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Parameter dependence for existence, nonexistence and multiplicity of nontrivial solutions for an Atıcı–Eloe fractional difference Lidstone BVP

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Abstract. Dependence on a parameter λ are established for existence, nonexistence and multiplicity results for nontrivial solutions to a nonlinear Atıcı–Eloe fractional difference equation

$$\Delta^{\nu} y(t-2) - \beta \Delta^{\nu-2} y(t-1) = \lambda f(t+\nu-1, y(t+\nu-1)),$$

with $3 < \nu \leq 4$ a real number, under Lidstone boundary conditions. In particular, the uniqueness of solutions and the continuous dependence of the unique solution on the parameter λ are also studied.

Keywords: parameter dependence, Atıcı–Eloe fractional difference, multiplicity, uniqueness.

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1 Introduction

Currently, there is increasing interest in Atıcı–Eloe fractional difference equations, with pioneering papers by Atıcı and Eloe [2–4] and Goodrich [6,7] driving much of this interest. It is natural to investigate questions for Atıcı–Eloe fractional difference equations devoted to the important results, such as those obtained in [1,5,9,10,12]. That is the goal of this paper for fractional difference equations involving Lidstone boundary conditions.

In 2008, Graef, Kong and Wang in [9] obtained periodic solutions for a boundary value problem for a second order nonlinear ordinary differential equation depending on a positive parameter λ. Under different combinations of superlinearity and sublinearity of the nonlinearity, the authors obtained various existence, multiplicity, and nonexistence results for positive solutions in terms of different values of λ. Following that paper, Anderson and Minhós [1] applied a symmetric Green's function approach to investigate the fourth-order discrete Lidstone

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problem with parameters:

$$\begin{cases} \Delta^4 y(t-2) - \beta\Delta^2 y(t-1) = \lambda f(t, y(t)), & t \in \{a+1, a+2, \ldots, b-1\}, \\ y(a) = 0 = \Delta^2 y(a-1) = 0, & y(b) = 0 = \Delta^2 y(b-1) = 0. \end{cases}$$

In a recent paper [10], under the same boundary conditions, Graef et al. studied a nonlinear discrete fourth-order equation with dependence on two parameters:

$$\Delta^4 u(t-2) - \beta\Delta^2 u(t-1) = \lambda[f(t, u(t), u(t)) + r(t, u(t))]$$

for $t \in \{a+1, a+2, \ldots, b-1\}$. Two sequences were constructed so that they converged uniformly to its unique solution.

Motivated by the above works, in this paper, for $b \in \mathbb{N}$ and $b \geq 3$, we are concerned with the parameter dependence for existence, nonexistence and multiplicity of nontrivial solutions, as well as the uniqueness of solutions, for the νth order Atıcı–Eloe fractional difference equation,

$$\Delta^\nu y(t-2) - \beta\Delta^{\nu-2} y(t-1) = \lambda f(t+\nu-1, y(t+\nu-1)) \tag{1.1}$$

for $t \in \{1, 2, \ldots, b\}$, satisfying the discrete Lidstone boundary conditions

$$\begin{cases} y(\nu-4) = 0, & y(\nu+b-2) = 0, \\ \Delta^{\nu-2} y(-1) = 0, & \Delta^{\nu-2} y(b) = 0, \end{cases} \tag{1.2}$$

where Δ^ν is the νth Atıcı–Eloe fractional difference with $3 < \nu \leq 4$ a real number, $\beta > 0$ and $\lambda > 0$ are parameters, and $f : \{\nu, \nu+1, \ldots, \nu+b-1\} \times [0, \infty) \to [0, \infty)$ is a continuous function with $f(\cdot, y) > 0$ for $y > 0$. By a positive solution of the BVP (1.1)–(1.2), we mean a function $y : \{\nu-4, \nu-3, \ldots, \nu+b-2\} \to \mathbb{R}$ that satisfies both the equation (1.1) and the boundary conditions (1.2), and is positive on $\{\nu-3, \nu-2, \ldots, \nu+b-3\}$.

The rest of this paper is organized as follows. In Section 2, we give some preliminary definitions and theorems from the theory of cones in Banach spaces that are employed to establish the main results. In Section 3, we give main results. We first construct some Green's functions, evaluate bounds for the Green's functions and define a suitable cone in a Banach space. Then, we derive existence, nonexistence and multiplicity results for nontrivial solutions to the BVP (1.1)–(1.2) in terms of different values of λ, as well as the unique solution for the BVP, which depends continuously on the parameter λ.

2 Preliminaries

We shall state some definitions from fractional difference equations along with some definitions and theorems from cone theory on which the paper's main results depend.

Definition 2.1 ([2,8]). Let $n-1 < \nu \leq n$ be a real number and $t \in \{a+\nu, a+\nu+1, \ldots\}$. The νth Atıcı–Eloe fractional sum of the function u is defined by

$$\Delta_a^{-\nu} u(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} u(s),$$

where $t^{(\nu)} = \Gamma(t+1)/\Gamma(t+1-\nu)$ is the falling function. If $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{(\nu)} = 0$. Also, the νth Atıcı–Eloe fractional difference of the function u is defined by

$$\Delta^\nu u(t) = \Delta^{n-(n-\nu)} u(t) = \Delta^n (\Delta_a^{-(n-\nu)} u(t)),$$

where Δ is the forward difference defined as $\Delta u(t) = u(t+1) - u(t)$, and $\Delta^i u(t) = \Delta(\Delta^{i-1} u(t))$, $i = 2, 3, \ldots$

Remark 2.2. We note that for u defined on $\{a, a+1, \ldots\}$, then $\Delta_a^{-\nu} u$ is defined on $\{a+\nu, a+\nu+1, \ldots\}$. We shall suppress the dependence on a in $\Delta_a^{-\nu} u(t)$ since domains will be clear by context.

