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Some remarks on a fractional differential inclusion with non-separated boundary conditions

Aurelian Cernea
Faculty of Mathematics and Informatics,
University of Bucharest,
Academiei 14, 010014 Bucharest, Romania,
e-mail: acernea@fmi.unibuc.ro

Abstract

We study a boundary value problem for a fractional differential inclusion of order $\alpha \in (1,2]$ with non-separated boundary conditions involving a nonconvex set-valued map. We establish a Filippov type existence theorem and we prove the arcwise connectedness of the solution set of the problem considered.

Key words. differential inclusion, fractional derivative, boundary value problem.

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1 Introduction

In this paper we study the following problem

$$D_c^{\alpha} x(t) \in F(t, x(t)) \quad a.e.\ ([0,T]), \tag{1.1}$$

$$x(0) - k_1 x(T) = c_1, \quad x'(0) - k_2 x'(T) = c_2, \tag{1.2}$$

where $\alpha \in (1,2]$, D_c^{α} is the Caputo fractional derivative of order α, $F : [0,T] \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map and $c_1, c_2, k_1, k_2 \in \mathbf{R}$, $k_1, k_2 \neq 1$.

The present paper is motivated by a recent paper of Ahmad and Ntouyas ([1]) where it is studied problem (1.1)-(1.2) and several existence results for this problem are obtained using nonlinear alternative of Leray Schauder type and some suitable theorems of fixed point theory. For motivation, examples and recent developments on differential inclusions of fractional order (in particular, for problem (1.1)-(1.2)) we refer the reader to [1] and the references therein.

The aim of our paper is twofold. On one hand, we show that Filippov's ideas ([5]) can be suitably adapted in order to obtain the existence of solutions for problem (1.1)-(1.2). We recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem ([5]) consists in proving the existence of a solution starting from a given almost solution. Moreover, the result provides an estimate between the starting almost solution and the solution of the differential inclusion.

On the other hand, following the approach in [10] we prove the arcwise connectedness of the solution set of problem (1.1)-(1.2). The proof is based on a result ([9, 10]) concerning the arcwise connectedness of the fixed point set of a class of set-valued contractions.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel, Section 3 is devoted to the Filippov type existence theorem and in Section 4 we obtain the arcwise connectedness of the solution set.

2 Preliminaries

In what follows we denote by I the interval $[0,T]$, $C(I,\mathbf{R})$ is the Banach space of all continuous functions from I to $\mathbf{R}$ with the norm $||x||_C = \sup_{t\in I}|x(t)|$ and $L^1(I,\mathbf{R})$ is the Banach space of integrable functions $u(.) : I \to \mathbf{R}$ endowed with the norm $||u||_1 = \int_0^T |u(t)|dt$.

Let (X,d) be a metric space. We recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$D(A,B) = \max\{d^*(A,B), d^*(B,A)\}, \quad d^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $d(x,B) = \inf_{y\in B} d(x,y)$.

Definition 2.1. ([7]) a) *The fractional integral of order* $\alpha > 0$ of a

Lebesgue integrable function $f : (0, \infty) \to \mathbf{R}$ is defined by

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds,$$

provided the right-hand side is pointwise defined on $(0, \infty)$ and $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$.

b) *The Caputo fractional derivative of order* $\alpha > 0$ of a function $f : [0, \infty) \to \mathbf{R}$ is defined by

$$D_c^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{-\alpha+n-1} f^{(n)}(s)ds,$$

where $n = [\alpha] + 1$. It is assumed implicitly that f is n times differentiable whose n-th derivative is absolutely continuous.

We recall (e.g., [7]) that if $\alpha > 0$ and $f \in C(I, \mathbf{R})$ or $f \in L^\infty(I, \mathbf{R})$ then $(D_c^\alpha I^\alpha f)(t) \equiv f(t)$.

A function $x \in C(I, \mathbf{R})$ is called a solution of problem (1.1)-(1.2) if there exists a function $f \in L^1(I, \mathbf{R})$ with $f(t) \in F(t, x(t))$, *a.e.* (I) such that $D_c^\alpha x(t) = f(t)$ *a.e.* (I) and conditions (1.2) are satisfied.

