Electronic Journal of Qualitative Theory of Differential Equations
2011, No. **17**, 1-10; **http://www.math.u-szeged.hu/ejqtde/**

Existence and multiplicity of solutions for a Neumann-type $p(x)$-Laplacian equation with nonsmooth potential

Bin Ge[1,*] Qingmei Zhou[2]

[1] Department of Applied Mathematics, Harbin Engineering University, Harbin, 150001, P. R. China
[2] Library, Northeast Forestry University, Harbin, 150040, P. R. China

Abstract: In this paper we study Neumann-type $p(x)$-Laplacian equation with nonsmooth potential. Firstly, applying a version of the non-smooth three-critical-points theorem we obtain the existence of three solutions of the problem in $W^{1,p(x)}(\Omega)$. Finally, we obtain the existence of at least two nontrivial solutions, when $\alpha^- > p^+$.

Key words: $p(x)$-Laplacian, Differential inclusion problem, Three critical points theorem, Neumann-type problem.

§1 Introduction

The study of differential equations and variational problems with variable exponent has been a new and interesting topic. It arises from nonlinear elasticity theory, electrorheological fluids, etc. (see [1, 2]). It also has wide applications in different research fields, such as image processing model (see e.g. [3, 4]), stationary thermorheological viscous flows (see [5]) and the mathematical description of the processes filtration of an idea barotropic gas through a porous medium (see [6]).

The study on variable exponent problems attracts more and more interest in recent years, many results have been obtained on this kind of problems, for example [7-14].

In this paper, we investigate the following Neumann-type differential equation with $p(x)$-Laplacian and a nonsmooth potential:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u \in \lambda\partial j(x,u), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \gamma} = 0, & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary, $\lambda > 0$ is a real number, $p(x) \in C(\overline{\Omega})$ with $1 < p^- := \min_{x\in\overline{\Omega}} p(x) \leq p^+ := \max_{x\in\overline{\Omega}} p(x) < +\infty$, $\partial j(x,u)$ is the Clarke subdifferential of $j(x,\cdot)$, γ is the outward unit normal to the boundary $\partial\Omega$.

*Corresponding author: gebin04523080261@163.com

Supported by the National Science Fund (grant 10971043, 11001063), Heilongjiang Province Foundation for Distinguished Young Scholars (JC200810), Program of Excellent Team in Harbin Institute of Technology and the Natural Science Foundation of Heilongjiang Province (No. A200803).

In [7], Dai studied the particular case $p(x) \in C(\overline{\Omega})$ with $N < p^-$. He established the existence of three solutions by using the non-smooth critical -points theorem [15]. In this paper we will study problem (P) in the case when $1 < p(x) < +\infty$ for any $x \in \overline{\Omega}$. We will prove that there also exist three weak solutions for problem (P), and existence of at least two nontrivial solutions, when $\alpha^- > p^+$.

This paper is organized as follows. We will first introduce some basic preliminary results and lemma. In Section 2, including the variable exponent Lebesgue, Sobolev spaces, generalized gradient of locally Lipschitz function and non-smooth three-critical-points theorem. In section 3, we give the main results and their proof.

§2 Preliminaries

In this part, we introduce some definitions and results which will be used in the next section.

Firstly, we introduce some theories of Lebesgue-Sobolev space with variable exponent. The detailed can be found in [8-13].

Assume that $p \in C(\overline{\Omega})$ and $p(x) > 1$, for all $x \in \overline{\Omega}$. Set $C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : h(x) > 1$ for any $x \in \overline{\Omega}\}$. Define

$$h^- = \min_{x \in \overline{\Omega}} h(x), \quad h^+ = \max_{x \in \overline{\Omega}} h(x) \text{ for any } h \in C_+(\overline{\Omega}).$$

For $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space:

$L^{p(x)}(\Omega) = \{u : u \text{ is a measurable real value function } \int_\Omega |u(x)|^{p(x)}dx < +\infty\}$,

with the norm $|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \leq 1\}$,

and define the variable exponent Sobolev space

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm $\|u\| = \|u\|_{W^{1,p(x)}(\Omega)} = |u|_{p(x)} + |\nabla u|_{p(x)}$.