Remark 2.3. From the definition of νth Atıcı–Eloe fractional difference, we have $\Delta^m \Delta^\nu u(t) = \Delta^{m+\nu} u(t)$ for $n-1 < \nu \leq n$, $m, n \in \mathbb{N}$, $m, n \geq 1$. However, in general, $\Delta^\mu \Delta^\nu u(t) \neq \Delta^{\mu+\nu} u(t)$ for $m-1 < \nu \leq m$, $n-1 < \nu \leq n$.

Remark 2.4. It is easy to check that $x^{(\nu)}$ is an increasing function for $x \in \{\nu, \nu+1, \ldots\}$.

We also require the following operational properties of fractional sum operator.

Lemma 2.5 ([2]). *Let $0 \leq n-1 < \nu \leq n$. Then*

$$\Delta^{-\nu} \Delta^\nu u(t) = u(t) + c_1 t^{(\nu-1)} + c_2 t^{(\nu-2)} + \cdots + c_n t^{(\nu-n)},$$

for some $c_i \in \mathbb{R}$, with $i = 1, 2, \ldots, n$.

Let $(\mathcal{B}, \|\cdot\|)$ be a real Banach space. $\mathcal{P} \subset \mathcal{B}$ is a *cone* provided (i) $\alpha u + \beta v \in \mathcal{P}$, for all $\alpha, \beta \geq 0$ and for all $u, v \in \mathcal{P}$, and (ii) $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$. A cone $\mathcal{P}$ in a real Banach space $\mathcal{B}$ induces a partial order on $\mathcal{B}$; namely, for $u, v \in \mathcal{B}$, $u \preceq v$ with respect to $\mathcal{P}$, if $v - u \in \mathcal{P}$.

For our existence results, we will employ the theorem below which is due to Krasnosel'skiĭ [11].

Theorem 2.6. *Let $\mathcal{B}$ be a Banach space, $\mathcal{P} \subset \mathcal{B}$ be a cone, and suppose that Ω_1, Ω_2 are bounded open balls of $\mathcal{B}$ centered at the origin, with $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $A : \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$ is a completely continuous operator such that either*

$$\|Au\| \leq \|u\|, \quad u \in \mathcal{P} \cap \partial\Omega_1 \quad \text{and} \quad \|Au\| \geq \|u\|, \quad u \in \mathcal{P} \cap \partial\Omega_2,$$

or

$$\|Au\| \geq \|u\|, \quad u \in \mathcal{P} \cap \partial\Omega_1 \quad \text{and} \quad \|Au\| \leq \|u\|, \quad u \in \mathcal{P} \cap \partial\Omega_2$$

holds. Then A has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Main results

First, let us consider the following boundary value problems

$$\begin{cases} -(\Delta^2 u(t-1) - \beta u(t)) = h(t+\nu-1), & t \in \{0, 1, \ldots, b+1\}, \\ u(0) = 0, \quad u(b+1) = 0 \end{cases} \tag{3.1}$$

and

$$\begin{cases} -\Delta^{\nu-2} y(t-1) = u(t), & t \in \{1, 2, \ldots, b\}, \\ y(\nu-4) = 0, \quad y(\nu+b-2) = 0, \end{cases} \tag{3.2}$$

respectively. Anderson and Minhós [1] derived the expression for the Green's function $G_1(t,s)$ for the BVP (3.1),

$$G_1(t,s) = \frac{1}{l(1,0) l(b+1,0)} \begin{cases} l(t,0) l(b+1,s), & t \leq s, \\ l(s,0) l(b+1,t), & s \leq t, \end{cases} \tag{3.3}$$

where $(t,s) \in \{0,1,\ldots,b+1\} \times \{0,1,\ldots,b+1\}$, with

$$l(t,s) = \chi^{t-s} - \chi^{s-t} \quad \text{for } \chi = \frac{1}{2}\left(\beta + 2 + \sqrt{\beta(\beta+4)}\right) > 1.$$

Also, by direct computation, we can get the Green's function $G_2(t,s)$ for the BVP (3.2),

$$G_2(t,s) = \frac{1}{\Gamma(\nu-2)} \begin{cases} \dfrac{t^{(\nu-3)}(\nu+b-s-2)^{(\nu-3)}}{(\nu+b-2)^{(\nu-3)}}, & (t,s) \in T_1, \\ \dfrac{t^{(\nu-3)}(\nu+b-s-2)^{(\nu-3)}}{(\nu+b-2)^{(\nu-3)}} - (t-s)^{(\nu-3)}, & (t,s) \in T_2, \end{cases} \tag{3.4}$$

where

$$T_1 := \big\{(t,s) \in \{\nu-4, \nu-3, \ldots, \nu+b-2\} \times \{0,1,\ldots,b+1\} : 0 \le t-\nu+4 \le s \le b+1\big\},$$
$$T_2 := \big\{(t,s) \in \{\nu-4, \nu-3, \ldots, \nu+b-2\} \times \{0,1,\ldots,b+1\} : 0 \le s \le t-\nu+3 \le b+1\big\}.$$

Next, we consider the Banach space $(\mathcal{B}, \|\cdot\|)$ of real-valued functions on $\{\nu-4, \nu-3, \ldots, \nu+b-2\}$ with the norm

$$\|y\| := \max\{|y(t)| : \ t \in \{\nu-4, \nu-3, \ldots, \nu+b-2\}\}.$$

From the following result we can see that the Green's function of the νth order boundary value problem is a convolution of (3.3) and (3.4).