Lemma 2.2. *For a given integrable function* $f(.) : [0, T] \to \mathbf{R}$*, the unique solution of the boundary problem*

$$D_c^\alpha x(t) = f(t) \quad a.e.\ ([0,T]), \quad x(0) - k_1 x(T) = c_1, \quad x'(0) - k_2 x'(T) = c_2$$

is given by

$$x(t) = P_c(t) + \int_0^T G(t,s) f(s)ds,$$

where, if $c = (c_1, c_2) \in \mathbf{R}^2$,

$$P_c(t) = \frac{c_2[k_1T + (1-k_1)t]}{(k_1-1)(k_2-1)} - \frac{c_1}{k_1-1}, \quad t \in I$$

and the Green function is given by

$$G(t,s) = \begin{cases} \frac{(k_1-1)(t-s)^{\alpha-1} - k_1(T-s)^{\alpha-1}}{(k_1-1)\Gamma(\alpha)} + \frac{k_2[k_1T+(1-k_1)t](T-s)^{\alpha-2}}{(k_1-1)(k_2-1)\Gamma(\alpha-1)}, & 0 \le s \le t \le T, \\ \frac{-k_1(T-s)^{\alpha-1}}{(k_1-1)\Gamma(\alpha)} + \frac{k_2[k_1T+(1-k_1)t](T-s)^{\alpha-2}}{(k_1-1)(k_2-1)\Gamma(\alpha-1)}, & 0 \le t \le s \le T. \end{cases}$$

For the proof of Lemma 2.2, see [1].

Taking into account the definition of the Green's function, using the fact that $\Gamma(\alpha)=(\alpha-1)\Gamma(\alpha-1)$ and the inequality $|k_1T+(1-k_1)t|\leq(1+|k_1|)T$ $\forall\, t\in I$ we obtain that

$$|G(t,s)|\leq\frac{T^{\alpha-1}}{\Gamma(\alpha)}(1+\frac{|k_1|}{|k_1-1|}+\frac{|k_2|(1+|k_1|)(\alpha-1)}{|(k_1-1)(k_2-1)|})\quad\forall\, t,s\in I.$$

For simplicity we denote $M:=\frac{T^{\alpha-1}}{\Gamma(\alpha)}(1+\frac{|k_1|}{|k_1-1|}+\frac{|k_2|(1+|k_1|)(\alpha-1)}{|(k_1-1)(k_2-1)|})$.

Finally, if $a=(a_1,a_2)\in\mathbf{R}^2$ we put $||a||=|a_1|+|a_2|$.

3 A Filippov type existence result

First we recall a selection result which is a version of the celebrated Kuratowski and Ryll-Nardzewski selection theorem ([8]).

Lemma 3.1. ([3]) *Consider X a separable Banach space, B is the closed unit ball in X, $H:I\to\mathcal{P}(X)$ is a set-valued map with nonempty closed values and $g:I\to X, L:I\to\mathbf{R}_+$ are measurable functions. If*

$$H(t)\cap(g(t)+L(t)B)\neq\emptyset\quad a.e.(I),$$

then the set-valued map $t\to H(t)\cap(g(t)+L(t)B)$ has a measurable selection.

In the sequel we assume the following conditions on F.

Hypothesis 3.2. i) *$F:I\times\mathbf{R}\to\mathcal{P}(\mathbf{R})$ has nonempty closed values and for every $x\in\mathbf{R}$ $F(.,x)$ is measurable.*

ii) *There exists $L\in L^1(I,\mathbf{R})$ such that for almost all $t\in I, F(t,.)$ is $L(t)$-Lipschitz in the sense that*

$$D(F(t,x),F(t,y))\leq L(t)|x-y|\quad\forall\ x,y\in\mathbf{R}.$$

We are now ready to prove the main result of this section.

Theorem 3.3. *Assume that Hypothesis 3.2 is satisfied, assume that $M||L||_1<1$ and let $y\in C(I,\mathbf{R})$ be such that there exists $q(.)\in L^1(I,\mathbf{R})$ with $d(D_c^\alpha y(t),F(t,y(t)))\leq q(t)$ a.e. (I). Denote $\tilde{c}_1=y(0)-k_1y(T)$, $\tilde{c}_2=y'(0)-k_2y'(T)$.*

Then there exists $x(.) \in C(I, \mathbf{R})$ a solution of problem (1.1)-(1.2) satisfying for all $t \in I$

$$|x(t) - y(t)| \leq \frac{1}{1 - M||L||_1}|P_c(t) - P_{\tilde{c}}(t)| + \frac{M}{1 - M||L||_1}||q||_1. \tag{3.1}$$