We remember that spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces. Denoting by $L^{q(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ with $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$, then the Hölder type inequality

$$\int_\Omega |uv|dx \leq (\frac{1}{p^-} + \frac{1}{q^-})|u|_{L^{p(x)}(\Omega)}|v|_{L^{q(x)}(\Omega)}, \; u \in L^{p(x)}(\Omega), v \in L^{q(x)}(\Omega) \tag{1}$$

holds. Furthermore, define mapping $\rho : W^{1,p(x)} \to \mathbb{R}$ by

$$\rho(u) = \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})dx,$$

then the following relations hold

$$\|u\| < 1 (= 1, > 1) \Leftrightarrow \rho(u) < 1 (= 1, > 1), \tag{2}$$

$$\|u\| > 1 \Rightarrow \|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}, \tag{3}$$

$$\|u\| < 1 \Rightarrow \|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}. \tag{4}$$

Hereafter, let $p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)}, & p(x) < N, \\ +\infty, & p(x) \geq N. \end{cases}$

Remark 2.1. If $h \in C_+(\overline{\Omega})$ and $h(x) \leq p^*(x)$ for any $x \in \overline{\Omega}$, by Theorem 2.3 in [11], we deduce that $W^{1,p(x)}(\Omega)$ is continuously embedded in $L^{h(x)}(\Omega)$. When $h(x) < p^*(x)$, the embedding is compact.

Let X be a Banach space and X^* be its topological dual space and we denote $< \cdot, \cdot >$ as the duality bracket for pair (X^*, X). A function $\varphi : X \mapsto \mathbb{R}$ is said to be locally lipschitz, if for every $x \in X$, we can find a neighbourhood U of x and a constant $k > 0$(depending on U), such that $|\varphi(y) - \varphi(z)| \leq k\|y - z\|, \forall y, z \in U$.

The generalized directional derivative of φ at the point $u \in X$ in the direction $h \in X$ is

$$\varphi^0(u; h) = \limsup_{u' \to u; \lambda \downarrow 0} \frac{\varphi(u' + \lambda h) - \varphi(u')}{\lambda}.$$

The generalized subdifferential of φ at the point $u \in X$ is defined by

$$\partial\varphi(u) = \{u^* \in X^*; < u^*, h > \leq \varphi^0(u; h), \ \forall\ h \in X\},$$

which is a nonempty, convex and w^*-compact set of X. We say that $u \in X$ is a critical point of φ, if $0 \in \partial\varphi(x)$. For further details, we refer the reader to [16].

Finally, for proving our results in the next section, we introduce the following lemma:

Lemma 2.1(see [15]). Let X be a separable and reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz functions. Assume that there exist $u_0 \in X$ such that $\Phi(u_0) = \Psi(u_0) = 0$ and $\Phi(u) \geq 0$ for every $u \in X$ and that there exists $u_1 \in X$ and $r > 0$ such that:

(1) $r < \Phi(u_1)$;

(2) $\sup_{\Phi(u)<r} \Psi(u) < r\frac{\Psi(u_1)}{\Phi(u_1)}$, and further, we assume that function $\Phi - \lambda\Psi$ is sequentially lower semicontinuous, satisfies the (PS)-condition, and

(3) $\lim_{\|u\|\to\infty} (\Phi(u) - \lambda\Psi(u)) = +\infty$

for every $\lambda \in [0, \overline{a}]$, where

$\overline{a} = \frac{hr}{r\frac{\Psi(u_1)}{\Phi(u_1)} - \sup_{\Phi(u)<r} \Psi(u)}$, with $h > 1$.

Then, there exits an open interval $\Lambda_1 \subseteq [0, \overline{a}]$ and a positive real number σ such that, for every $\lambda \in \Lambda_1$, the function $\Phi(u) - \lambda\Psi(u)$ admits at least three critical points whose norms are less than σ.

§3 Existence theorems

In this section, we will prove that there also exist three weak solutions for problem (P).

Our hypotheses on nonsmooth potential $j(x, t)$ as follows.

$\mathbf{H(j)}$: $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(x, 0) = 0$ a.e. on Ω and satisfies the following facts:

(i) for all $t \in \mathbb{R}$, $x \mapsto j(x, t)$ is measurable;

(ii) for almost all $x \in \Omega$, $t \mapsto j(x, t)$ is locally Lipschitz;

(iii) there exist $\alpha \in C_+(\overline{\Omega})$ with $\alpha^+ < p^-$ and positive constants c_1, c_2, such that

$|w| \leq c_1 + c_2|t|^{\alpha(x)-1}$

for every $t \in \mathbb{R}$, almost all $x \in \Omega$ and all $w \in \partial j(x, t)$;

(iv) there exists a $t_0 \in \mathbb{R}^+$, such that $j(x, t_0) > 0$ for all $x \in \overline{\Omega}$;

(v) there exist $q \in C(\overline{\Omega})$ such that $p^+ < q^- \leq q(x) < p^*(x)$ and

$\lim_{|t|\to 0} \dfrac{j(x, t)}{|t|^{q(x)}} = 0$ uniformly a.e. $x \in \Omega$.

