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Strictly localized bounding functions and Floquet boundary value problems*

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Abstract

Semilinear multivalued equations are considered, in separable Banach spaces with the Radon-Nikodym property. An effective criterion for the existence of solutions to the associated Floquet boundary value problem is showed. Its proof is obtained combining a continuation principle with a Liapunov-like technique and a Scorza-Dragoni type theorem. A strictly localized transversality condition is assumed. The employed method enables to localize the solution values in a not necessarily invariant set; it allows also to introduce nonlinearities with superlinear growth in the state variable.

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1 Introduction

The paper deals with the Floquet boundary value problem (b.v.p.) associated to a semilinear multivalued differential equation

$$\begin{cases} x'(t) \in A(t)x(t) + F(t, x(t)), \ t \in [a, b], \ x(t) \in E \\ x(b) = Mx(a). \end{cases} \tag{1}$$

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in a separable Banach space E, with norm $\|\cdot\|$, satisfying the Radon-Nikodym property (in particular in a separable and reflexive Banach space E). We assume that

(A) $A : [a, b] \to \mathcal{L}(E)$ is Bochner integrable, where $\mathcal{L}(E)$ denotes the space of linear bounded operators from E into itself;

(F1) $F : [a, b] \times E \multimap E$ is a upper-Carathéodory (u-Carathéodory) multivalued map, i.e.,

  (i) $F(t, x)$ is nonempty, compact and convex for any $t \in [a, b]$, $x \in E$;

  (ii) the multifunction $F(\cdot, x) : [a, b] \multimap E$ is measurable for all $x \in E$;

  (iii) the multimap $F(t, \cdot) : E \multimap E$ is upper semicontinuous (u.s.c.) for a.a. $t \in [a, b]$;

(F2) for every bounded $\Omega \subset E$, there exists $\nu_\Omega \in L^1([a, b], \mathbb{R})$ such that $\|y\| \le \nu_\Omega(t)$, for a.a. $t \in [a, b]$, every $x \in \Omega$, and $y \in F(t, x)$;

(M) $M \in \mathcal{L}(E)$.

The measurability is intended with respect to the Lebesgue σ-algebra in $[a, b]$ and the Borel σ-algebra in E. We denote with τ the Lebesgue measure on $[a, b]$.

We search for strong Carathéodory solutions of problem (1). Namely, by a solution of (1) we mean an absolutely continuous function $x : [a, b] \to E$ such that its derivative satisfies (1) for a.a. $t \in [a, b]$. We remark that, in a Banach space E with the Radon-Nikodym property, each absolutely continuous function $x : [a, b] \to E$ has the derivative $x'(t)$ for a.a. $t \in [a, b]$, x' is Bochner integrable in $[a, b]$ and x satisfies the integral formula.

We obtain a solution of (1) as the limit of a sequence of solutions of approximating problems, denoted by (P_m), that we construct by means of a Scorza-Dragoni type result (cfr. Theorem 2.1). We solve each problem (P_m) with a continuation principle proved in [2] (see also Theorem 2.3) and relative to the case of condensing solution operators. To this aim we have, in particular, to show the so called *transversality condition* (see e.g. condition (d) in Theorem 2.3), i.e. the lack of solutions on the boundary of a suitable set for all the parametrized problems associated to each (P_m). So we introduce a Frechét differentiable Liapunov-like function $V : E \to \mathbb{R}$ and denote with K its zero sublevel set. Under suitable conditions on V' we are able to guarantee that all the functions in $C([a, b], \overline{K})$ satisfy the required transversality. This approach originates by *Gaines and Mawhin* [7] and we refer to [2] and [4] for an updated list of contributions on this topic. A not completely satisfactory condition on V' in a neighborhood of the boundary ∂K of K, was proposed in [2, Theorem 5.2], for getting the required transversality. Indeed, as a consequence of the proof of [2, Proposition 4.1], it is not difficult to see that such a condition implies the positive invariance of K, which is not necessary for having the transversality. Under additional regularities, i.e. when A is continuous on $[a, b]$ and F is globally u.s.c., a strictly localized transversality condition on ∂K was proved in [2], which does not imply the invariance of K (see e.g. [2, Example 1]). If $|V'_x(x)| \neq 0$ and x belongs to an Hilbert space H, a straightforward consequence of the Riesz representation Theorem is the existence of a bounded and lipschitzian function $\phi : H \to \mathbb{R}$ satisfying $V'_x(\phi(x)) = \|V'_x\|$. In [4] such a ϕ is the key point for the construction of a sequence of approximating problems. This lead to an existence result for (1) ([4, Theorem 3.4]) in a separable Hilbert space, when $A(t)$ satisfies (A) and F is a Carathéodory nonlinearity, which is based on a strictly localized transversality condition. In Theorem 2.2, we assume that K is open, bounded, convex and $0 \in K$, and we prove the existence of a function with similar properties as the mentioned ϕ but in an arbitrary Banach space. Thanks to it, we are able to solve the b.v.p. (1) in an arbitrary separable Banach space with the Radon-Nikodym property and we assume the strictly localized transversality condition (V4); this is the main result in the paper and it is contained in Theorem 3.1. We remark that Theorem 3.1 is more general than the quoted result in [2]. Moreover, also in a Hilbert space, it is an improvement of the quoted one in [4]. In fact, in Example 3.1 we discuss a b.v.p. in $\mathbb{R}$ which can be investigated by means of Theorem 3.1 but that it is not possible to study with the quoted results in [2] and [4]. Finally in Example 3.1 we show that condition (V4) does not imply either the positive or the negative invariance of the sublevel set K. The employed technique enables to localize the solution values in $\overline{K}$. Moreover, due to condition (F2), the nonlinearity F can also have a superlinear growth in its variable x.

A spatial dispersal process where the classical diffusion term is replaced by a non-local type one can be modeled with an equation of the type

$$u_t(t, x) = \gamma(t, x)u(t, x) + \int_\Omega k(x, y)u(t, y)dy, \quad \text{for a.a. } t \in [a, b] \qquad (2)$$

where $x \in \Omega \subset \mathbb{R}^n$, $\gamma : [a, b] \times \Omega \to \mathbb{R}$ and the function $k : \Omega \times \Omega \to \mathbb{R}$ represents the dispersal kernel. Equation (2) can be viewed as a special case of the integro-differential inclusion

$$u_t(t, x) \in \gamma(t, x)u(t, x) + F(t, x, Su(t, \cdot)), \quad \text{for a.a. } t \in [a, b] \qquad (3)$$

where $Sv(x) = \int_\Omega k(x,y)v(y)dy$ and F is a suitable multivalued map. Nonlinear dynamics as (3) also appear in the study of viscoelasticity properties, in transport problems and in the theory of phase transitions (cfr. [3] and the references there contained). We remark that both (2) and (3) can be reformulated as equations or inclusions in Banach spaces of the type appearing in problem (1) and studied with the techniques developed in this paper. This is showed in details in [3].