Proof. The set-valued map $t \to F(t, y(t))$ is measurable with closed values and the hypothesis that $d(D_c^\alpha y(t), F(t, y(t))) \leq q(t)$ a.e. (I) is equivalent to

$$F(t, y(t)) \cap \{D_c^\alpha y(t) + q(t)[-1, 1]\} \neq \emptyset \quad a.e. \ (I).$$

It follows from Lemma 3.1 that there exists a measurable selection $f_1(t) \in F(t, y(t))$ *a.e.* (I) such that

$$|f_1(t) - D_c^\alpha y(t)| \leq q(t) \quad a.e. \ (I) \tag{3.2}$$

Define $x_1(t) = P_c(t) + \int_0^T G(t, s) f_1(s) ds$ and one has

$$\begin{aligned} &|x_1(t) - y(t)| = |P_c(t) - P_{\tilde{c}}(t) + \textstyle\int_0^T G(t, s)(f_1(s) - D_c^\alpha y(s)) ds| \leq \\ &|P_c(t) - P_{\tilde{c}}(t)| + \textstyle\int_0^T |G(t, s)| q(s) ds \leq |P_c(t) - P_{\tilde{c}}(t)| + M||q||_1. \end{aligned}$$

We claim that it is enough to construct the sequences $x_n(.) \in C(I, \mathbf{R})$, $f_n(.) \in L^1(I, \mathbf{R})$, $n \geq 1$ with the following properties

$$x_n(t) = P_c(t) + \int_0^T G(t, s) f_n(s) ds, \quad t \in I, \tag{3.3}$$

$$f_n(t) \in F(t, x_{n-1}(t)) \quad a.e. \ (I), \ n \geq 1, \tag{3.4}$$

$$|f_{n+1}(t) - f_n(t)| \leq L(t)|x_n(t) - x_{n-1}(t)| \quad a.e. \ (I), n \geq 1. \tag{3.5}$$

If this construction is realized then from (3.2)-(3.5) we have for almost all $t \in I$

$$|x_{n+1}(t) - x_n(t)| \leq \int_0^T |G(t, t_1)|.|f_{n+1}(t_1) - f_n(t_1)| dt_1 \leq$$

$$M \int_0^T L(t_1)|x_n(t_1) - x_{n-1}(t_1)| dt_1 \leq M \int_0^T L(t_1) \int_0^T |G(t_1, t_2)|.$$

$$|f_n(t_2) - f_{n-1}(t_2)| dt_2 \leq M^2 \int_0^T L(t_1) \int_0^T L(t_2)|x_{n-1}(t_2) - x_{n-2}(t_2)| dt_2 dt_1$$

$$\leq M^n \int_0^T L(t_1) \int_0^T L(t_2) ... \int_0^T L(t_n)|x_1(t_n) - y(t_n)| dt_n ... dt_1 \leq$$

$$\leq (M||L||_1)^n(|P_c(t) - P_{\tilde{c}}(t)| + M||q||_1).$$

Therefore $\{x_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in the Banach space $C(I,\mathbf{R})$, hence converging uniformly to some $x \in C(I,\mathbf{R})$. Therefore, by (3.5), for almost all $t \in I$, the sequence $\{f_n(t)\}_{n\in\mathbf{N}}$ is Cauchy in $\mathbf{R}$. Let f be the pointwise limit of f_n.

Moreover, one has

$$\begin{aligned}|x_n(t) - y(t)| &\leq |x_1(t) - y(t)| + \textstyle\sum_{i=1}^{n-1} |x_{i+1}(t) - x_i(t)| \leq |P_c(t) - P_{\tilde{c}}(t)| \\ &+M||q||_1 + \textstyle\sum_{i=1}^{n-1}(|P_c(t) - P_{\tilde{c}}(t)| + M||q||_1)(M||L||_1)^i = \frac{|P_c(t)-P_{\tilde{c}}(t)|+M||q||_1}{1-M||L||_1}.\end{aligned} \tag{3.6}$$

On the other hand, from (3.2), (3.5) and (3.6) we obtain for almost all $t \in I$

$$\begin{aligned}&|f_n(t) - D_c^\alpha y(t)| \leq \textstyle\sum_{i=1}^{n-1} |f_{i+1}(t) - f_i(t)| + |f_1(t) - D_c^\alpha y(t)| \\ &\leq L(t)\frac{|P_c(t)-P_{\tilde{c}}(t)|+M||q||_1}{1-M||L||_1} + q(t).\end{aligned}$$

Hence the sequence f_n is integrably bounded and therefore $f \in L^1(I,\mathbf{R})$.