Remark 3.1. It is easy to give examples satisfying all conditions in $\mathbf{H(j)}$. For example, the following nonsmooth locally Lipschitz function $j : \Omega \times \mathbb{R} \to \mathbb{R}$, satisfies hypotheses $\mathbf{H(j)}$:

$$j(x,t) = \begin{cases} \frac{1}{\beta(x)}|t|^{\beta(x)}, & \text{if } |t| \le 1, \\ \frac{1}{\alpha(x)}|t|^{\alpha(x)} + \frac{\alpha(x)-\beta(x)}{\alpha(x)\beta(x)}\,t, & \text{if } |t| > 1, \end{cases}$$

where $\alpha, \beta \in C_+(\overline{\Omega})$ with $\alpha^+ < p^- \le p^+ < q^- \le q^+ < \beta^- \le \beta(x) < p^*(x)$.

In order to use Lemma 2.1, we define the function $\Phi, \Psi : W^{1,p(x)}(\Omega) \to \mathbb{R}$ by

$$\Phi(u) = \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx, \quad \Psi(u) = \int_\Omega j(x,u)dx.$$

and let $\varphi(u) = \Phi(u) - \lambda\Psi(u)$, by Fan [14, Theorem 3.1], we know that Φ is continuous and convex, hence locally Lipschitz on $W^{1,p(x)}(\Omega)$. On the other hand, because of hypotheses $\mathbf{H(j)}$(i),(ii),(iii), Ψ is locally Lipschitz (see Clarke [16], p.83)). Therefore $\varphi(u)$ is locally Lipschitz. We state below our main results

Theorem 3.1. If hypotheses $\mathbf{H(j)}$ hold, Then there are an open interval $\Lambda \subseteq [0.+\infty)$ and a number σ such that, for each $\lambda \in \Lambda$ the problem (P) possesses at least three weak solutions in $W^{1,p(x)}(\Omega)$ whose norms are less than σ.

Proof: The proof is divided into the following three Steps.

Step 1. We will show that φ is coercive in the step.

Firstly, for almost all $x \in \Omega$, by $t \mapsto j(x,t)$ is differentiable almost everywhere on $\mathbb{R}$ and we have $\frac{d}{dt}j(x,t) \in \partial j(z,t)$. Moveover, from $\mathbf{H(j)(iii)}$, there exist positive constants c_3, c_4, such that

$$j(x,t) = j(x,0) + \int_0^t \frac{d}{dy}j(x,y)dy \le c_1|t| + \frac{c_2}{\alpha(x)}|t|^{\alpha(x)} \le c_3 + c_4|t|^{\alpha(x)} \tag{5}$$

for almost all $x \in \Omega$ and $t \in \mathbb{R}$.

Note that $1 < \alpha(x) \le \alpha^+ < p^- < p^*(x)$, then by Remark 2.1, we have $W^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$(compact embedding). Furthermore, there exists a c such that $|u|_{\alpha(x)} \le c\|u\|$ for any $u \in W^{1,p(x)}(\Omega)$.

So, for any $|u|_{\alpha(x)} > 1$ and $\|u\| > 1$, $\int_\Omega |u|^{\alpha(x)}dx \le |u|^{\alpha^+}_{\alpha(x)} \le c^{\alpha^+}\|u\|^{\alpha^+}$.

Hence, from (3) and (5), we have

$$\begin{aligned} \varphi(u) &= \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx - \lambda\int_\Omega j(x,u)dx \\ &\ge \frac{1}{p^+}\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})dx - \lambda\int_\Omega j(x,u)dx \\ &\ge \frac{1}{p^+}\|u\|^{p^-} - \lambda\int_\Omega j(x,u)dx \\ &\ge \frac{1}{p^+}\|u\|^{p^-} - \lambda c_3\text{meas}(\Omega) - \lambda c_4 c^{\alpha^+}\|u\|^{\alpha^+} \to +\infty, \end{aligned}$$

as $\|u\| \to +\infty$.

Step 2. We show that (PS)-condition holds.

Suppose $\{u_n\}_{n\ge1} \subseteq W^{1,p(x)}(\Omega)$ such that $|\varphi(u_n)| \le c$ and $m(u_n) \to 0$ as $n \to +\infty$. Let $u_n^* \in \partial\varphi(u_n)$ be such that $m(u_n) = \|u_n^*\|_{(W^{1,p(x)}(\Omega))^*}, n \ge 1$, then we know that

$$u_n^* = \Phi'(u_n) - \lambda w_n$$

where the nonlinear operator $\Phi' : W^{1,p(x)}(\Omega) \to (W^{1,p(x)}(\Omega))^*$ defined as

$$< \Phi'(u), v >= \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_\Omega |u|^{p(x)-2}uvdx,$$

for all $v \in W^{1,p(x)}(\Omega)$ and $w_n \in \partial\Psi(u_n)$. From Chang [17] we know that $w_n \in L^{\alpha'(x)}(\Omega)$, where $\frac{1}{\alpha'(x)} + \frac{1}{\alpha(x)} = 1$.