Lemma 3.1. *Let $h : \{\nu, \nu+1, \ldots, \nu+b-1\} \to [0,+\infty)$ be a function. Then the linear discrete Lidstone BVP*

$$\begin{cases} \Delta^{\nu} y(t-2) - \beta \Delta^{\nu-2} y(t-1) = h(t+\nu-1), \quad t \in \{1,2,\ldots,b\}, \\ y(\nu-4) = 0 = y(\nu+b-2), \quad \Delta^{\nu-2}y(-1) = 0 = \Delta^{\nu-2}y(b) \end{cases} \tag{3.5}$$

has the solution

$$y(t) = \sum_{s=0}^{b+1}\sum_{z=0}^{b+1} G_2(t,s)G_1(s,z)h(z+\nu-1) \quad \textit{for } t \in \{\nu-4, \nu-3, \ldots, \nu+b-2\}.$$

Moreover, $y(t) \ge \sigma\|y\|$ for $t \in \{\nu-4, \nu-3, \ldots, \nu+b-2\}$, where

$$\sigma = \frac{l^2(1,0)l(b,0)}{l^2((b+1)/2,0)l(b+1,0)} \cdot \frac{M_1}{M_2}, \tag{3.6}$$

$$M_1 = \min\{G_2(v-3,b),\ G_2(\nu+b-3,1)\},$$
$$M_2 = \max\big\{G_2([\![(b+1)/2]\!]+\nu-4, [\![(b+1)/2]\!]),\ G_2([\![b/2]\!]+\nu-3, [\![b/2]\!]),$$
$$G_2([\![(b+1)/2]\!]+\nu-5, [\![(b+1)/2]\!]-1),\ G_2([\![b/2]\!]+\nu-4, [\![b/2]\!]-1)\big\}$$

with $[\![r]\!]$ denoting the smallest integer larger than or equal to r.

Proof. Since $G_2(t,0)G_1(0,z) = 0 = G_2(t,b+1)G_1(b+1,z)$ and $G_1(s,0) = 0 = G_1(s,b+1)$, the solution of BVP (3.5) can be written as

$$y(t) = \sum_{s=1}^{b}\sum_{z=1}^{b} G_2(t,s)G_1(s,z)h(z+\nu-1) \quad \text{for } t \in \{\nu-4, \nu-3, \ldots, \nu+b-2\}.$$

Since $y(\nu-4) = y(\nu+b-2) = 0$, the maximum of y occurs on $\{\nu-3, \nu-2, \ldots, \nu+b-3\}$. Applying the methods used in [4, Theorem 3.2], we can show that $G_2(t+1,s) > G_2(t,s)$ for $(t,s) \in T_1$ and $G_2(t+1,s) < G_2(t,s)$ for $(t,s) \in T_2$. So, for $t \in \{\nu-3, \nu-2, \ldots, \nu+b-3\}$, we have

$$G_2(t,s) \geq \min\{G_2(v-3,s),\ G_2(\nu+b-3,s)\}$$

and

$$G_2(t,s) \leq \max\{G_2(s+\nu-4,s),\ G_2(s+\nu-3,s)\}.$$

Well,

$$G_2(v-3,s) = \frac{1}{\Gamma(\nu-2)} \frac{(\nu-3)^{(\nu-3)}(\nu+b-s-2)^{(\nu-3)}}{(\nu+b-2)^{(\nu-3)}} = \frac{(\nu+b-s-2)^{(\nu-3)}}{(\nu+b-2)^{(\nu-3)}},$$

which is decreasing with respect to s according to Remark 2.3. So, for $s \in \{1, 2, \ldots, b\}$, we have $G_2(\nu-3,s) \geq G_2(\nu-3,b)$. Also,

$$\begin{aligned}
&G_2(\nu+b-3,s)\\
&\quad= \frac{1}{\Gamma(\nu-2)}\left[\frac{(\nu+b-3)^{(\nu-3)}(\nu+b-s-2)^{(\nu-3)}}{(\nu+b-2)^{(\nu-3)}} - (\nu+b-s-3)^{(\nu-3)}\right]\\
&\quad= \frac{(\nu+b-3)^{(\nu-3)}(\nu+b-s-2)^{(\nu-3)}}{\Gamma(\nu-2)(\nu+b-2)^{(\nu-3)}} - \frac{(\nu+b-2)^{(\nu-3)}(\nu+b-s-3)^{(\nu-3)}}{\Gamma(\nu-2)(\nu+b-2)^{(\nu-3)}}\\
&\quad= \frac{(\nu+b-3)^{(\nu-4)}(\nu+b-s-3)^{(\nu-4)}}{\Gamma(\nu-2)(\nu+b-2)^{(\nu-3)}} \cdot [(b+1)(\nu+b-s-2) - (\nu+b-2)(b-s+1)]\\
&\quad= \frac{(\nu+b-s-3)^{(\nu-4)}}{(\nu+b-2)\Gamma(\nu-2)} \cdot (\nu-3)s\\
&\quad= \frac{(\nu+b-s-3)^{(\nu-4)}s}{(\nu+b-2)\Gamma(\nu-3)}.
\end{aligned}$$

Let $g(s) := (\nu+b-s-3)^{(\nu-4)}s$. Then

$$\begin{aligned}
\Delta g(s) &= (\nu+b-s-4)^{(\nu-4)}(s+1) - (\nu+b-s-3)^{(\nu-4)}s\\
&= (\nu+b-s-4)^{(\nu-5)}[(b-s+1)(s+1) - (\nu+b-s-3)s]\\
&= (\nu+b-s-4)^{(\nu-5)}[(b+1-s) + (4-\nu)s]\\
&> 0,
\end{aligned}$$

for $s \in \{1, 2, \ldots, b\}$, that is, $G_2(\nu+b-3,s)$ is increasing with respect to the variable s. So, $G_2(\nu+b-3,s) \geq G_2(\nu+b-3,1)$. Hence, $G_2(t,s) \geq \min\{G_2(v-3,b),\ G_2(\nu+b-3,1)\} := M_1$ on $\{\nu-3, \nu-2, \ldots, \nu+b-3\} \times \{1, 2, \ldots, b\}$.