We denote by $U(t,s), (t,s) \in \Delta = \{(t,s) \in [a,b] \times [a,b] \,:\, a \le s \le t \le b\}$, the evolution system generated by $\{A(t)\}_{t\in[a,b]}$ (see [9] for details). It is well known that

$$\|U(t,s)\| \le \mathrm{e}^{\int_a^b \|A(t)\|\,dt}, \quad \text{for all } (t,s) \in \Delta. \tag{4}$$

Moreover, the map $M - U(b,a)$ is invertible if and only if, for any $f \in L^1([a,b],E)$ the b.v.p.

$$\begin{cases} x' = A(t)x + f(t), & \text{for a.a. } t \in [a,b], \\ x(b) = Mx(a) \end{cases}$$

is uniquely solvable and, in this case, its solution can be written as follows

$$x(t) = U(t,a)\big(M - U(b,a)\big)^{-1} \int_a^b U(b,s)f(s)\,ds + \int_a^t U(t,s)f(s)\,ds \tag{5}$$

(see e.g. [2, Lemma 5.1], where the result is proved assuming the invertibility of M, but indeed this condition is not necessary).

We denote by γ the Hausdorff measure of non-compactness (m.n.c.) on E. It is well known that, if $V : E \to E$ is a Lipschitz function of constant L and $\Omega \subset E$, then

$$\gamma(V(\Omega)) \le L\gamma(\Omega). \tag{6}$$

Let $\{f_n\}_n \subset L^1([a,b],E)$. If there exist $\nu, c \in L^1[a,b]$ such that $\|f_n(t)\| \le \nu(t)$ and $\gamma(\{f_n(t)\}_n) \le c(t)$ for a.a. $t \in [a,b]$ and $n \in \mathbb{N}$, then

$$\gamma\Big(\Big\{\int_a^b f_n(t)dt\Big\}_n\Big) \le \int_a^b c(t)dt. \tag{7}$$

For any subset Ω of E and $\delta > 0$ it follows that (see e.g. [2])

$$\gamma\Big(\bigcup_{\lambda\in[0,\delta]} \lambda\,\Omega\Big) = \delta\gamma(\Omega). \tag{8}$$

In a space of continuous functions, an important example of monotone and non-singular m.n.c. is the modulus of equicontinuity:

$$\mathrm{mod}_C(\Omega) = \lim_{\delta\to 0} \sup_{x\in\Omega} \max_{|t_1-t_2|\le\delta} |x(t_1) - x(t_2)|.$$

It is easy to see that the modulus of equicontinuity of a set is equal to zero if and only if the set is equicontinuous. We refer to [8] for a wide presentation of the theory of m.n.c.
If X is a subset of E and Λ is a space of parameters, a family of compact valued multimaps $G : \Lambda \times X \multimap E$ is called condensing with respect to a m.n.c. β (shortly β-condensing) if, for every $\Omega \subseteq X$ that is not relatively compact, we have

$$\beta(G(\Lambda \times \Omega)) < \beta(\Omega) \ .$$

Given the topological spaces X and Y, the multimap $F : X \multimap Y$ is said to be quasi-compact if it maps compact sets of X into relatively compact sets in Y.

Let B be the open unit ball in E. Given $\varepsilon > 0$ and $H \subset E$ bounded, define $B_H^\varepsilon = H + \varepsilon B$ and $\|H\| = \sup\limits_{x \in H} \|x\|$. Finally we denote with $\| \cdot \|_1$ the norm in $L^1([a, b], \mathbb{R})$.

2 Preliminaries

The technique that we use in order to prove the existence result in Theorem 3.1, consists into associating to the b.v.p. (1) a sequence of approximating problems. Each one of them is obtained by means of a Scorza-Dragoni type result for u-Carathéodory multimaps. It is well known that, for a single-valued map, the measurability in t for every x and the continuity in x for a.a. t implies the almost continuity. This result was extended to set valued function under the same assumptions (see [8, Theorem 1.3.2.]), but a straightforward generalizations to the case of upper-semicontinuity is not possible (see, e.g., [8, Example 1.3.1.]). So, we introduce the following notion.

Definition 2.1 *An u-Carathéodory map $F : [a, b] \times E \multimap E$ is said to have the Scorza-Dragoni property if there exists a multivalued mapping $F_0 : [a, b] \times E \multimap E \cup \{\emptyset\}$ with compact, convex values having the following properties:*

(i) $F_0(t, x) \subset F(t, x)$, *for all* $(t, x) \in [a, b] \times E$;

(ii) *if* $u, v : [a, b] \to E$ *are measurable functions with* $v(t) \in F(t, u(t))$ *a.e. on* $[a, b]$, *then* $v(t) \in F_0(t, u(t))$ *a.e. on* $[a, b]$;

(iii) *for every* $\varepsilon > 0$ *there exists a closed* $I_\varepsilon \subset [a, b]$ *such that* $\tau([a, b] \setminus I_\varepsilon) < \varepsilon$, $F_0(t, x) \neq \emptyset$ *when* $(t, x) \in I_\varepsilon \times E$ *and* F_0 *is u.s.c. on* $I_\varepsilon \times E$.

Trivially, every almost-usc multimap (see [6, Definition 3.3]) has the Scorza-Dragoni property. Notice that, if E is separable, an u-Carathédory map is almost-usc if and only if it is globally measurable (see [11, Theorems 1 and 2]). Moreover, if E is separable, every quasi-compact u-Carathédory multimap has the Scorza-Dragoni property (see [5, Theorem 1], see also [10, Thoerem 1] and [8, Theorem 1.1.12]). We remark that (see [5]) an u-Carathédory map F is quasi-compact if there exists $g \in L^1([a,b],\mathbb{R})$ such that for any bounded $\Omega \subset E$ and $t \in [a,b]$

$$\lim_{h\to 0^+} \gamma(F((t-h,t+h)\cap[a,b],\Omega)) \le g(t)\gamma(\Omega). \tag{9}$$

Hence the following theorem holds.

Theorem 2.1 *Let E be a separable Banach space and $F : [a,b]\times E \multimap E$ be an u-Carathéodory map. If F is globally measurable or quasi-compact, then F has the Scorza-Dragoni property.*

We prove now the existence of a function with the necessary properties needed in order to construct a sequence of problems which approximate (1).

Theorem 2.2 *Let E be a Banach space and $K \subset E$ be nonempty, open, bounded, convex and such that $0 \in K$. Assume that $V : E \to \mathbb{R}$ is Fréchet differentiable with V' Lipschitzian in $\overline{B^{\varepsilon}_{\partial K}}$, for some $\varepsilon > 0$, and*

$(V1)$ $V\lfloor_{\partial K} \equiv 0$*;*

$(V2)$ $V\lfloor_{\overline{K}} \le 0$*;*

$(V3)$ $\|V'_x\| \ge \delta$ *for all $x \in \partial K$, where $\delta > 0$ is given.*

Then there exists a bounded Lipschitzian function $\phi : \overline{B^{\varepsilon}_{\partial K}} \to E$ such that $V'_x(\phi(x)) = 1$ for every $x \in \overline{B^{\varepsilon}_{\partial K}}$.