Since, φ is coercive, $\{u_n\}_{n\geq 1}$ is bounded in $W^{1,p(x)}(\Omega)$ and there exists $u \in W_0^{1,p(x)}(\Omega)$ such that a subsequence of $\{u_n\}_{n\geq 1}$, which is still be denoted as $\{u_n\}_{n\geq 1}$, satisfies $u_n \rightharpoonup u$ weakly in $W^{1,p(x)}(\Omega)$. Next we will prove that $u_n \to u$ in $W^{1,p(x)}(\Omega)$.

By $W^{1,p(x)}(\Omega) \to L^{\alpha(x)}(\Omega)$, we have $u_n \to u$ in $L^{\alpha(x)}(\Omega)$. Moreover, since $\|u_n^*\|_* \to 0$, we get $| < u_n^*, u_n > | \leq \varepsilon_n$.

Note that $u_n^* = \Phi'(u_n) - \lambda w_n$, we obtain

$$< \Phi'(u_n), u_n - u > -\lambda \int_\Omega w_n(u_n - u)dx \leq \varepsilon_n, \forall n \geq 1.$$

Moreover, $\int_\Omega w_n(u_n - u)dx \to 0$, since $u_n \to u$ in $L^{\alpha(x)}(\Omega)$ and $\{w_n\}_{n\geq 1}$ are bounded in $L^{\alpha'(x)}(\Omega)$, where $\frac{1}{\alpha(x)} + \frac{1}{\alpha'(x)} = 1$. Therefore,

$$\limsup_{n\to\infty} < \Phi'(u_n), u_n - u >\leq 0.$$

But we know Φ' is a mapping of type (S_+)(see [14, Theorem 3.1]). Thus we obtain

$$u_n \to u \text{ in } W^{1,p(x)}(\Omega).$$

Step 3. We show that Φ, Ψ satisfy the conditions (1) and (2) in Lemma 2.1.

Consider $u_0, u_1 \in W^{1,p(x)}(\Omega)$, $u_0(x) = 0$ and $u_1(x) = t_0$ for any $x \in \overline{\Omega}$. A simple computation implies $\Phi(u_0) = \Psi(u_0) = 0$ and $\Psi(u_1) > 0$.

From (3) and (4), we have

if $\|u\| \geq 1$, then

$$\frac{1}{p^+}\|u\|^{p^-} \leq \Phi(u) \leq \frac{1}{p^-}\|u\|^{p^+}; \tag{6}$$

if $\|u\| < 1$, then

$$\frac{1}{p^+}\|u\|^{p^+} \leq \Phi(u) \leq \frac{1}{p^-}\|u\|^{p^-}. \tag{7}$$

From $\mathbf{H(j)}$(v), there exist $\eta \in [0, 1]$ and $c_5 > 0$ such that

$$j(x, t) \leq c_5|t|^{q(x)} \leq c_5|t|^{q^-}, \forall t \in [-\eta, \eta], x \in \Omega.$$

In view of $\mathbf{H(j)}$(iii), if we put

$$c_6 = \max\{c_5, \sup_{\eta\leq|t|<1} \frac{c_3 + c_4|t|^{\alpha^-}}{|t|^{q^-}}, \sup_{|t|\geq 1} \frac{a_1 + a_2|t|^{\alpha^+}}{|t|^{q^-}}\},$$

then we have

$$j(x, t) \leq c_6|t|^{q^-}, \forall t \in \mathbb{R}, x \in \Omega.$$

Fix r such that $0 < r < 1$. And when $\frac{1}{p^+}\max\{\|u\|^{p^-}, \|u\|^{p^+}\} < r < 1$, by Sobolev Embedding Theorem $(W^{1,p(x)}(\Omega) \hookrightarrow L^{q^-}(\Omega))$, we have (for suitable positive constants c_7, c_8)

$$\Psi(u) = \int_\Omega j(x, u)dx \leq c_6 \int_\Omega |u|^{q^-} dx \leq c_7\|u\|^{q^-} < c_8 r^{\frac{q^-}{p^-}} (\text{or } c_8 r^{\frac{q^-}{p^+}}).$$

Since $q^- > p^+$, we have

$$\lim_{r\to 0^+} \frac{\sup\limits_{\frac{1}{p^+}\max\{\|u\|^{p^-},\|u\|^{p^+}\}<r} \Psi(u)}{r} = 0. \tag{8}$$

Fix r_0 such that $r_0 < \frac{1}{p^+}\min\{\|u_1\|^{p^-}, \|u_1\|^{p^+}, 1\}$.