Let

$$\begin{aligned}
p(s) = G_2(s+\nu-4,s) &= \frac{(s+\nu-4)^{(\nu-3)}(\nu+b-s-2)^{(\nu-3)}}{\Gamma(\nu-2)(\nu+b-2)^{(\nu-3)}},\\
q(s) = G_2(s+\nu-3,s) &= \frac{(s+\nu-3)^{(\nu-3)}(\nu+b-s-2)^{(\nu-3)}}{\Gamma(\nu-2)(\nu+b-2)^{(\nu-3)}} - 1.
\end{aligned}$$

Then,

$$\begin{aligned}\Delta p(s) &= p(s+1) - p(s)\\ &= \frac{(s+\nu-3)^{(\nu-3)}(\nu+b-s-3)^{(\nu-3)}}{\Gamma(\nu-2)(\nu+b-2)^{(\nu-3)}} - \frac{(s+\nu-4)^{(\nu-3)}(\nu+b-s-2)^{(\nu-3)}}{\Gamma(\nu-2)(\nu+b-2)^{(\nu-3)}}\\ &= \frac{(s+\nu-4)^{(\nu-4)}(\nu+b-s-3)^{(\nu-4)}}{\Gamma(\nu-2)(\nu+b-2)^{(\nu-3)}} \cdot [(\nu+s-3)(b-s+1) - (\nu+b-s-2)s]\\ &= \frac{(s+\nu-4)^{(\nu-4)}(\nu+b-s-3)^{(\nu-4)}}{\Gamma(\nu-3)(\nu+b-2)^{(\nu-3)}} \cdot (b+1-2s).\end{aligned}$$

So, $\Delta p(s) \geq 0$ for $s \leq (b+1)/2$ and $\Delta p(s) \leq 0$ for $s \geq (b+1)/2$. Then,

$$p(s) \leq \max\{p([\![(b+1)/2]\!]),\ p([\![(b+1)/2]\!]-1)\}.$$

Similarly,

$$\begin{aligned}\Delta q(s) &= q(s+1) - q(s)\\ &= \frac{(s+\nu-2)^{(\nu-3)}(\nu+b-s-3)^{(\nu-3)}}{\Gamma(\nu-2)(\nu+b-2)^{(\nu-3)}} - \frac{(s+\nu-3)^{(\nu-3)}(\nu+b-s-2)^{(\nu-3)}}{\Gamma(\nu-2)(\nu+b-2)^{(\nu-3)}}\\ &= \frac{(s+\nu-3)^{(\nu-3)}(\nu+b-s-3)^{(\nu-3)}}{\Gamma(\nu-2)(\nu+b-2)^{(\nu-3)}} \cdot [(\nu+s-2)(b-s+1) - (\nu+b-s-2)(s+1)]\\ &= \frac{(s+\nu-3)^{(\nu-4)}(\nu+b-s-3)^{(\nu-4)}}{\Gamma(\nu-3)(\nu+b-2)^{(\nu-3)}} \cdot (b-2s).\end{aligned}$$

So, we obtain $q(s) \leq \max\{q([\![b/2]\!]),\ q([\![b/2]\!]-1)\}$. Hence, we have

$$G_2(t,s) \leq \max\{p([\![(b+1)/2]\!]),\ p([\![(b+1)/2]\!]-1),\ q([\![b/2]\!]),\ q([\![b/2]\!]-1)\} =: M_2.$$

At the same time, for $(s,z) \in \{1,2,\dots,b\} \times \{1,2,\dots,b\}$, it is straightforward that

$$G_1(s,z) \geq \frac{\min\{l(s,0), l(b+1,s)\}}{l(b+1,0)} G_1(z,z) \geq \frac{l(1,0)}{l(b+1,0)} G_1(z,z) \geq \frac{l(1,0)l(b,0)}{l^2(b+1,0)} =: m_1.$$

Likewise,

$$G_1(s,z) \leq G_1(z,z) \leq \frac{l(\frac{b+1}{2},0)l(b+1,\frac{b+1}{2})}{l(1,0)l(b+1,0)} = \frac{l^2(\frac{b+1}{2},0)}{l(1,0)l(b+1,0)} =: m_2,$$

where we are allowing l to be evaluated as a function over the real numbers, not just over the integers.

Then,

$$y(t) \geq \frac{M_1 m_1}{M_2 m_2}\|y\| = \sigma\|y\|, \qquad t \in \{1,2,\dots,b\}. \qquad \square$$

For $\sigma > 0$ as in (3.6), define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} := \{y \in \mathcal{B} : y(\nu-4) = y(\nu+b-2) = 0,\ y(t) \geq \sigma\|y\|,\ t \in \{\nu-3, \nu-2, \dots, \nu+b-3\}\}.$$

Define for $t \in \{\nu-4, \nu-3, \dots, \nu+b-2\}$ the functional operator $A : \mathcal{B} \to \mathcal{B}$ as

$$Ay(t) := \sum_{s=1}^{b}\sum_{z=1}^{b} G_2(t,s)G_1(s,z)f(z+\nu-1, y(z+\nu-1)).$$

By Lemma 3.1, the fixed points of λA are solutions of the BVP (1.1)–(1.2).

Now, we deduce the following four existence results by employing Theorem 2.6 due to Krasnosel'skiĭ.