Using Lebesque's dominated convergence theorem and taking the limit in (3.3), (3.4) we deduce that x is a solution of (1.1). Finally, passing to the limit in (3.6) we obtained the desired estimate on x.

It remains to construct the sequences x_n, f_n with the properties in (3.3)-(3.5). The construction will be done by induction.

Since the first step is already realized, assume that for some $N \geq 1$ we already constructed $x_n \in C(I,\mathbf{R})$ and $f_n \in L^1(I,\mathbf{R}), n = 1,2,...N$ satisfying (3.3),(3.5) for $n = 1,2,...N$ and (3.4) for $n = 1,2,...N-1$. The set-valued map $t \to F(t, x_N(t))$ is measurable. Moreover, the map $t \to L(t)|x_N(t) - x_{N-1}(t)|$ is measurable. By the lipschitzianity of $F(t,.)$ we have that for almost all $t \in I$

$$F(t, x_N(t)) \cap \{f_N(t) + L(t)|x_N(t) - x_{N-1}(t)|[-1,1]\} \neq \emptyset.$$

From Lemma 3.1 there exists a measurable selection $f_{N+1}(.)$ of $F(., x_N(.))$ such that

$$|f_{N+1}(t) - f_N(t)| \leq L(t)|x_N(t) - x_{N-1}(t)| \quad a.e.\ (I).$$

We define x_{N+1} as in (3.3) with $n = N+1$. Thus f_{N+1} satisfies (3.4) and (3.5) and the proof is complete.

Remark 3.4. Several remarks are in order.

i) If $k_1 = k_2 = 0$, Theorem 3.3 yields an existence result of Filippov type for the Cauchy problem associated to fractional differential inclusion (1.1)

ii) A less powerful Filippov type existence result for problem (1.1)-(1.2) may be obtained using fixed point techniques. More exactly, by applying the set-valued contraction principle in the space of derivatives of trajectories instead of the space of solutions (as usual, for example [1]) one may obtain (see, for example, [4] for this technique) that for any $\varepsilon > 0$ there exists $x_\varepsilon(.)$ a solution of (1.1)-(1.2) satisfying for all $t \in I$

$$|x_\varepsilon(t) - y(t)| \leq \frac{1}{1 - M||L||_1}|P_c(t) - P_{\tilde{c}}(t)| + \frac{M}{1 - M||L||_1}||q||_1 + \varepsilon. \quad (3.7)$$

Obviously, the estimation in (3.1) is better than the one in (3.7).

iii) If the assumptions of of Theorem 3.3 are satisfied with $y = 0$, $q = L$, then Theorem 3.3 improves Theorem 3.3 in [1], since in addition our result provides an a priori estimate of the solution of the form

$$|x(t)| \leq \frac{1}{1 - M||L||_1}|P_c(t)| + \frac{M}{1 - M||L||_1}||q||_1, \quad \forall t \in I. \quad (3.8)$$

4 Arcwise connectedness of the solution set

In this section we are concerned with the more general problem

$$D_c^\alpha x(t) \in F(t, x(t), H(t, x(t))) \quad a.e. \ ([0, T]), \quad (4.1)$$

$$x(0) - k_1 x(T) = c_1, \quad x'(0) - k_2 x'(T) = c_2, \quad (4.2)$$

where $F : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ and $H : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$.

We assume that F and H are closed-valued multifunctions Lipschitzian with respect to the second variable and F is contractive in the third variable. Obviously, the right-hand side of the differential inclusion in (4.1) is in general neither convex nor closed. We prove the arcwise connectedness of the solution set to (4.1)-(4.2). When F does not depend on the last variable (4.1) reduces to (1.1) and the result remains valid for problem (1.1)-(1.2).

Let Z be a metric space with the distance d_Z. In what follows, when the product $Z = Z_1 \times Z_2$ of metric spaces $Z_i, i = 1, 2$, is considered, it is assumed that Z is equipped with the distance $d_Z((z_1, z_2), (z_1', z_2')) = \sum_{i=1}^2 d_{Z_i}(z_i, z_i')$.