Proof. The proof splits into three steps.
STEP 1. $V'_x(y-x) < 0$ *for every $x \in \partial K, y \in K$.* Given $x \in \partial K$, let us suppose that there exists $y_0 \in K$ such that $V'_x(y_0 - x) \ge 0$. Then one of the following three conditions holds:

1) $\exists y \in K : V'_x(y-x) > 0$.

2) $V'_x(y-x) = 0$ for all $y \in K$.

3) $V'_x(y-x) \le 0$ for all $y \in K$ and there exist $y_1, y_2 \in K : V'_x(y_1 - x) = 0$ and $V'_x(y_2 - x) < 0$.

Assume 1). The convexity of K and the linearity of V'_x imply that $z_\lambda = (1-\lambda)x + \lambda y \in \overline{K}$ and $V'_x(z_\lambda - x) = \lambda V'_x(y-x) \geq 0$ for every $\lambda \in [0,1]$. According to Taylor's formula and (V1) we then have, for every $\lambda \in (0,1]$,

$$\begin{aligned}\frac{V(z_\lambda)}{\|z_\lambda - x\|} &= \frac{V(z_\lambda) - V(x)}{\|z_\lambda - x\|} = \frac{V'_x(z_\lambda - x) + o(\|z_\lambda - x\|)}{\|z_\lambda - x\|} \\ &= \frac{V'_x(y-x)}{\|y-x\|} + \frac{o(\|z_\lambda - x\|)}{\|z_\lambda - x\|}.\end{aligned}$$

Therefore

$$\lim_{\lambda \to 0^+} \frac{V(z_\lambda)}{\|z_\lambda - x\|} = \frac{V'_x(y-x)}{\|y-x\|} > 0,$$

in contradiction with (V2).
Assume 2). Then $V'_x(y) \equiv V'_x(x)$ in K. Since K is open and V'_x is linear, it follows that $V'_x \equiv 0$, in contradiction with (V3).
Finally assume 3). For $\lambda \in \mathbb{R}$ put $w_\lambda = (1-\lambda)y_1 + \lambda y_2$. Since K is open and $y_1 \in K$, there is $r > 0$ such that $y_1 + rB \subset K$. Since $w_0 = y_1$, there exists $\overline{\lambda} > 0$ such that $\|w_\lambda - y_1\| \leq r$, i.e. $w_\lambda \in K$, for $|\lambda| \leq \overline{\lambda}$. Take now $\lambda \in (-\overline{\lambda}, 0)$. Since $V'_x(y_1 - x) = 0$ and $V'_x(y_2 - x) < 0$, according to the linearity of V'_x we have that $V'_x(w_\lambda - x) = \lambda V'_x(y_2 - x) > 0$, in contradiction with 3).
STEP 2. *The function $x \to V'_x(x)$ is strictly positive and Lipschitzian in $\overline{B^\varepsilon_{\partial K}}$*. Since $0 \in K$ then, for every $x \in \partial K$, $V'_x(x) = -V'_x(0-x) > 0$. Moreover, since K is open, there exists $r \in (0, \frac{\inf_{x\in\partial K}\|x\|}{2})$ such that $rB \subset K$. Let $\rho \in (0, \frac{1}{2}\delta r)$. Given $x \in \partial K$, since $\|V'_x\| \geq \delta > \frac{2\rho}{r}$, there exists w such that $\|w\| = 1$ and $V'_x(w) > \frac{\rho}{r}$. Consider $z = rw$. Then $\|z\| = r$ and $V'_x(z) > \rho$. Hence $\lambda_0 = \frac{V'_x(x)}{V'_x(z)} > 0$. Since $V'_x(\lambda_0 z - x) = 0$ and $V'_x(y-x) < 0$ for every $y \in K$, we have that $\lambda_0 z \notin K$. Hence $\|\lambda_0 z\| \geq \inf_{x\in\partial K}\|x\|$, i.e. $\lambda_0 \geq \frac{\inf_{x\in\partial K}\|x\|}{r} > 2$. Therefore, for every $x \in \partial K$,

$$V'_x(x) = \lambda_0 V'_x(z) > 2\rho. \tag{10}$$

Denoted by L the Lipschitz constant of V' in $\overline{B^\varepsilon_{\partial K}}$ and fixed $y_0 \in \partial K$, for every $x, y \in \overline{B^\varepsilon_{\partial K}}$ it holds

$$\begin{aligned}|V'_x(x) - V'_y(y)| &\leq \|V'_x - V'_y\|\|x\| + \|V'_y\|\|x-y\| \\ &\leq L\|\overline{B^\varepsilon_{\partial K}}\|\|x-y\| + (\|V'_y - V'_{y_0}\| + \|V'_{y_0}\|)\|x-y\| \\ &\leq [L(\|\partial K\| + \varepsilon) + L\|y - y_0\| + \|V'_{y_0}\|]\|x-y\| \\ &\leq [2L(\|\partial K\| + \varepsilon) + \|V'_{y_0}\|]\|x-y\| \\ &:= \overline{L}\|x-y\|.\end{aligned}$$

Hence the map $x \to V'_x(x)$ is Lipschitizian in $\overline{B^\varepsilon_{\partial K}}$ of Lipschitz constant $\overline{L} = 2L(\|\partial K\| + \varepsilon) + \|V'_{y_0}\|$ and $\overline{L} = \overline{L}(\epsilon)$ is increasing in ε. According to

(10), we then have $V_x'(x) \geq 2\rho - \varepsilon\overline{L}$ for every $x \in \overline{B_{\partial K}^{\varepsilon}}$. We can then take ε sufficiently small to have $V_x'(x) \geq \rho$ for every $x \in \overline{B_{\partial K}^{\varepsilon}}$.
STEP 3. *Definition and properties of ϕ.* Let us define now $\phi : \overline{B_{\partial K}^{\varepsilon}} \to E$ as $\phi(x) = \frac{x}{V_x'(x)}$. Then $\|\phi(x)\| \leq \frac{\|\partial K\|+\varepsilon}{\delta r}$ and $V_x'(\phi(x)) = 1$ for every x. Moreover, fixed $y_0 \in \partial K$, for every $x, y \in \overline{B_{\partial K}^{\varepsilon}}$

$$\begin{aligned}
\|\phi(x) - \phi(y)\| &= \|\tfrac{x}{V_x'(x)} - \tfrac{y}{V_y'(y)}\| = \tfrac{1}{V_x'(x)V_y'(y)}\|xV_y'(y) - yV_x'(x)\| \\
&\leq \tfrac{1}{\rho^2}\left(|V_x'(x) - V_y'(y)|\|x\| + |V_x'(x)|\|x-y\|\right) \\
&\leq \tfrac{1}{\rho^2}\left(\overline{L}\|\overline{B_{\partial K}^{\varepsilon}}\| + |V_x'(x) - V_{y_0}'(y_0)| + |V_{y_0}'(y_0)|\right)\|x-y\| \\
&\leq \tfrac{1}{\rho^2}\left(2\overline{L}\|\overline{B_{\partial K}^{\varepsilon}}\| + |V_{y_0}'(y_0)|\right)\|x-y\|,
\end{aligned}$$

which implies that ϕ is Lipschitzian.