Theorem 3.2. *Suppose that there exist positive numbers $0 < r < R < \infty$ such that for all $t \in \{\nu-3, \nu-2, \dots, \nu+b-3\}$, the nonlinearity f satisfies*

(H1) $$f(t,y) \le \frac{y}{\lambda M_2 m_2 b^2} \quad \text{for } y \in [0,r] \quad \text{and} \quad f(t,y) \ge \frac{y}{\lambda M_1 m_1 \sigma b^2} \quad \text{for } y \in [R,+\infty).$$

Then the BVP (1.1)–(1.2) has a nontrivial solution y such that

$$\sigma r \le y(t) \le \frac{R}{\sigma} \quad \text{for } t \in \{\nu-3, \nu-2, \dots, \nu+b-3\}.$$

Proof. If $y \in \mathcal{P}$, then $Ay(\nu-4) = 0 = Ay(\nu+b-2)$ and $Ay(t) \ge \sigma\|Ay\|$ for $t \in \{\nu-3, \nu-2, \dots, \nu+b-3\}$ by Lemma 3.1. So $A(\mathcal{P}) \subset \mathcal{P}$. Moreover, A is completely continuous using standard arguments. Define bounded open balls centered at the origin by

$$\Omega_1 := \{y \in \mathcal{P} : \|y\| < r\} \quad \text{and} \quad \Omega_2 := \left\{y \in \mathcal{P} : \|y\| < \frac{R}{\sigma}\right\}.$$

Then $0 \in \Omega_1 \subset \Omega_2$. For $y \in \mathcal{P} \cap \partial\Omega_1$, $\|y\| = r$, we have

$$\begin{aligned}
\lambda Ay(t) &= \lambda \sum_{s=1}^{b}\sum_{z=1}^{b} G_2(t,s)G_1(s,z)f(z+\nu-1, y(z+\nu-1)) \\
&\le \lambda M_2 m_2 \sum_{s=1}^{b}\sum_{z=1}^{b} f(z+\nu-1, y(z+\nu-1)) \\
&\le \frac{1}{b^2}\sum_{s=1}^{b}\sum_{z=1}^{b} y(z+\nu-1) \\
&\le \|y\|, \qquad t \in \{\nu-3, \nu-2, \dots, \nu+b-3\}.
\end{aligned}$$

Thus, $\|\lambda Ay\| \le \|y\|$ for $y \in \mathcal{P} \cap \partial\Omega_1$. Similarly, let $y \in \mathcal{P} \cap \partial\Omega_2$, so that $\|y\| = R/\sigma$. Then, $y(t) \ge \sigma\|y\| = R$, $t \in \{\nu-3, \nu-2, \dots, \nu+b-3\}$, and

$$\lambda Ay(t) \ge \lambda M_1 m_1 \sum_{s=1}^{b}\sum_{z=1}^{b} f(z+\nu-1, y(z+\nu-1)) \ge \|y\|.$$

So, $\|\lambda Ay\| \ge \|y\|$ for $y \in \mathcal{P} \cap \partial\Omega_2$. By Krasnosel'skiĭ's theorem, λA has a fixed point $y \in \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$, which is a nontrivial solution of the BVP (1.1)–(1.2), such that $r \le \|y\| \le R/\sigma$. From the fact that $y \in \mathcal{P}$ and the definition of σ in Lemma 3.1, we have

$$\sigma r \le y(t) \le \|y\| \le \frac{R}{\sigma}.$$ □

The proof of next theorem is similar to that just completed.

Theorem 3.3. *Suppose that there exist positive numbers $0 < r < R < \infty$ such that for all $t \in \{\nu-3, \nu-2, \dots, \nu+b-3\}$, the nonlinearity f satisfies*

(H2) $$f(t,y) \le \frac{y}{\lambda M_2 m_2 b^2} \quad \text{for } y \in [R,+\infty) \quad \text{and} \quad f(t,y) \ge \frac{y}{\lambda M_1 m_1 \sigma b^2} \quad \text{for } y \in [0,r].$$

Then the BVP (1.1)–(1.2) has a nontrivial solution y such that

$$\sigma r \le y(t) \le \frac{R}{\sigma} \quad \text{for } t \in \{\nu-3, \nu-2, \dots, \nu+b-3\}.$$

With an additional assumption one can show the existence of at least two nontrivial solutions to the BVP (1.1)–(1.2). The proofs of next two theorems are modifications of that in Theorem 3.2, so we omit them here.

Theorem 3.4. *Suppose that there exist positive numbers $0 < r < N < R < \infty$ such that for all $t \in \{\nu-3, \nu-2, \dots, \nu+b-3\}$, the nonlinearity f satisfies*

$$\text{(H3)} \qquad \begin{aligned} f(t,y) &< \frac{y}{\lambda M_2 m_2 b^2} && \text{for } y \in [\sigma N, N] \quad \text{and} \\ f(t,y) &\geq \frac{y}{\lambda M_1 m_1 \sigma b^2} && \text{for } y \in [0,r] \cup [R, +\infty). \end{aligned}$$

Then the BVP (1.1)–(1.2) has at least two nontrivial solutions y_1 and y_2 such that $\|y_1\| < N < \|y_2\|$ and

$$\sigma r \leq y_1(t) < N, \qquad \sigma N < y_2(t) \leq \frac{R}{\sigma} \quad \text{for } t \in \{\nu-3, \nu-2, \dots, \nu+b-3\}.$$

Theorem 3.5. *Suppose that there exist positive numbers $0 < r < N < R < \infty$ such that for all $t \in \{\nu-3, \nu-2, \dots, \nu+b-3\}$, the nonlinearity f satisfies*

$$\text{(H4)} \qquad \begin{aligned} f(t,y) &> \frac{y}{\lambda \sigma M_1 m_1 b^2} && \text{for } y \in [\sigma N, N] \quad \text{and} \\ f(t,y) &\leq \frac{y}{\lambda M_2 m_2 b^2} && \text{for } y \in [0,r] \cup [R, +\infty). \end{aligned}$$

Then the BVP (1.1)–(1.2) has at least two nontrivial solutions y_1 and y_2 such that $\|y_1\| < N < \|y_2\|$ and

$$\sigma r \leq y_1(t) < N, \qquad \sigma N < y_2(t) \leq \frac{R}{\sigma} \quad \text{for } t \in \{\nu-3, \nu-2, \dots, \nu+b-3\}.$$

We summarize the above results in the following theorem in terms of the parameter λ.