Let X be a nonempty set and let $F : X \to \mathcal{P}(Z)$ be a set-valued map with nonempty closed values. The range of F is the set $F(X) = \cup_{x \in X} F(x)$. The multifunction F is called Hausdorff continuous if for any $x_0 \in X$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $x \in X, d_X(x, x_0) < \delta$ implies $D_Z(F(x), F(x_0)) < \epsilon$.

Let $(T, \mathcal{F}, \mu)$ be a finite, positive, nonatomic measure space and let $(X, |.|_X)$ be a Banach space. We recall that a set $A \in \mathcal{F}$ is called atom of μ if $\mu(A) \neq 0$ and for any $B \in \mathcal{F}$, $B \subset A$ one has $\mu(B) = 0$ or $\mu(B) = \mu(A)$. μ is called nonatomic measure if $\mathcal{F}$ does not contains atoms of μ. For example, Lebesgue's measure on a given interval in $\mathbf{R}^n$ is a nonatomic measure.

We denote by $L^1(T, X)$ the Banach space of all (equivalence classes of) Bochner integrable functions $u : T \to X$ endowed with the norm

$$|u|_{L^1(T,X)} = \int_T |u(t)|_X d\mu$$

A nonempty set $K \subset L^1(T, X)$ is called decomposable if, for every $u, v \in K$ and every $A \in \mathcal{F}$, one has

$$\chi_A.u + \chi_{T \setminus A}.v \in K$$

where $\chi_B, B \in \mathcal{F}$ indicates the characteristic function of B.

Next we recall some preliminary results that are the main tools in the proof of our result.

To simplify the notation we write E in place of $L^1(T, X)$.

Lemma 4.1. ([9]) *Assume that* $\phi : S \times E \to \mathcal{P}(E)$ *and* $\psi : S \times E \times E \to \mathcal{P}(E)$ *are Hausdorff continuous multifunctions with nonempty, closed, decomposable values, satisfying the following conditions*

a) There exists $L \in [0, 1)$ *such that, for every* $s \in S$ *and every* $u, u' \in E$,

$$D_E(\phi(s, u), \phi(s, u')) \leq L|u - u'|_E.$$

b) There exists $M \in [0, 1)$ *such that* $L + M < 1$ *and for every* $s \in S$ *and every* $(u, v), (u', v') \in E \times E$,

$$D_E(\psi(s, u, v), \psi(s, u', v')) \leq M(|u - u'|_E + |v - v'|_E).$$

Set $Fix(\Gamma(s, .)) = \{u \in E; u \in \Gamma(s, u)\}$, *where* $\Gamma(s, u) = \psi(s, u, \phi(s, u))$, $(s, u) \in S \times E$. *Then*

1) For every $s \in S$ the set $Fix(\Gamma(s,.))$ is nonempty and arcwise connected.

2) For any $s_i \in S$, and any $u_i \in Fix(\Gamma(s,.)), i = 1,...,p$ there exists a continuous function $\gamma : S \to E$ such that $\gamma(s) \in Fix(\Gamma(s,.))$ for all $s \in S$ and $\gamma(s_i) = u_i, i = 1,...,p$.

Lemma 4.2. ([9]) *Let $U : T \to \mathcal{P}(X)$ and $V : T \times X \to \mathcal{P}(X)$ be two nonempty closed-valued multifunctions satisfying the following conditions*

a) U is measurable and there exists $r \in L^1(T)$ such that $D_X(U(t), \{0\}) \leq r(t)$ for almost all $t \in T$.

b) The multifunction $t \to V(t,x)$ is measurable for every $x \in X$.

c) The multifunction $x \to V(t,x)$ is Hausdorff continuous for all $t \in T$.

Let $v : T \to X$ be a measurable selection from $t \to V(t, U(t))$.

Then there exists a selection $u \in L^1(T,X)$ of $U(.)$ such that $v(t) \in V(t,u(t))$, $t \in T$.