Remark 2.1 Notice that the function $x \to \phi(x)\|V_x'\|$ is Lipschitizian and bounded in $\overline{B_{\partial K}^{\varepsilon}}$.

The following continuation principle was proved in [2, Theorem 3.1] in the case when the r.h.s. is sublinear in x. It is not difficult to see that the same result is true under the more general condition (F2).

Theorem 2.3 *Consider an u-Carathéodory map $P : [a,b] \times E \multimap E$ satisfying (F2) (with P instead of F) and a subset S of absolutely continuous functions $x : [a,b] \to E$. Let $H : [a,b] \times E \times E \times [0,1] \multimap E$ be an u-Carathéodory map. Assume that, for every bounded $\Omega \subset E$, there exists $\nu_\Omega \in L^1([a,b], \mathbb{R})$ such that $\|w\| \leq \nu_\Omega(t)$, for a.a. $t \in [a,b]$, every $x, y \in \Omega$, $\lambda \in [0,1]$ and $w \in H(t,x,y,\lambda)$ and let*

$$H(t,c,c,1) \subset P(t,c), \quad \textit{for all } (t,c) \in [a,b] \times E. \tag{11}$$

Furthermore, assume that

(a) *There exists a closed and convex subset $Q \subseteq C([a,b],E)$, with $\mathring{Q} \neq \emptyset$, and a closed subset S_1 of S such that the problem*

$$\begin{cases} x'(t) \in H(t, x(t), q(t), \lambda), & \textit{for a.a. } t \in [a,b], \\ x \in S_1 \end{cases}$$

is solvable with a convex set $T(q,\lambda)$ of solutions, for each $(q,\lambda) \in Q \times [0,1]$;

(b) *T is quasi-compact and β-condensing with respect to a monotone and non-singular m.n.c. β defined on $C([a,b],E)$;*

(c) $T(Q \times \{0\}) \subset Q$;

(d) *The map* $T(\cdot, \lambda)$ *has no fixed points on the boundary* ∂Q *of* Q *for every* $\lambda \in [0, 1)$.

Then the b.v.p.

$$\begin{cases} x' \in P(t, x), & \text{for a.a. } t \in [a, b], \\ x \in S, \end{cases}$$

has a solution in Q.

3 Existence Result

In this section we show the solvability of the b.v.p. (1). Our proof involves a sequence of approximating problems that we obtain combining the Scorza-Dragoni type result in Theorem 2.1 with the result in Theorem 2.2. The approximating problems are treated by means of the continuation principle in the form of Theorem 2.3. A standard limit argument is then applied to complete the proof. Strict transversality conditions are assumed.

Theorem 3.1 *Consider the b.v.p.* (1) *under assumptions (A), (F1), (F2) and (M) and suppose that* F *has the Scorza-Dragoni property. Let us assume the following hypotheses:*

(i) $(M - U(b, a))$ *is invertible;*

(ii) *there exists* $g \in L^1([a, b], \mathbb{R})$ *such that* $\gamma(F(t, \Omega)) \leq g(t)\gamma(\Omega)$ *for any bounded* $\Omega \subset E$ *and a.a.* $t \in [a, b]$ *and*

$$\|g\|_1 \left(e^{\int_a^b \|A(t)\|\, dt} \|[M - U(b, a)]^{-1}\| + 1 \right) e^{\int_a^b \|A(t)\|\, dt} < 1; \qquad (12)$$

(iii) *there exist a nonempty, open, bounded, convex set* $K \subset E$*, such that* $0 \in K$ *and* $M\partial K \subset \partial K$*, positive constants* δ, ε *and a Fréchet differentiable function* $V : E \to \mathbb{R}$ *with* V' *Lipschitzian in* $\overline{B^{\varepsilon}_{\partial K}}$*, satisfying (V1)-(V2)-(V3) as well as*

   $(V4)$ $V'_x(A(t)x + \lambda w) \leq 0$ *for a.a.* $t \in (a, b]$ *and for every* $x \in \partial K, \lambda \in (0, 1)$ *and* $w \in F(t, x)$.

Then (1) *has at least a solution* x *with* $x(t) \in \overline{K}$ *for all* $t \in [a, b]$.

Proof The proof splits into three steps
STEP 1. *Introduction of a sequence of approximating problems.* According to Urisohn lemma, there exists a continuous function $\mu : E \to [0,1]$ such that $\mu \equiv 0$ on $E \setminus B^{\varepsilon}_{\partial K}$ and $\mu \equiv 1$ on $\overline{B^{\varepsilon/2}_{\partial K}}$. Theorem 2.2 then implies that $\hat{\phi} : E \to \mathbb{R}$ defined by

$$\hat{\phi}(x) = \begin{cases} \mu(x)\phi(x)\|V'_x\| & x \in \overline{B^{\varepsilon}_{\partial K}} \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

is well-defined, continuous and bounded on all E. Since $(t,x) \longmapsto A(t)x$ is a Carathéodory map on $[a,b] \times E$, it is also almost continuous. Hence the multimap $(t,x) \multimap A(t)x + F(t,x)$ has the Scorza-Dragoni property (cfr. e.g. Theorem 2.1). We can then find a decreasing sequence $\{J_m\}_m$ of sets and a multimap $F_0 : [a,b] \times E \multimap E$ such that, for each $m \in \mathbb{N}$,

- $J_m \subset [a,b]$ and $\tau(J_m) < \frac{1}{m}$;
- $[a,b] \setminus J_m$ is closed;
- $(t,x) \multimap A(t)x + F_0(t,x) - \frac{p(t)\|V'_x\|\phi(x)}{m}$ is u.s.c. on $[a,b] \setminus J_m \times E$.

