f(d_1, d_2; \mathbf{u}), \qquad h(\mathbf{u}; 0) = f(\hat{d}, \hat{d}; \mathbf{u}), \tag{4.10}$$

and

$$\begin{cases} D_{\mathbf{u}} f(d_1, d_2; \mathbf{u}^*) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\{\mathbf{D}^{-1}\mathbf{A} + \mathbf{I}\}, \\ D_{\mathbf{u}} f(\hat{d}, \ \hat{d}; \ \mathbf{u}^*) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\{\widetilde{\mathbf{D}}^{-1}\mathbf{A} + \mathbf{I}\}, \end{cases} \tag{4.11}$$

where $f(\cdot, \cdot; \cdot)$ was defined by (4.5) and

$$\widetilde{\mathbf{D}} = \begin{pmatrix} \hat{d} & 0 \\ 0 & \hat{d} \end{pmatrix}.$$

It is obvious that $\mathbf{u}^*$ is the only positive constant solution of (1.4) and by the choice of $\hat{d}$, (4.9) has no non-constant positive solution for $t = 0, 1$.

From Proposition 4.1, it immediately follows that

$$\begin{cases} \text{index}(h(\cdot, 1), \mathbf{u}^*) = \text{index}(f(\cdot, d_1, d_2), \mathbf{u}^*) = (-1)^{\sum_{j=1}^{s} m(\mu_j)} = -1, \\ \text{index}(h(\cdot, 0), \mathbf{u}^*) = \text{index}(f(\cdot, \hat{d}, \hat{d}), \mathbf{u}^*) = (-1)^0 = 1. \end{cases} \tag{4.12}$$

By Theorem 3.1 and 3.2, there exists a positive constant C such that (1.4) has no solution on $\partial\Theta$, where

$$\Theta = \Big\{\mathbf{u} \in [C(\overline{\Omega})]^2 | \ \frac{1}{2}C < u(x), v(x) < \frac{a_2 + \sqrt{a_2^2 + 4a_1 a_3}}{a_3}\Big\}.$$

Since $h(\mathbf{u}; t) : \Theta \times [0, 1] \to C(\overline{\Omega}) \times C(\overline{\Omega})$ is compact, the degree $\deg(h(\mathbf{u}; t), \Theta, 0)$ is well defined. By the homotopy invariance of degree, we can conclude

$$\deg((h(\cdot; 0), \Theta, 0) = \deg((h(\cdot; 1), \Theta, 0). \tag{4.13}$$

However, as both equations $h(\mathbf{u}; 0) = 0$ and $h(\mathbf{u}; 1) = 0$ have the unique positive solution $\mathbf{u}^*$ in Θ, we get from (4.12) that,

$$\deg((h(\cdot; 1), \Theta, 0) = \mathrm{index}(h(\cdot, 1), \mathbf{u}^*) = (-1)^{\sum_{j=1}^{s} m(\mu_j)} = -1,$$

$$\deg((h(\cdot; 0), \Theta, 0) = \mathrm{index}(h(\cdot, 0), \mathbf{u}^*) = (-1)^0 = 1.$$

This contradicts (4.13). The proof is complete. □

Similarly, we have the following result, whose proof is similar to the above and thus is omitted.

Theorem 4.3 *Assume that $a_1 + a_2 > a_3$ and (4.7) hold. Let $\mu_*(d_1, d_2) < \mu^*(d_1, d_2)$ be the two positive roots of $H(\mu; d_1, d_2) = 0$. If*

$$\mu_*(d_1, d_2) \in (\mu_l, \mu_{l+1}) \quad \text{and} \quad \mu^*(d_1, d_2) \in (\mu_q, \mu_{q+1}) \quad for\ some \quad 0 \le l < q,$$

and $\sum_{k=l+1}^{q} m(\mu_k)$ is odd, then (1.4) has at least one non-constant positive solution.

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