Theorem 3.6. *For $t \in \{\nu-3, \nu-2, \dots, \nu+b-3\}$, define*

$$f_0(t) := \lim_{y \to 0^+} \frac{f(t,y)}{y} \quad \text{and} \quad f_\infty(t) := \lim_{y \to \infty} \frac{f(t,y)}{y}. \tag{3.7}$$

Then, for $t \in \{\nu-3, \nu-2, \dots, \nu+b-3\}$, we have the following statements.

(i) *If $f_0(t) = 0$ and $f_\infty(t) = \infty$, then the BVP (1.1)–(1.2) has a nontrivial solution for all $\lambda \in (0, \infty)$.*

(ii) *If $f_0(t) = \infty$ and $f_\infty(t) = 0$, then the BVP (1.1)–(1.2) has a nontrivial solution for all $\lambda \in (0, \infty)$.*

(iii) *If $f_0(t) = f_\infty(t) = \infty$, then there exists $\lambda_0 > 0$ such that the BVP (1.1)–(1.2) has at least two nontrivial solutions for $0 < \lambda < \lambda_0$.*

(iv) *If $f_0(t) = f_\infty(t) = 0$, then there exists $\lambda_0 > 0$ such that the BVP (1.1)–(1.2) has at least two nontrivial solutions for $\lambda > \lambda_0$.*

(v) *If $f_0(t), f_\infty(t) < \infty$, then there exists $\lambda_0 > 0$ such that the BVP (1.1)–(1.2) has no nontrivial solution for $0 < \lambda < \lambda_0$.*

(vi) *If $f_0(t), f_\infty(t) > 0$, then there exists $\lambda_0 > 0$ such that the BVP (1.1)–(1.2) has no nontrivial solution for $\lambda > \lambda_0$.*

Proof. If $f_0(t) = 0$ and $f_\infty(t) = \infty$ for all $t \in \{\nu-3, \nu-2, \dots, \nu+b-3\}$, then (H1) is satisfied for sufficiently small $r > 0$ and sufficiently large $R > 0$.

If $f_0(t) = \infty$ and $f_\infty(t) = 0$ for all $t \in \{\nu-3, \nu-2, \dots, \nu+b-3\}$, then (H2) holds.

Likewise, if $f_0(t) = f_\infty(t) = \infty$ for all $t \in \{\nu-3, \nu-2, \dots, \nu+b-3\}$, then (H3) is satisfied for $\lambda > 0$ sufficiently small, and if $f_0(t) = f_\infty(t) = 0$ for all $t \in \{\nu-3, \nu-2, \dots, \nu+b-3\}$, then (H4) holds if λ is sufficiently large.

To see (v), since $f_0(t)$, $f_\infty(t) < \infty$ for all $t \in \{\nu-3, \nu-2, \dots, \nu+b-3\}$, there exist positive constants η_1, η_2, r and R such that $r < R$ and

$$f(t,y) \le \eta_1 y \quad \text{for } y \in [0,r] \quad \text{and} \quad f(t,y) \le \eta_2 y \quad \text{for } y \in [R,\infty).$$

Let $\eta > 0$ be given by

$$\eta = \max\left\{\eta_1,\ \eta_2,\ \max\left\{\frac{f(t,y)}{y} : t \in \{\nu-3, \nu-2, \dots, \nu+b-3\},\ y \in [r,R]\right\}\right\}.$$

Then $f(t,y) \le \eta y$ for all $y \in (0,\infty)$ and $t \in \{\nu-3, \nu-2, \dots, \nu+b-3\}$. If x is a nontrivial solution of the BVP (1.1)–(1.2), then $\lambda Ax = x$. We have

$$\|x\| = \|\lambda Ax\| \le \lambda \eta m_2 M_2 \sum_{s=1}^{b} \sum_{z=1}^{b} x(z) \le \lambda \eta m_2 M_2 b^2 \|x\| < \|x\|$$

for $0 < \lambda < 1/(\eta m_2 M_2 b^2)$, which is a contradiction.

The proof of part (vi) is similar to (v) and thus omitted. □

The final theorem in this section is obtained for the uniqueness of the solutions for the BVP (1.1)–(1.2) and the continuous dependence on the parameter λ under specialized conditions when the nonlinear term f is a separable form.

Theorem 3.7. *Assume $f(t,y) = g(t)w(y)$, where $g : \{\nu, \nu+1, \dots, \nu+b-1\} \to [0,\infty)$ with $\sum_{t=1}^{b} g(t+\nu-1) > 0$, and $w : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing, and there exists $\theta \in (0,1)$ such that $w(ky) \ge k^\theta w(y)$ for $k \in (0,1)$ and $y \in [0,\infty)$.*

Then, for any $\lambda \in (0,\infty)$, the BVP (1.1)–(1.2) has a unique solution y_λ. Furthermore, such a solution y_λ satisfies the following properties:

(i) *y_λ is nondecreasing in λ;*

(ii) *$\lim_{\lambda\to 0^+} \|y_\lambda\| = 0$ and $\lim_{\lambda\to\infty} \|y_\lambda\| = \infty$;*

(iii) *y_λ is continuous in λ, i.e., if $\lambda \to \lambda_0$, then $\|y_\lambda - y_{\lambda_0}\| \to 0$.*