Put $J = \cap_{m=1}^{\infty} J_m$. We remark that $\tau(J) = 0, F_0(t,x) \neq \emptyset$ whenever $t \notin J$ and the multimap $(t,x) \multimap A(t)x + F_0(t,x)$ is u.s.c. on $[a,b] \setminus J \times E$. Let

$$p(t) := \|A(t)\| \left(\|\partial K\| + \frac{\varepsilon}{2} \right) + \nu_{B^{\varepsilon/2}_{\partial K}}(t) + 1, \tag{14}$$

with $\nu_{B^{\varepsilon/2}_{\partial K}} \in L^1([a,b],\mathbb{R})$ obtained by condition (F2). For each $m \in \mathbb{N}$, we define the nonempty, compact, convex valued multimap

$$F_m(t,x) = \begin{cases} F_0(t,x) - p(t)\left[\chi_{J_m}(t) + \frac{1}{m}\right]\hat{\phi}(x) & (t,x) \in [a,b] \setminus J \times E \\ -p(t)\left[\chi_{J_m}(t) + \frac{1}{m}\right]\hat{\phi}(x) & (t,x) \in J \times E \end{cases}$$

and introduce the b.v.p.

$$\begin{cases} x'(t) \in A(t)x(t) + F_m(t,x(t)), & \text{for a.a. } t \in [a,b] \\ \\ x(b) = Mx(a). \end{cases} \tag{P_m}$$

STEP 2. *Solvability of problems* (P_m). Fix $m \in \mathbb{N}$. Since F_0 is globally u.s.c. in $([a,b] \setminus J) \times E$, hence $F_m(\cdot,x)$ is measurable, for each $x \in E$, and according to the continuity of $\hat{\phi}$, $F_m(t,\cdot)$ is u.s.c. for all $t \in [a,b] \setminus J$. Consequently F_m satisfies (F1). Take $\Omega \subset E$ bounded. According to (F2), there exists

$\hat{J} \subset [a,b]$, with $\tau(\hat{J}) = 0$, such that when $t \in [a,b] \setminus (J \cup \hat{J})$ and $y \in F_m(t,\Omega)$, since $y = y_0 - p(t)\left[\chi_{J_m}(t) + \frac{1}{m}\right]\hat{\phi}(x)$ for some $y_0 \in F_0(t,x)$, we have that

$$\|y\| \leq \nu_\Omega(t) + 2p(t) \max_{x \in \overline{B^\varepsilon_{\partial K}}} \|\hat{\phi}(x)\|.$$

Hence F_m satisfies condition (F2). Now we prove that, whenever m is sufficiently large, all the assumptions, from *(a)* to *(d)*, of Theorem 2.3 are satisfied.

Property (a). Introduce the nonempty, compact, convex valued multimap

$$G_m(t,y,\lambda) = \begin{cases} \lambda F_0(t,y) - p(t)\left[\chi_{J_m}(t) + \frac{1}{m}\right]\hat{\phi}(y), & (t,y,\lambda) \in ([a,b] \setminus J) \times E \times [0,1] \\ -p(t)\left[\chi_{J_m}(t) + \frac{1}{m}\right]\hat{\phi}(y), & (t,y,\lambda) \in J \times E \times [0,1] \end{cases}$$

which is clearly u-Caratheodory and trivially $A(t)x + G_m(t,y,\lambda)$ satisfies (11). Consider the closed set $Q = C([a,b], \overline{K})$. Since K is convex and open, with $0 \in K$, we have that also Q is convex and it has a nonempty interior. Define the multivalued map $T_m(q,\lambda)$ which associates to each $(q,\lambda) \in Q \times [0,1]$ the set of all solutions of the problem

$$\begin{cases} x'(t) \in A(t)x(t) + G_m(t,q(t),\lambda), & \text{for a.a. } t \in [a,b] \\ x(b) = Mx(a). \end{cases} \tag{15}$$

Since (15) is a linear problem, then T_m is a well-defined, convex valued multimap on $Q \times [0,1]$, so (a) is satisfied.

Property (b). Given $\{q_n\}_n \subset Q$ and $\{\lambda_n\}_n \subset [0,1]$, let $\{x_n\}_n$ be such that $x_n \in T_m(q_n,\lambda_n)$ for all n. According to condition (i) and (5), there exists $\{k_n\}_n \subset L^1([a,b],E)$, with $k_n(t) \in F_0(t,q_n(t))$ for a.a. $t \in [a,b]$ and every n, such that

$$x_n(t) = U(t,a)\,(M - U(b,a))^{-1} \int_a^b U(b,s) f_n(s)\,ds + \int_a^t U(t,s) f_n(s)\,ds \tag{16}$$

where where $f_n(t) = \lambda_n k_n(t) - p(t)\left[\chi_{J_m}(t) + \frac{1}{m}\right]\hat{\phi}(q_n(t))$. Put

$$\tilde{D} := \left(e^{\int_a^b \|A(t)\|\,dt}\|[M - U(b,a)]^{-1}\| + 1\right) e^{\int_a^b \|A(t)\|\,dt},$$

Condition (F2) implies that,

$$\|x_n(t)\| \leq \tilde{D}\left[\|\nu_{\overline{K}}\|_1 + 2\|p\|_1 \max_{x \in \overline{B^\varepsilon_{\partial K}}} \|\hat{\phi}(x)\|\right], \quad \text{for all } t \in [a,b], n \in \mathbb{N},$$

implying that $\{x_n\}_n$ is equibounded. For each $t \in [a,b]$, the properties of the Hausdorff m.n.c. yield

$$\begin{aligned}\gamma\left(\{f_n(t)\}_n\right) \le\ & \gamma\left(\{\lambda_n k_n(t)\}_n\right) + p(t)\left(\chi_{J_m}(t) + \tfrac{1}{m}\right)\gamma\left(\{\hat{\phi}(q_n(t))\}_n\right) \le \\ & \gamma\left(\cup_{\lambda\in[0,1]}\{\lambda k_n(t)\}_n\right) + \\ & p(t)\left(\chi_{J_m}(t) + \tfrac{1}{m}\right)\gamma\left(\{\phi(q_n(t))\|V'_{q_n(t)}\| \,:\, q_n(t) \in \overline{B^{\varepsilon}_{\partial K}}\}\right) = \\ & \gamma\left(\{k_n(t)\}_n\right) + p(t)\left(\chi_{J_m}(t) + \tfrac{1}{m}\right)\gamma\left(\{\phi(q_n(t))\|V'_{q_n(t)}\| \,:\, q_n(t) \in \overline{B^{\varepsilon}_{\partial K}}\}\right).\end{aligned}$$