Hypothesis 4.3. *Let $F : I \times \mathbf{R}^2 \to \mathcal{P}(\mathbf{R})$ and $H : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ be two set-valued maps with nonempty closed values, satisfying the following assumptions*

i) The set-valued maps $t \to F(t,u,v)$ and $t \to H(t,u)$ are measurable for all $u, v \in \mathbf{R}$.

ii) There exists $l \in L^1(I, \mathbf{R}_+)$ such that, for every $u, u' \in \mathbf{R}$,

$$D(H(t,u), H(t,u')) \leq l(t)|u - u'| \quad a.e.\ (I).$$

iii) There exist $m \in L^1(I, \mathbf{R}_+)$ and $\theta \in [0,1)$ such that, for every u, v, u', $v' \in \mathbf{R}$,

$$D(F(t,u,v), F(t,u',v')) \leq m(t)|u - u'| + \theta|v - v'| \quad a.e.(I).$$

iv) There exist $f, g \in L^1(I, \mathbf{R}_+)$ such that

$$d(0, F(t,0,0)) \leq f(t), \quad d(0, H(t,0)) \leq g(t) \quad a.e.(I).$$

For $c = (c_1, c_2) \in \mathbf{R}^2$ we denote by $S(c)$ the solution set of (4.1)-(4.2).

In what follows $N(t) := \max\{l(t), m(t)\}$, $t \in I$.

Theorem 4.4. *Assume that Hypothesis 4.3. is satisfied and $2M \int_0^T N(s)ds + \theta < 1$. Then*

1) For every $c \in \mathbf{R}^2$, the solution set $S(c)$ of (4.1)-(4.2) is nonempty and arcwise connected in the space $C(I, \mathbf{R})$.

2) For any $c_i \in \mathbf{R}^2$ and any $u_i \in S(c_i)$, $i = 1, ..., p$, there exists a continuous function $s : \mathbf{R}^2 \to C(I, \mathbf{R})$ such that $s(c) \in S(c)$ for any $c \in \mathbf{R}^2$ and $s(c_i) = u_i, i = 1, ..., p$.

3) The set $S = \cup_{c\in\mathbf{R}^2} S(c)$ is arcwise connected in $C(I, \mathbf{R})$.

Proof. 1) For $c \in \mathbf{R}^2$ and $u \in L^1(I, \mathbf{R})$, set

$$u_c(t) = P_c(t) + \int_0^T G(t, s)u(s)ds, \quad t \in I.$$

We prove that the multifunctions $\phi : \mathbf{R}^2 \times L^1(I, \mathbf{R}) \to \mathcal{P}(L^1(I, \mathbf{R}))$ and $\psi : \mathbf{R}^2 \times L^1(I, \mathbf{R}) \times L^1(I, \mathbf{R}) \to \mathcal{P}(L^1(I, \mathbf{R}))$ given by

$$\phi(c, u) = \{v \in L^1(I, \mathbf{R}); \quad v(t) \in H(t, u_c(t)) \quad a.e.\ (I)\},$$

$$\psi(c, u, v) = \{w \in L^1(I, \mathbf{R}); \quad w(t) \in F(t, u_c(t), v(t)) \quad a.e.\ (I)\},$$

$c \in \mathbf{R}^2$, $u, v \in L^1(I, \mathbf{R})$ satisfy the hypotheses of Lemma 4.1.

Since u_c is measurable and H satisfies Hypothesis 4.3 i) and ii), the multifunction $t \to H(t, u_c(t))$ is measurable and nonempty closed valued, hence it has a measurable selection. Therefore due to Hypothesis 4.3 iv), the set $\phi(c, u)$ is nonempty. The fact that the set $\phi(c, u)$ is closed and decomposable follows by simple computation. In the same way we obtain that $\psi(c, u, v)$ is a nonempty closed decomposable set.

Pick $(c, u), (c_1, u_1) \in \mathbf{R}^2 \times L^1(I, \mathbf{R})$ and choose $v \in \phi(c, u)$. For each $\varepsilon > 0$ there exists $v_1 \in \phi(c_1, u_1)$ such that, for every $t \in I$, one has

$$|v(t) - v_1(t)| \le D(H(t, u_c(t)), H(t, u_{c_1}(t))) + \frac{\varepsilon}{T} \le N(t)[|P_c(t) - P_{c_1}(t)| +$$

$$\int_0^T |G(t, s)|.|u(s) - u_1(s)|ds] + \frac{\varepsilon}{T}.$$

Hence there exists $M_0 \ge 0$ such that

$$||v - v_1||_1 \le M_0||c - c_1||. \int_0^T N(t)dt + M \int_0^T N(t)dt||u - u_1||_1 + \varepsilon$$

for any $\varepsilon > 0$.