Case 1. When $\|u_1\| \geq 1$, from (6), we have

$$\frac{1}{p^-}\|u_1\|^{p^+} \geq \Phi(u_1) \geq \frac{1}{p^+}\|u_1\|^{p^-}. \tag{9}$$

From (8) and (9), we know that when $0 < r < r_0$, $\Phi(u_1) > r$ and

$$\sup_{\frac{1}{p^+}\|u\|^{p^-} < r} \Psi(u) \leq \frac{r}{2}\frac{\Psi(u_1)}{\frac{1}{p^-}\|u_1\|^{p^+}} \leq \frac{r}{2}\frac{\Psi(u_1)}{\Phi(u_1)} < r\frac{\Psi(u_1)}{\Phi(u_1)}.$$

From (6), we have

$$\{u \in W^{1,p(x)}(\Omega) : \Phi(u) < r\} \subseteq \{u \in W^{1,p(x)}(\Omega) : \frac{1}{p^+}\|u\|^{p^-} < r\}.$$

Hence,

$$\sup_{\Phi(u)<r} \Psi(u) < r\frac{\Psi(u_1)}{\Phi(u_1)}.$$

Case 2. When $\|u_1\| \geq 1$, fixing r as above, with the role of $\|u_1\|^{p^+}$ above now assumed by $\|u_1\|^{p^-}$, we can analogously get

$$\sup_{\frac{1}{p^+}\|u\|^{p^+} < r} \Psi(u) \leq \frac{r}{2}\frac{\Psi(u_1)}{\frac{1}{p^-}\|u_1\|^{p^-}} \leq \frac{r}{2}\frac{\Psi(u_1)}{\Phi(u_1)} < r\frac{\Psi(u_1)}{\Phi(u_1)}.$$

From (7), we have

$$\{u \in W^{1,p(x)}(\Omega) : \Phi(u) < r\} \subseteq \{u \in W^{1,p(x)}(\Omega) : \frac{1}{p^+}\|u\|^{p^+} < r\}.$$

Hence,

$$\sup_{\Phi(u)<r} \Psi(u) < r\frac{\Psi(u_1)}{\Phi(u_1)}.$$

Thus, Φ and Ψ satisfy all the assumptions of Lemma 2.1, and the proof is complect. $\square$

Thus far the results involved potential functions exhibiting $p(x)$-sublinear. The next theorem concerns problems where the potential function is $p(x)$-superlinear. The hypotheses on the nonsmooth potential are the following:

$\mathbf{H(j)_1}$: $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(x, 0) = 0$ a.e. on Ω and satisfies the following facts:

(i) for all $t \in \mathbb{R}$, $x \mapsto j(x, t)$ is measurable;

(ii) for almost all $x \in \Omega$, $t \mapsto j(x, t)$ is locally Lipschitz;

(iii) there exist $\alpha \in C_+(\overline{\Omega})$ with $\alpha^- > p^+$ and positive constants c_1, c_2, such that

$$|w| \leq c_1 + c_2|t|^{\alpha(x)-1}$$

for every $t \in \mathbb{R}$, almost all $x \in \Omega$ and all $w \in \partial j(x, t)$;

(iv) There exist $\gamma \in C(\overline{\Omega})$ with $p^+ < \gamma(x) < p^*(x)$ and $\mu \in L^\infty(\Omega)$, such that

$$\limsup_{t\to 0} \frac{< w, t >}{|t|^{\gamma(x)}} < \mu(x),$$

uniformly for almost all $x \in \Omega$ and all $w \in \partial j(x, t)$;

(v) There exist $\xi_0 \in \mathbb{R}, x_0 \in \Omega$ and $r_0 > 0$, such that

$$j(x, \xi_0) > \delta_0 > 0, \text{a.e. } x \in B_{r_0}(x_0)$$

where $B_{r_0}(x_0) := \{x \in \Omega : |x - x_0| \leq r_0\} \subset \Omega$;

(vi) For almost all $x \in \Omega$, all $t \in \mathbb{R}$ and all $w \in \partial j(x,t)$, we have

$$j(x,t) \leq \nu(x) \text{ with } \nu \in L^{\beta(x)}(\Omega), 1 \leq \beta(x) < p^-.$$