Therefore, according to condition (ii),

$$\begin{aligned}\gamma\left(\{f_n(t)\}_n\right) \ & \le g(t)\gamma\left(\{q_n(t)\}_n\right) \\ & + p(t)\left(\chi_{J_m}(t) + \tfrac{1}{m}\right)\gamma\left(\{\phi(q_n(t))\|V'_{q_n(t)}\| \,:\, q_n(t) \in \overline{B^{\varepsilon}_{\partial K}}\}\right)\end{aligned}$$

for a.a. $t \in [a,b]$. Since the function $x \longmapsto \phi(x)\|V'_x\|$ is Lipschitzian on $\overline{B^{\varepsilon}_{\partial K}}$, of some Lipschitz constant $\hat{L} > 0$ (see Remark 2.1), (6) finally implies that

$$\begin{aligned}\gamma\left(\{f_n(t)\}_n\right) \le\ & \left(g(t) + \hat{L}p(t)(\chi_{J_m}(t) + \tfrac{1}{m})\right)\gamma\left(\{q_n(t)\}_n\right) \\ \le\ & \left(g(t) + \hat{L}p(t)(\chi_{J_m}(t) + \tfrac{1}{m})\right)\sup_{t\in[a,b]}\gamma\left(\{q_n(t)\}_n\right)\end{aligned} \tag{17}$$

for a.a. $t \in [a,b]$. According to (F2), (4) and (7) we have that

$$\gamma\left(\{x_n(t)\}_n\right) \le \tilde{D}\sup_{t\in[a,b]}\gamma\left(\{q_n(t)\}_n\right)\int_a^b [g(s) + \hat{L}(\chi_{J_m}(s) + \frac{1}{m})p(s)]\,ds \tag{18}$$

If we assume in addition that $q_n \to q$ in $C([a,b],\overline{K})$ and $\lambda_n \to \lambda$ as $n \to \infty$, we obtain that $\gamma\left(\{q_n(t)\}_n\right) \equiv 0$ and (18) implies that $\gamma\left(\{x_n(t)\}_n\right) \equiv 0$. Hence $\{x_n(t)\}_n$ is relatively compact for each $t \in [a,b]$. Moreover, since $\{x_n\}_n$ is an equibounded set of solutions of (15), it is not difficult to show that $\{x'_n\}_n$ is equibounded in $L^1([a,b],E)$. Consequently, according to a classical convergence result (see e.g. [1, Lemma 1.30]), there exist $x \in C([a,b],E)$ with $x'(t)$ defined for a.a. t and a subsequence, denoted again as the sequence, such that $x_n \to x$ in $C([a,b],E)$ and $x'_n \rightharpoonup x'$ weakly in $L^1([a,b],E)$ as $n \to \infty$. A classical closure theorem (see e.g. [8, Lemma 5.1.1]) then implies that $x \in T_m(q,\lambda)$ hence T_m is quasi-compact.

Now we show that T_m is also β-condensing with respect to the monotone and non-singular m.n.c.

$$\beta(\Omega) := \max_{\{q_n\}_n\subset\Omega}\left(\sup_{t\in[a,b]}\gamma(\{q_n(t)\}_n), \mathrm{mod}_C(\{q_n\}_n)\right),$$

where the ordering is induced by the positive cone in $\mathbb{R}^2$ (see [8, Example 2.1.4]). Indeed, let $\Omega \subseteq Q$ be such that $\beta\left(T_m(\Omega \times [0,1])\right) \geq \beta(\Omega)$ and take $x_n \in T_m(q_n, \lambda_n)$ satisfying

$$\beta\left(\{x_n\}_n\right) = \beta\left(T_m(\Omega \times [0,1])\right) \geq \beta(\Omega) \geq \beta\left(\{q_n\}_n\right).$$

According to (18), we obtain that

$$\begin{aligned}
\sup_{t\in[a,b]} \gamma\left(\{q_n(t)\}_n\right) &\leq \sup_{t\in[a,b]} \gamma\left(\{x_n(t)\}_n\right) \\
&\leq \tilde{D}\left(\|g\|_1 + (\|p\|_{L^1(J_m)} + \tfrac{1}{m}\|p\|_1)\hat{L}\right) \sup_{t\in[a,b]} \gamma\left(\{q_n(t)\}_n\right).
\end{aligned}$$

Condition (12) and the definition of $\tilde{D}$ then implies the contradictory conclusion

$$\sup_{t\in[a,b]} \gamma\left(\{q_n(t)\}_n\right) < \sup_{t\in[a,b]} \gamma\left(\{q_n(t)\}_n\right),$$

whenever m is sufficiently large. Hence T_m is β-condensing.

Property (c). The set $T_m(q,0)$, for each $q \in Q$, coincides with the unique solution x_m of the linear system

$$\begin{cases} x'(t) = A(t)x(t) - p(t)(\chi_{J_m}(t) + \frac{1}{m})\hat{\phi}(t) & t \in [a,b] \\ x(b) = Mx(a) \end{cases} \tag{19}$$

Condition (i) and (5) then implies that, for all $t \in [a,b]$,

$$x_m(t) = U(t,a)\left(M - U(b,a)\right)^{-1} \int_a^b U(b,s)\varphi_m(s)\,ds + \int_a^t U(t,s)\varphi_m(s)\,ds$$

with $\varphi_m(t) = -p(t)(\chi_{J_m}(t) + \frac{1}{m})\hat{\phi}(t)$. We also have that

$$\|\varphi_m\|_1 \leq \max_{x\in\overline{B^\varepsilon_{\partial K}}} \|\hat{\phi}(x)\| \left(\|p\|_{L^1(J_m)} + \frac{\|p\|_1}{m}\right).$$

According to condition (4), it implies that

$$\|x_m(t)\| \leq \tilde{D} \max_{x\in\overline{B^\varepsilon_{\partial K}}} \|\hat{\phi}(x)\| \left(\|p\|_{L^1(J_m)} + \frac{\|p\|_1}{m}\right)$$

for all $t \in [a,b]$. Let $r > 0$ be such that $rB \subset K$; if we assume a sufficiently large m, we have that $\|x_m(t)\| \leq r$ for all $t \in [a,b]$, implying that $T_m(Q \times \{0\}) \subset \mathring{Q}$. Hence condition (c) is satisfied.

Property (d). Since we already showed that $T_m(\cdot,0)$ has no fixed points on ∂Q, it remains to prove this property for $T_m(\cdot,\lambda)$ with $\lambda \in (0,1)$. We reason by a contradiction and assume the existence of $\lambda \in (0,1)$, $q \in \partial Q$ and $t_0 \in [a,b]$ such that $q \in T_m(q,\lambda)$ and $q(t_0) \in \partial K$. Since, when $q(a) \in \partial K$, it follows that $q(b) = Mq(a) \in M\partial K \subset \partial K$, we can assume, with no loss of generality, that $t_0 \in (a,b]$. Hence there is $h > 0$ such that $q(t) \in B^{\varepsilon/2}_{\partial K}$ for all $t \in [t_0 - h, t_0]$. Moreover, according to the continuity of $t \longrightarrow \|V'_{q(t)}\|$ in $[a,b]$ and (V3), with no loss of generality, we can assume that $\|V'_{q(t)}\| \geq \delta/2$ in $[t_0 - h, t_0]$. Since J_m is open in $[a,b]$, if in addition $t_0 \in J_m$, we can take h in such a way that $[t_0 - h, t_0] \subseteq J_m$. Since $\tau(J) = 0$, with no loss of generality, we can assume the existence of $g_0 \in L^1([a,b],E)$ with $g_0(t) \in F_0(t,q(t))$ for a.a. $t \in [a,b]$ such that $q'(t) = A(t)q(t) + \lambda g_0(t) - p(t)(\chi_{J_m}(t) + \frac{1}{m})\hat{\phi}(q(t))$ for a.a. $t \in [a,b]$. Consequently, conditions (V1)-(V2) imply that