This implies

$$d_{L^1(I,\mathbf{R})}(v, \phi(c_1, u_1)) \le M_0||c - c_1||. \int_0^T N(t)dt + M \int_0^T N(t)dt||u - u_1||_1$$

for all $v \in \phi(c,u)$. Consequently,

$$D_{L^1(I,\mathbf{R})}(\phi(c,u),\phi(c_1,u_1)) \le M_0||c-c_1||.\int_0^T N(t)dt + M\int_0^T N(t)dt||u-u_1||_1$$

which shows that ϕ is Hausdorff continuous and satisfies the assumptions of Lemma 4.1.

Pick $(c,u,v),(c_1,u_1,v_1) \in \mathbf{R}^2 \times L^1(I,\mathbf{R}) \times L^1(I,\mathbf{R})$ and choose $w \in \psi(c,u,v)$. Then, as before, for each $\varepsilon > 0$ there exists $w_1 \in \psi(c_1,u_1,v_1)$ such that for every $t \in I$

$$|w(t)-w_1(t)| \le D(F(t,u_c(t),v(t)),F(t,u_{c_1}(t),v_1(t))) + \frac{\varepsilon}{T} \le N(t)|u_c(t)-$$

$$u_{c_1}(t)| + \theta|v(t)-v_1(t)| + \frac{\varepsilon}{T} \le N(t)[|P_c(t)-P_{c_1}(t)| + \int_0^1 ||G(t,s)||.|u(s)-$$

$$u_1(s)|ds] + \theta|v(t)-v_1(t)| + \frac{\varepsilon}{T} \le N(t)[M_0||c-c_1|| + M||u-u_1||_1]$$

$$+\theta|v(t)-v_1(t)| + \frac{\varepsilon}{T}.$$

Hence

$$||w-w_1||_1 \le M_0||c-c_1||.\int_0^T N(t)dt + M\int_0^T N(t)dt||u-u_1||_1$$

$$+\theta||v-v_1||_1 + \varepsilon \le M_0||c-c_1||.\int_0^T N(t)dt+$$

$$(M\int_0^T N(t)dt+\theta)d_{L^1(I,\mathbf{R})\times L^1(I,\mathbf{R})}((u,v),(u_1,v_1)) + \varepsilon.$$

As above, we deduce that

$$D_{L^1(I,\mathbf{R})}(\psi(c,u,v),\psi(c_1,u_1,v_1)) \le M_0|c-c_1|.\int_0^T N(t)dt+$$

$$(M\int_0^T N(t)dt+\theta)d_{L^1(I,\mathbf{R})\times L^1(I,\mathbf{R})}((u,v),(u_1,v_1)),$$

namely, the multifunction ψ is Hausdorff continuous and satisfies the hypothesis of Lemma 4.1.

Define $\Gamma(c,u) = \psi(c,u,\phi(c,u)), (c,u) \in \mathbf{R}^2 \times L^1(I,\mathbf{R})$. According to Lemma 4.1, the set $Fix(\Gamma(c,.)) = \{u \in L^1(I,\mathbf{R}); u \in \Gamma(c,u)\}$ is nonempty and arcwise connected in $L^1(I,\mathbf{R})$. Moreover, for fixed $c_i \in \mathbf{R}^2$ and $v_i \in$

$Fix(\Gamma(c_i,.)), i=1,...,p$, there exists a continuous function $\gamma : \mathbf{R}^2 \to L^1(I, \mathbf{R})$ such that

$$\gamma(c) \in Fix(\Gamma(c,.)), \quad \forall c \in \mathbf{R}^2, \tag{4.3}$$

$$\gamma(c_i) = v_i, \quad i=1,...,p. \tag{4.4}$$

We shall prove that

$$Fix(\Gamma(c,.)) = \{u \in L^1(I,\mathbf{R}); \quad u(t) \in F(t,u_c(t),H(t,u_c(t))) \quad a.e.\ (I)\}. \tag{4.5}$$

Denote by $A(c)$ the right-hand side of (4.5). If $u \in Fix(\Gamma(c,.))$ then there is $v \in \phi(c,v)$ such that $u \in \psi(c,u,v)$. Therefore, $v(t) \in H(t,u_c(t))$ and

$$u(t) \in F(t,u_c(t),v(t)) \subset F(t,u_c(t),H(t,u_c(t))) \quad a.e.\ (I),$$

so that $Fix(\Gamma(c,.)) \subset A(c)$.

Let now $u \in A(c)$. By Lemma 4.2, there exists a selection $v \in L^1(I,\mathbf{R})$ of the multifunction $t \to H(t,u_c(t)))$ satisfying

$$u(t) \in F(t,u_c(t),v(t)) \quad a.e.\ (I).$$

Hence, $v \in \phi(c,v)$, $u \in \psi(c,u,v)$ and thus $u \in \Gamma(c,u)$, which completes the proof of (4.5).