Remark 3.2. It is easy to give examples satisfying all conditions in $\mathbf{H(j)_1}$. For example, the following nonsmooth locally Lipschitz function $j : \Omega \times \mathbb{R} \to \mathbb{R}$, satisfies hypotheses $\mathbf{H(j)_1}$:

$$j(x,t) = \begin{cases} -\sin(\frac{\pi}{2}|t|^{\gamma(x)}), & |t| \leq 1, \\ \frac{1}{2\sqrt{|t|}} - \frac{3}{2}, & |t| > 1, \end{cases}$$

Theorem 3.2. If hypotheses $\mathbf{H(j)_1}$ hold, then there exists a $\lambda_0 > 0$ such that for each $\lambda > \lambda_0$, the problem (P) has at least two nontrivial solutions.

Proof: The proof is divided into the following five Steps.

Step 1. We will show that φ is coercive in the step.

By $\mathbf{H(j)_1}$(vi), for all $u \in W^{1,p(x)}(\Omega)$, $\|u\| > 1$, we have

$$\begin{aligned} \varphi(u) &= \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx - \lambda \int_\Omega j(x,u)dx \\ &\geq \frac{1}{p^+}\|u\|^{p^-} - \lambda \int_\Omega \nu(x)dx \to \infty, \quad \text{as } \|u\| \to \infty. \end{aligned}$$

Step 2. We will show that the φ is weakly lower semi-continuous.

Let $u_n \rightharpoonup u$ weakly in $W^{1,p(x)}(\Omega)$, by Remark 2.1, we obtain the following results:

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega);$$
$$u_n \to u \text{ in } L^{p(x)}(\Omega);$$
$$u_n \to u \text{ for a.e. } x \in \Omega;$$
$$j(x, u_n(x)) \to j(x, u(x)) \text{ for a.e. } x \in \Omega.$$

By Fatou's Lemma,

$$\limsup_{n\to\infty} \int_\Omega j(x, u_n(x))dx \leq \int_\Omega j(x, u(x))dx.$$

Thus,

$$\begin{aligned} \liminf_{n\to\infty} \varphi(u_n) &= \liminf_{n\to\infty} \int_\Omega \frac{1}{p(x)}(|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx - \limsup_{n\to\infty} \lambda \int_\Omega j(x, u_n)dx \\ &\geq \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx - \lambda \int_\Omega j(x,u)dx = \varphi(u). \end{aligned}$$

Hence, by The Weierstrass Theorem, we deduce that there exists a global minimizer $u_0 \in W^{1,p(x)}(\Omega)$ such that

$$\varphi(u_0) = \min_{u \in W^{1,p(x)}(\Omega)} \varphi(u).$$

Step 3. We will show that there exists $\lambda_0 > 0$ such that for each $\lambda > \lambda_0$, $\varphi(u_0) < 0$.

By the condition $\mathbf{H(j)_1}$(v), there exists $\xi_0 \in \mathbb{R}$ such that $j(x, \xi_0) > \delta_0 > 0$, a.e. $x \in B_{r_0}(x_0)$. It is clear that

$$0 < M_1 := \max_{|t|\leq|\xi_0|} \{c_1|t| + c_2|t|^{\alpha^+}, c_1|t| + c_2|t|^{\alpha^-}\} < +\infty.$$

Now we denote

$$t_0 = (\frac{M_1}{\delta_0 + M_1})^{\frac{1}{N}}, \quad K(t) := \max\{(\frac{\xi_0}{r_0(1-t)})^{p^-}, (\frac{\xi_0}{r_0(1-t)})^{p^+}\}$$

and

$$\lambda_0 = \max_{t\in[t_1,t_2]} \frac{K(t)(1-t^N) + \max\{\xi_0^{p^-}, \xi_0^{p^+}\}}{[\delta_0 t^N - M_1(1-t^N)]},$$

where $t_0 < t_1 < t_2 < 1$ and δ_0 is given in the condition $\mathbf{H(j)_1}$(v). A simple calculation shows that the function $t \mapsto \delta_0 t^N - M_1(1-t^N)$ is positive whenever $t > t_0$ and $\delta_0 t_0^N - M_1(1-t_0^N) = 0$. Thus λ_0 is well defined and $\lambda_0 > 0$.