$$\begin{aligned}
0 \ &\leq -V(q(t_0 - h)) = \textstyle\int_{t_0-h}^{t_0} V'_{q(t)}(q'(t))\, dt \\
&= \textstyle\int_{t_0-h}^{t_0} V'_{q(t)} \left(A(t)q(t) + \lambda g_0(t) - p(t)(\chi_{J_m}(t) + \frac{1}{m})\hat{\phi}(q(t)) \right) dt \\
&= \textstyle\int_{[t_0-h,t_0]\cap J_m} \left[V'_{q(t)} \left(A(t)q(t) + \lambda g_0(t) \right) - p(t)(1 + \frac{1}{m})\|V'_{q(t)}\| \right] dt \\
&+ \textstyle\int_{[t_0-h,t_0]\setminus J_m} \left[V'_{q(t)} \left(A(t)q(t) + \lambda g_0(t) - \frac{p(t)\|V'_{q(t)}\|\phi(q(t))}{m} \right) \right] dt \\
&\leq \textstyle\int_{[t_0-h,t_0]\cap J_m} \|V'_{q(t)}\| \left(\|A(t)\|(\|\partial K\| + \frac{\varepsilon}{2}) + \nu_{B^{\varepsilon/2}_{\partial K}}(t) - p(t) \right) dt \\
&+ \textstyle\int_{[t_0-h,t_0]\setminus J_m} \left[V'_{q(t)} \left(A(t)q(t) + \lambda g_0(t) - \frac{p(t)\|V'_{q(t)}\|\phi(q(t))}{m} \right) \right] dt.
\end{aligned}$$

Therefore, denoted

$$\Lambda_m = \int_{[t_0-h,t_0]\cap J_m} \|V'_{q(t)}\| \left(\|A(t)\|(\|\partial K\| + \frac{\varepsilon}{2}) + \nu_{B^{\varepsilon/2}_{\partial K}}(t) - p(t) \right) dt,$$

and

$$\Gamma_m = \int_{[t_0-h,t_0]\setminus J_m} \left[V'_{q(t)} \left(A(t)q(t) + \lambda g_0(t) - \frac{p(t)\|V'_{q(t)}\|\phi(q(t))}{m} \right) \right] dt$$

it holds $\Gamma_m + \Lambda_m \geq 0$. Since $t_0 \in J_m$ implies $[t_0 - h, t_0] \setminus J_m = \emptyset$ and, according to (14), $\Lambda_m < 0$, it is clear that $t_0 \notin J_m$. The assumption $q(t_0) \in \partial K$, condition (V4) and the positivity of $p(t)$ on all $[a,b]$ then imply that

$$\begin{aligned}
V'_{q(t_0)} \left(A(t_0)q(t_0) + \lambda w_0 - \tfrac{p(t_0)\|V'_{q(t_0)}\|\phi(q(t_0))}{m} \right) \ &\leq \\
&\leq -\tfrac{p(t_0)\|V'_{q(t_0)}\|}{2m} \leq -\tfrac{\delta p(t_0)}{2m} < 0
\end{aligned}$$

for all $w_0 \in F(t_0, q(t_0))$ and since F is compact valued and the operator $V'_{q(t_0)} : E \to \mathbb{R}$ is continuous, we can find $\sigma > 0$ satisfying

$$V'_{q(t_0)} \left(A(t_0)q(t_0) + \lambda w_0 - \frac{p(t_0)\|V'_{q(t_0)}\|\phi(q(t_0))}{m} \right) \leq -2\sigma,$$

for all $w_0 \in F(t_0, q(t_0))$. In $[a,b] \backslash J_m$ the multimap $t \multimap A(t)q(t)+\lambda F_0(t,q(t)) - \frac{p(t)\|V'_{q(t)}\|\phi(q(t))}{m}$ is u.s.c.; therefore

$$\begin{aligned} \Phi\ : [a,b] \setminus J_m &\multimap \mathbb{R}, \\ t &\multimap \{V'_{q(t)}\left(A(t)q(t) + \lambda w - \tfrac{p(t)\|V'_{q(t)}\|\phi(q(t))}{m}\right)\ :\ w \in F_0(t,q(t))\} \end{aligned}$$

is u.s.c. When h is sufficiently small, we have then $\Phi(t) \subset (-\infty, -\sigma]$ implying $V'_{q(t)}\left(A(t)q(t) + \lambda g_0(t) - \frac{p(t)\|V'_{q(t)}\|\phi(q(t))}{m}\right) < 0$ on all $[t_0-h, t_0] \setminus J_m$. Recalling (14) we then obtain $0 \le \Gamma_m + \Lambda_m < 0$, a contradiction.
Since every problem (P_m), with m sufficiently large, satisfies all the assumptions of Theorem 2.3, it has a solution x_m such that $x_m(t) \in \overline{K}$ for all $t \in [a,b]$.
STEP 3. *Conclusions.* There exists $\{f_m\}_m \subset L^1([a,b],E)$ with $f_m(t) \in F_0(t, x_m(t))$ for a.a. $t \in [a,b]$ such that, when putting $h_m(t) := f_m(t) - p(t)(\chi_{J_m}(t) + \frac{1}{m})\hat{\phi}(x_m(t))$, we obtain

$$x'_m(t) = A(t)x_m(t) + h_m(t), \qquad \text{for a.a. } t \in [a,b] \tag{20}$$

and

$$\begin{aligned} x_m(t) &= U(t,a)\left(M - U(b,a)\right)^{-1} \textstyle\int_a^b U(b,s)h_m(s)\,ds \\ &+ \textstyle\int_a^t U(t,s)h_m(s)\,ds. \end{aligned} \tag{21}$$

If $t \notin J$ there is m_0, depending on t, satisfying $t \notin J_m$ for all $m > m_0$ and according to (ii) we have that

$$\begin{aligned} \gamma\left(\{h_m(t)\}_m\right) &\le g(t)\gamma\left(\{x_m(t)\}_m\right) \\ &+\gamma\left(\{p(t)(\chi_{J_m}(t)+\tfrac{1}{m})\hat{\phi}(x_m(t))\ :\ m=1,2,...,m_0\} \cup \{0\}\right) \\ &\le g(t) \sup_{t\in[a,b]} \gamma\left(\{x_m(t)\}_m\right), \end{aligned} \tag{22}$$

with $\sup_{t\in[a,b]} \gamma(\{x_m(t)\}_m) < +\infty$ for the boundedness of K. The sequence $\{h_m\}_m$ is integrably bounded; indeed $x_m(t) \in \overline{K}$ for all $t \in [a,b]$ and $m \in \mathbb{N}$ and this yields

$$\|h_m(t)\| \le \nu_{\overline{K}}(t) + 2p(t) \sup_{x\in\overline{B^{\varepsilon}_{\partial K}}} \|\hat{\phi}(x)\|, \quad \text{for a.a. } t \in [a,b].$$