We next note that the function $\mathcal{T} : L^1(I,\mathbf{R}) \to C(I,\mathbf{R})$,

$$\mathcal{T}(u)(t) := \int_0^T G(t,s)u(s)ds, \quad t \in I$$

is continuous and one has

$$S(c) = P_c(.) + \mathcal{T}(Fix(\Gamma(c,.))), \quad c \in \mathbf{R}^2. \tag{4.6}$$

Since $Fix(\Gamma(c,.))$ is nonempty and arcwise connected in $L^1(I,\mathbf{R})$, the set $S(c)$ has the same properties in $C(I,\mathbf{R})$.

2) Let $c_i \in \mathbf{R}^2$ and let $u_i \in S(c_i), i=1,...,p$ be fixed. By (4.6) there exists $v_i \in Fix(\Gamma(c_i,.))$ such that

$$u_i = P_{c_i}(.) + \mathcal{T}(v_i), \quad i=1,...,p.$$

If $\gamma : \mathbf{R}^2 \to L^1(I,\mathbf{R})$ is a continuous function satisfying (4.3) and (4.4) we define, for every $c \in \mathbf{R}$,

$$s(c) = P_c(.) + \mathcal{T}(\gamma(c)).$$

Obviously, the function $s : \mathbf{R}^2 \to C(I, \mathbf{R})$ is continuous, $s(c) \in S(c)$ for all $c \in \mathbf{R}^2$, and

$$s(c_i) = P_{c_i}(.) + \mathcal{T}(\gamma(c_i)) = P_{c_i}(.) + \mathcal{T}(v_i) = u_i, \quad i = 1, ..., p.$$

3) Let $u_1, u_2 \in S = \cup_{c\in\mathbf{R}^2} S(c)$ and choose $c_i \in \mathbf{R}^2$, $i = 1, 2$ such that $u_i \in S(c_i)$, $i = 1, 2$. From the conclusion of 2) we deduce the existence of a continuous function $s : \mathbf{R}^2 \to C(I, \mathbf{R})$ satisfying $s(c_i) = u_i$, $i = 1, 2$ and $s(c) \in S(c)$, $c \in \mathbf{R}^2$. Let $h : [0, 1] \to \mathbf{R}$ be a continuous mapping such that $h(0) = c_1$ and $h(1) = c_2$. Then the function $s \circ h : [0, 1] \to C(I, \mathbf{R})$ is continuous and verifies

$$s \circ h(0) = u_1, \quad s \circ h(1) = u_2, \quad s \circ h(\tau) \in S(h(\tau)) \subset S, \quad \tau \in [0, 1].$$

Remark 4.5. We point out the fact that the results in Theorems 3.3 and 4.4 take care of a fractional differential inclusion with antiperiodic boundary conditions for $c_1 = 0, c_2 = 0, k_1 = -1, k_2 = -1$ studied in [2].

Example 4.6. Consider the following problem

$$D_c^\alpha x(t) = \frac{1}{2e^{t+1}(1 + |x(t)|)} \quad a.e. \ ([0, T]), \tag{4.7}$$

$$x(0) - k_1 x(T) = c_1, \quad x'(0) - k_2 x'(T) = c_2, \tag{4.8}$$

where $T > 0$ is chosen such that $\frac{MT}{e} < 1$. In this case $F(t, x) = \{\frac{1}{2e^{t+1}(1+|x|)}\}$. A straightforward computation shows that $L(t) \equiv \frac{1}{2e}$, $d(0, F(t, 0)) = \frac{1}{2e^{t+1}} \leq \frac{1}{2e}$.

Since $\frac{MT}{2e} < 1$, by Theorem 3.3, we obtain the existence of a solution of problem (4.7)-(4.8) which, according to (3.8), satisfies

$$|x(t)| \leq \frac{2e}{2e - MT} |P_c(t)| + \frac{MT}{2e - MT} \quad \forall t \in [0, T].$$

For every $c \in \mathbf{R}^2$ denote by $S(c)$ the solution set of (4.7)-(4.8). Since $2\frac{MT}{2e} < 1$, by Theorem 4.4, $S(c)$ is arcwise connected in the space $C(I, \mathbf{R})$.

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