Proof. We first show that for any $\lambda \in (0,\infty)$, the BVP (1.1)–(1.2) has a solution. It is easy to see that A is nondecreasing. For $k \in (0,1)$, there exists $\theta \in (0,1)$ such that

$$\begin{aligned}
\lambda A(ky(t)) &= \lambda \sum_{s=1}^{b} \sum_{z=1}^{b} G_2(t,s) G_1(s,z) g(z+\nu-1) w(ky(z+\nu-1)) \\
&\ge \lambda k^\theta \sum_{s=1}^{b} \sum_{z=1}^{b} G_2(t,s) G_1(s,z) g(z+\nu-1) w(y(z+\nu-1))
\end{aligned}$$

for $y \in \mathcal{P}$ with $y(t) \ge 0$ for $t \in \{\nu-3, \nu-2, \dots, \nu+b-3\}$. Let $L = b\sum_{z=1}^{b} g(z+\nu-1)$, and

$$\overline{y}(t) = \begin{cases} 0, & t = \nu-4, \nu+b-2, \\ \lambda L, & t \in \{\nu-3, \nu-4, \dots, \nu+b-3\}. \end{cases}$$

Then $\overline{y} \in \mathcal{P}$ and $\overline{y}(t) > 0$ for $t \in \{\nu-3, \nu-4, \ldots, \nu+b-3\}$, and

$$A\overline{y}(t) \geq m_1 M_1 w(0) \sum_{s=1}^{b}\sum_{z=1}^{b} g(z+\nu-1) = m_1 M_1 w(0) L,$$
$$A\overline{y}(t) \leq m_2 M_2 w(\lambda L) \sum_{s=1}^{b}\sum_{z=1}^{b} g(z+\nu-1) = m_2 M_2 w(\lambda L) L.$$

Thus,

$$m_1 M_1 w(0) L \leq A\overline{y} \leq m_2 M_2 w(\lambda L) L.$$

Define $\overline{c}$ and $\overline{d}$ by

$$\overline{c} := \sup\{x : Lx \leq A\overline{y}(t)\} \quad \text{and} \quad \overline{d} := \inf\{x : A\overline{y}(t) \leq Lx\}.$$

Clearly, $\overline{c} \geq m_1 M_1 w(0)$ and $\overline{d} \leq m_2 M_2 w(\lambda L)$. Choose c and d such that

$$0 < c < \min\{1, \overline{c}^{\frac{1}{1-\theta}}\} \quad \text{and} \quad \max\{1, \overline{d}^{\frac{1}{1-\theta}}\} < d < \infty.$$

Define two sequences $\{u_k(t)\}_{k=1}^{\infty}$ and $\{v_k(t)\}_{k=1}^{\infty}$ by

$$u_1(t) = \begin{cases} 0, & t = \nu-4, \nu+b-2, \\ c\lambda L, & t \in \{\nu-3, \nu-4, \ldots, \nu+b-3\}, \end{cases}$$
$$u_{k+1}(t) = \lambda A u_k(t), \qquad t \in \{\nu-4, \nu-3, \ldots, \nu+b-2\}, \quad k = 1, 2, \ldots,$$

and

$$v_1(t) = \begin{cases} 0, & t = \nu-4, \nu+b-2, \\ d\lambda L, & t \in \{\nu-3, \nu-4, \ldots, \nu+b-3\}, \end{cases}$$
$$v_{k+1}(t) = \lambda A v_k(t), \qquad t \in \{\nu-4, \nu-3, \ldots, \nu+b-2\}, \quad k = 1, 2, \ldots$$

From the monotonicity of A, we have

$$c\lambda L = u_1 \leq u_2 \leq \cdots \leq u_k \leq \cdots \leq v_k \leq \cdots \leq v_2 \leq v_1 = d\lambda L.$$

Let $\delta = c/d \in (0,1)$. We claim that

$$u_k(t) \geq \delta^{\theta^{k-1}} v_k(t) \quad \text{for } t \in \{\nu-4, \nu-3, \ldots, \nu+b-2\}. \tag{3.8}$$

In fact, it is clear that $u_1 = \delta v_1$ on $\{\nu-4, \nu-3, \ldots, \nu+b-2\}$. Assume (3.8) holds for $k = n$, i.e., $u_n(t) \geq \delta^{\theta^{n-1}} v_n(t)$ for $t \in \{\nu-4, \nu-3, \ldots, \nu+b-2\}$. Then, from the monotonicity of A, we can obtain

$$u_{n+1}(t) = \lambda A u_n(t) \geq \lambda A(\delta^{\theta^{n-1}} v_n(t)) \geq \lambda(\delta^{\theta^{n-1}})^{\theta} A v_n(t) = \lambda \delta^{\theta^n} A v_n(t) = \delta^{\theta^n} v_{n+1}(t),$$

for $t \in \{\nu-4, \nu-3, \ldots, \nu+b-2\}$. It follows from mathematical induction that (3.8) holds. Then, for a nonnegative integer l, we have

$$0 \leq u_{k+l}(t) - u_k(t) \leq v_k(t) - u_k(t) \leq (1 - \delta^{\theta^{k-1}}) v_k(t) \leq \lambda(1 - \delta^{\theta^{k-1}}) dL$$

for $t \in \{\nu-4, \nu-3, \ldots, \nu+b-2\}$. Hence,

$$\|u_{k+l} - u_k\| \leq \|v_k - u_k\| \leq \lambda(1 - \delta^{\theta^{k-1}}) dL.$$

Then, there exists a function $y \in \mathcal{P}$ such that

$$\lim_{k\to\infty} u_k(t) = \lim_{k\to\infty} v_k(t) = y(t) \quad \text{for } t \in \{\nu-4, \nu-3, \dots, \nu+b-2\}.$$

Clearly, $y(t)$ is a positive solution of the BVP (1.1)–(1.2).