Consequently, since $\tau(J) = 0$, from (7) and (21) we have that

$$\gamma\left(\{x_m(t)\}_m\right) \le \tilde{D} \sup_{t\in[a,b]} \gamma\left(\{x_m(t)\}_m\right)\|g\|_1.$$

According to (12) and the definition of $\tilde{D}$, we then obtain that $\gamma(\{x_m(t)\}_m) = 0$, implying the relative compactness of $\{x_m(t)\}_m$ and from (22) also of $\{h_m(t)\}$ for a.a. $t \in [a,b]$. Moreover it follows that condition (20) implies

$$\|x'_m(t)\| \leq \|A(t)\|\|\overline{K}\| + \nu_{\overline{K}}(t) + 2p(t) \sup_{x \in \overline{B^{\varepsilon}_{\partial K}}} \|\hat{\phi}(x)\|$$

and $\{x'_m(t)\}_m$ is relatively compact for a.a. $t \in [a,b]$ and all $m \in \mathbb{N}$. According to a classical compactness result (see e.g. [1, Lemma 1.30]) there is $x \in C([a,b],E)$ with x' defined for a.a. t and a subsequence of $\{x_m\}_m$, again denoted as the sequence, such that $x_m \to x$ in $C([a,b],E)$ and $x'_m \rightharpoonup x'$ weakly in $L^1([a,b],E)$. Since $p(t)(\chi_{J_m}(t) + \frac{1}{m})\hat{\phi}(x_m(t)) \to 0$, as $m \to \infty$, for a.a. t, we have that also $x'_m + p(t)(\chi_{J_m}(t) + \frac{1}{m})\hat{\phi}(x_m) \rightharpoonup x'$ weakly in $L^1([a,b],E)$ and since $x'_m(t) + p(t)(\chi_{J_m}(t) + \frac{1}{m})\hat{\phi}(x_m(t)) \in A(t)x_m(t) + F_0(t, x_m(t))$ for a.a. $t \in [a,b]$, we can apply a classical closure theorem (see e.g. [8, Lemma 5.1.1]) to have that x is a solution of (1) with $x(t) \in \overline{K}$ for all $t \in [a,b]$ and the proof is complete.

Remark 3.1 Notice that condition (9) *implies the* $\gamma-$ *regularity of* F *required in assumption (ii) of Theorem 3.1.*

The following example deals with an anti-periodic problem in $\mathbb{R}$. Thanks to its very simple nature we are able to complete all its computations. We show, in particular, the existence of a unique solution in the interval $(-1,1)$. This solution can not be detected either by means of any result in [2] or by [4, Theorem 3.4]. Indeed, the nonlinearity is not globally u.s.c. and the required transversality fails to be satisfied here. Instead, according to the very general transversality condition (V4), the solution can be obtained by means of Theorem 3.1. The example hence motivates our analysis.

Example 3.1 *Consider the antiperiodic value problem*

$$\begin{cases} x' = \alpha(t)\sqrt{|x-1|}, & t \in [0,1] \\ x(1) = -x(0) \end{cases} \tag{23}$$

where $\alpha \in L^1([0,1])$ *satisfies* $\alpha(t) > 0$ *for a.a.* t *and* $\|\alpha\|_1 < 2\sqrt{2}$. *Problem* (23) *can be rewritten as* (1) *with* $E = \mathbb{R}, A \equiv 0, F(t,x) = \alpha(t)\sqrt{|x-1|}$ *and* $M = -I$. *Define* $\gamma(t) = \int_0^t \alpha(s)ds$. *Given* $c \in R$, *we put* $t_c = 1$ *if* $c \leq -\frac{1}{2}\|\alpha\|_1$, t_c *equal to the unique solution of the equation* $\frac{1}{2}\gamma(t) + c = 0$ *if* $-\frac{1}{2}\|\alpha\|_1 < c < 0$ *and* $t_c = 0$ *if* $c \geq 0$. *It is then easy to prove that all strictly increasing solutions of the equation in* (23) *belong to the family of functions* $x_c : \mathbb{R} \to \mathbb{R}$ *defined as*

$x_c(t) = 1-[\frac{1}{2}\gamma(t)+c]^2$ if $t \le t_c$ and $x_c(t) = 1+[\frac{1}{2}\gamma(t)+c]^2$ if $t > t_c$, for some $c \in \mathbb{R}$. Consider the nonempty, open, bounded, convex and symmetric with respect to the origin subset $K = (-1,1)$ of $\mathbb{R}$. Then K is neither positively nor negatively invariant for the equation in (23). *In fact, for $-\frac{1}{2}\|\alpha\|_1 < c < 0, x_c$ satisfies $x_c(0) \in K$ and $x_c(1) \notin K$, while if $-\frac{1}{2}\|\alpha\|_1 - \sqrt{2} < c < -\sqrt{2}, x_c$ satisfies $x_c(0) \notin K$ and $x_c(1) \in K$. It is easy to see that problem* (23) *has a unique solution which is the function $x_{\hat{c}}(t) = 1 - [\frac{1}{2}\gamma(t) + \hat{c}]^2$, with $\hat{c} = -\frac{\|\alpha\|_1+\sqrt{16-\|\alpha\|_1^2}}{4}$, and $x_{\hat{c}}(t) \in K$ for all t. We remark that it is possible to detect $x_{\hat{c}}$ by means of Theorem 3.1. In fact, the evolution operator associated to A is $U \equiv I$, hence condition (i) holds. Moreover F is a Carathéodory single valued map, thus it is almost continuous and satisfies (ii) with $g \equiv 0$. Consider the function $V(x) = |x|^2 - 1$. Trivially V' is Lipschitzian in $\mathbb{R}$ and (V1)-(V2)-(V3) hold for $\delta = 1$. Finally, according to the positivity of α almost everywhere, $V_x'(\lambda F(t,x)) \le 0$ for a.a. t, every $x = \pm 1 \in \partial K$ and $\lambda \in (0,1)$. On the other hand, it is not possible to apply [2, Theorem 5.2] to the same aim, since the transversality required there implies the positive invariance of K, which is not satisfied here. The transversality condition in [4, Theorem 3.4] is strictly localized on ∂K, but it is not satisifed here as well. In order to apply such result, in fact, we would need to show that $V_1'(\lambda F(t,1)) < 0$ for all $\lambda \in (0,1)$ and a.a. $t \in [0,1]$. This is not possible since every C^1- function $V : \mathbb{R} \to \mathbb{R}$ satisfies $V_1'(\lambda F(t,1)) = 0$ for a.a. t and $\lambda \in (0,1)$.*

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