We will show that for each $\lambda > \lambda_0$, the problem (P) has two nontrivial solutions. In order to do this, for $t \in [t_1, t_2]$, let us define

$$\eta_t(x) = \begin{cases} 0, & \text{if } x \in \Omega\backslash B_{r_0}(x_0), \\ \xi_0, & \text{if } x \in B_{tr_0}(x_0), \\ \dfrac{\xi_0}{r_0(1-t)}(r_0 - |x-x_0|), & \text{if } x \in B_{r_0}(x_0)\backslash B_{tr_0}(x_0). \end{cases}$$

By conditions $\mathbf{H(j)_1}$(iii) and (v) we have

$$\begin{aligned} \int_\Omega j(x,\eta_t(x))dx =& \int_{B_{tr_0}(x_0)} j(x,\eta_t(x))dx + \int_{B_{r_0}(x_0)\backslash B_{tr_0}(x_0)} j(x,\eta_t(x))dx \\ \geq& w_N r_0^N t^N \delta_0 - M_1(1-t^N) w_N r_0^N \\ =& w_N r_0^N (\delta_0 t^N - M_1(1-t^N)). \end{aligned}$$

Hence, for $t \in [t_1, t_2]$,

$$\begin{aligned} \varphi(\eta_t) =& \int_\Omega \frac{1}{p(x)} |\nabla \eta_t|^{p(x)} dx + \int_\Omega \frac{1}{p(x)} |\eta_t|^{p(x)} dx - \lambda \int_\Omega j(x,\eta_t(x))dx \\ \leq& \frac{1}{p^-} \int_\Omega (|\nabla\eta_t|^{p(x)} + |\eta_t|^{p(x)})dx - \lambda w_N r_0^N (\delta_0 t^N - M_1(1-t^N)) \\ \leq& \max\{[\frac{\xi_0}{r_0(1-t)}]^{p^-}, [\frac{\xi_0}{r_0(1-t)}]^{p^+}\} w_N r_0^N (1-t^N) \\ &+ \max\{\xi_0^{p^-}, \xi_0^{p^+}\} w_N r_0^N - \lambda w_N r_0^N (\delta_0 t^N - M_1(1-t^N)) \\ =& w_N r_0^N [K(t)(1-t^N) + \max\{\xi_0^{p^-}, \xi_0^{p^+}\} - \lambda(\delta_0 t^N - M_1(1-t^N))], \end{aligned}$$

so that $\varphi(\eta_t) < 0$ whenever $\lambda > \lambda_0$.

Step 4. We will check the C-condition in the following.

Suppose $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p(x)}(\Omega)$ such that $\varphi(u_n) \to c$ and $(1 + \|u_n\|)m(u_n) \to 0$.

Since, φ is coercive, $\{u_n\}_{n\geq1}$ is bounded in $W^{1,p(x)}(\Omega)$ and passed to a subsequence, still denote $\{u_n\}_{n\geq1}$, we may assume that there exists $u \in W^{1,p(x)}(\Omega)$, such that $u_n \rightharpoonup u$ weakly in $W^{1,p(x)}(\Omega)$. Next we will prove that $u_n \to u$ in $W^{1,p(x)}(\Omega)$.

By $W^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$, we have $u_n \to u$ in $L^{p(x)}(\Omega)$. Moreover, since $\|u_n^*\|_* \to 0$, we get $| < u_n^*, u_n > | \leq \varepsilon_n$.

Note that $u_n^* = \Phi'(u_n) - \lambda w_n$, we have

$$< \Phi'(u_n), u_n - u > -\lambda \textstyle\int_\Omega w_n(u_n - u)dx \leq \varepsilon_n, \forall n \geq 1.$$

Moreover, $\int_\Omega w_n(u_n - u)dx \to 0$, since $u_n \to u$ in $L^{p(x)}(\Omega)$ and $\{w_n\}_{n\geq1}$ in $L^{p'(x)}(\Omega)$ are bounded, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Therefore,

$$\limsup_{n\to\infty} < \Phi'(u_n), u_n - u > \leq 0.$$

From [14, Theorem 3.1], we have $u_n \to u$ as $n \to \infty$. Thus φ satisfies the nonsmooth C-condition.

Step 5. We will show that there exists another nontrivial weak solution of problem (P).

From Lebourg Mean Value Theorem, we obtain

$$j(x,t) - j(x,0) = \langle w, t\rangle$$

for some $w \in \partial j(x, \vartheta t)$ and $0 < \vartheta < 1$. Thus, from $\mathbf{H(j)_1}$(iv), there exists $\beta \in (0,1)$ such that

$$|j(x,t)| \leq |\langle w,t\rangle| \leq \mu(x)|t|^{\gamma(x)}, \quad \forall |t| < \beta \text{ and a.e. } x \in \Omega. \tag{10}$$

On the other hand, by the condition $\mathbf{H(j)_1}$(iii), we have

$$\begin{aligned} j(x,t) \leq & c_1|t| + c_2|t|^{\alpha(x)} \\ \leq & c_1|\frac{t}{\beta}|^{\alpha(x)-1}|t| + c_2|t|^{\alpha(x)} \\ = & c_1|\frac{1}{\beta}|^{\alpha^+-1}|t|^{\alpha(x)} + c_2|t|^{\alpha(x)} \\ = & c_5|t|^{\alpha(x)} \end{aligned} \tag{11}$$

for a.e. $x \in \Omega$, all $|t| \geq \beta$ with $c_5 > 0$.