Next, we show the uniqueness of solutions for BVP (1.1)–(1.2). Assume, to the contrary, that there exist two positive solutions $y_1(t)$ and $y_2(t)$ of BVP (1.1)–(1.2). Then $\lambda A y_1(t) = y_1(t)$ and $\lambda A y_2(t) = y_2(t)$ for $t \in \{\nu-4, \nu-3, \dots, \nu+b-2\}$. We note that there exists $\alpha > 0$ such that $y_1(t) \geq \alpha y_2(t)$ on $\{\nu-4, \nu-3, \dots, \nu+b-2\}$. Let $\alpha_0 = \sup\{\alpha :\ y_1(t) \geq \alpha y_2(t)\}$. Then $\alpha_0 \in (0,\infty)$ and $y_1(t) \geq \alpha_0 y_2(t)$ for $t \in \{\nu-4, \nu-3, \dots, \nu+b-2\}$. If $\alpha_0 < 1$, then there exists $\theta \in (0,1)$ such that $w(\alpha_0 y_2(t)) \geq \alpha_0^\theta w(y_2(t)) > \alpha_0 w(y_2(t))$ on $\{\nu-4, \nu-3, \dots, \nu+b-2\}$. This, together with the monotonicity of f, implies that

$$y_1(t) = \lambda A y_1(t) \geq \lambda A(\alpha_0 y_2(t)) \geq \alpha_0^\theta \lambda A(y_2(t)) > \alpha_0 y_2(t)$$

for $t \in \{\nu-4, \nu-3, \dots, \nu+b-2\}$. Thus, we can find $\tau > 0$ such that $y_1(t) \geq (\alpha_0+\tau) y_2(t)$ on $\{\nu-4, \nu-3, \dots, \nu+b-2\}$, which contradicts the definition of α_0. Hence, $y_1(t) \geq y_2(t)$ for $t \in \{\nu-4, \nu-3, \dots, \nu+b-2\}$. Similarly, we can show that $y_2(t) \geq y_1(t)$ for $t \in \{\nu-4, \nu-3, \dots, \nu+b-2\}$. Therefore, the BVP (1.1)–(1.2) has a unique solution.

In the following, we give the proof for (i)–(iii). Assume that $\lambda_1 > \lambda_2 > 0$. Let y_{λ_1} and y_{λ_2} be the unique solutions of the BVP (1.1)–(1.2) in $\mathcal{P}$ corresponding to $\lambda = \lambda_1$ and $\lambda = \lambda_2$, respectively. Let

$$\overline{\gamma} := \sup\{\gamma :\ y_{\lambda_1} \geq \gamma y_{\lambda_2}\}.$$

We assert that $\overline{\gamma} \geq 1$. In fact, if $\overline{\gamma} \in (0,1)$, we have

$$y_{\lambda_1} = \lambda_1 A y_{\lambda_1} \geq \lambda_1 A(\overline{\gamma} y_{\lambda_2}) \geq \lambda_1 \overline{\gamma}^\theta A y_{\lambda_2} = \frac{\lambda_1}{\lambda_2}\overline{\gamma}^\theta y_{\lambda_2}.$$

From the definition of $\overline{\gamma}$, we have $\overline{\gamma} \geq \frac{\lambda_1}{\lambda_2}\overline{\gamma}^\theta$, i.e., $\overline{\gamma} \geq (\frac{\lambda_1}{\lambda_2})^{\frac{1}{1-\theta}} > 1$, that is a contradiction. So, $y_{\lambda_1} \geq \overline{\gamma} y_{\lambda_2} \geq y_{\lambda_2}$. This proves (i).

Now, we show (ii). Set $\lambda_1 = \lambda$ and fix λ_2 in (i), we have $y_\lambda \geq (\frac{\lambda}{\lambda_2})^{\frac{1}{1-\theta}} y_{\lambda_2}$ for $\lambda > \lambda_2$. Further, $\|y_\lambda\| \geq (\frac{\lambda}{\lambda_2})^{\frac{1}{1-\theta}} \|y_{\lambda_2}\|$ for $\lambda > \lambda_2$. Recalling that $\theta \in (0,1)$, we have $\lim_{\lambda\to\infty} \|y_\lambda\| = \infty$. Let $\lambda_2 = \lambda$ and fix λ_1, again we obtain $y_\lambda \leq (\frac{\lambda}{\lambda_1})^{\frac{1}{1-\theta}} y_{\lambda_1}$. Then, $\lim_{\lambda\to 0^+} \|y_\lambda\| = 0$.

Finally, we prove the continuity of $y_\lambda(t)$ corresponding to λ. For given $\lambda_0 > 0$, by (i), $y_{\lambda_0} \geq y_\lambda$ for any $\lambda_0 > \lambda$. Let $\lambda_0 = \lambda_1$ and $\lambda = \lambda_2$ as in the proof of (i). Then,

$$y_{\lambda_0} \geq \frac{\lambda_0}{\lambda}\overline{\gamma}^\theta y_\lambda, \quad \text{i.e.,} \quad y_\lambda \leq \frac{\lambda}{\lambda_0}\overline{\gamma}^{-\theta} y_{\lambda_0} \leq \left(\frac{\lambda}{\lambda_0}\right)^{\frac{1}{1-\theta}} y_{\lambda_0}.$$

So,

$$\|y_\lambda - y_{\lambda_0}\| \leq \left[\left(\frac{\lambda}{\lambda_0}\right)^{\frac{1}{1-\theta}} - 1\right] \|y_{\lambda_0}\| \to 0 \quad \text{as } \lambda \to \lambda_0 - 0.$$

Similarly, we can obtain

$$\|y_\lambda - y_{\lambda_0}\| \to 0 \quad \text{as } \lambda \to \lambda_0 + 0.$$

Consequently, (iii) holds. □

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