Combining (10) and (11), it follows that

$$|j(x,t)| \leq \mu(x)|t|^{\gamma(x)} + c_5|t|^{\alpha(x)}$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$.

Thus, For all $\lambda > \lambda_0$, $\|u\| < 1$, $|u|_{\gamma(x)} < 1$ and $|u|_{\alpha(x)} < 1$, we have

$$\begin{aligned} \varphi(u) = & \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \lambda \int_\Omega j(x,u(x))dx \\ \geq & \frac{1}{p^+}\|u\|^{p^+} - \lambda \int_\Omega \mu(x)|u|^{\gamma(x)}dx - \lambda c_5 \int_\Omega |u|^{\alpha(x)}dx \\ \geq & \frac{1}{p^+}\|u\|^{p^+} - \lambda c_6\|u\|^{\gamma^-} - \lambda c_7\|u\|^{\alpha^-}. \end{aligned}$$

So, for $\rho > 0$ small enough, there exists a $\nu > 0$ such that

$$\varphi(u) > \nu, \text{ for } \|u\| = \rho$$

and $\|u_0\| > \rho$. So by the Nonsmooth Mountain Pass Theorem, we can get $u_1 \in W^{1,p(x)}(\Omega)$ satisfies

$$\varphi(u_1) = c > 0 \text{ and } m(u_1) = 0.$$

Therefore, u_1 is second nontrivial critical point of φ. □

References

[1] M. Ruzicka, Electrorheological Fluids: Modeling and Mathematical Theory, Springer-Verlag, Berlin, 2000.

[2] V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory. Math. USSR. Izv, 1987, 29(1): 33–66.

[3] P. Harjulehto, P. Hästö, V. Latvala, Minimizers of the variable exponent, non-uniformly convex Dirichlet energy. J. Math. Pures Appl, 2008, 89: 174-197.

[4] Y. Chen, S. Levine, M. Rao, Variable exponent linear growth functionals in image restoration. SIAM J. Appl. Math, 2006, 66(4): 1383-1406.

[5] S. N. Antontsev, J. F. Rodrigues, On stationary thermo-rheological viscous flows. Ann. Univ. Ferrara Sez. VII, Sci. Mat, 2006, 52: 19-36.

[6] S. N. Antontsev, S. I. Shmarev, A model porous medium equation with variable exponent of nonlinearity: Existence uniqueness and localization properties of solutions. Nonlinear Anal. TMA, 2005, 60: 515-545.

[7] G. Dai, Three solutions for a Neumann-type differential inclusion problem involving the $p(x)$-Laplacian. Nonlinear Anal, 2009, 70(10):3755-3760.

[8] D. E. Edmunds, J. Lang, A. Nekvinda, On $L^{p(x)}(\Omega)$ norms. Proc. R. Soc. Ser. A, 1999, 455:219-225.

[9] D. E. Edmunds, J. Rákosník, Sobolev embedding with variable exponent. Studia Math, 2000, 143:267-293.

[10] X. L. Fan, J. Shen, D. Zhao, Sobolev embedding theorems for spaces $W^{1,p(x)}$. J. Math. Anal. Appl, 2001, 262(2):749-760.

[11] X. L. Fan, D. Zhao, On the space $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. J. Math. Anal. Appl, 2001, 263(2):424-446.

[12] X. L. Fan, D. Zhao, On the generalized Orlicz-Sobolev space $W^{1,p(x)}(\Omega)$. J. Gansu Educ. College, 1998, 12(1):1-6.

[13] O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ $W^{1,p(x)}$. Czechoslovak Math. J, 1991, 41:592-618.

[14] X. L. Fan, Q. H Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problem. Nonlinear Anal, 2003, 52(8):1843-1852.

[15] G. Bonanno, P. Candito, On a class of nonlinear variational-hemivariational inequalities. Appl. Anal, 2004, 83:1229-1244.

[16] F. H. Clarke, Optimization and Nonsmooth Analysis. Wiley, New York, 1983.

[17] K. C. Chang, Variational methods for nondifferentiable functionals and their applications to partial differential equations. J. Math. Anal. Appl, 1981, 80(1):102-129.

(Received December 14, 2010)