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Existence of radially symmetric patterns for a diffusion problem with variable diffusivity

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Abstract. We give a sufficient condition for the existence of radially symmetric stable stationary solution of the problem $u_t = \operatorname{div}(a^2\nabla u) + f(u)$ on the unit ball whose border is supplied with zero Neumann boundary condition. Such a condition involves the diffusivity function a and the technique used here is inspired by the work of E. Yanagida.

Keywords: diffusion problem, stable solutions, radially symmetric solution, variable diffusivity, patterns.

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1 Introduction

In this paper we consider the radially symmetric stationary solutions of the problem

$$\begin{cases} u_t = \operatorname{div}[a^2(x)\nabla u] + f(u), & (t,x) \in (0,\infty) \times B, \\ \dfrac{\partial u}{\partial \nu} = 0, & (t,x) \in (0,\infty) \times \partial B, \end{cases} \tag{1.1}$$

where $B = \{x \in \mathbb{R}^2 : \|x\| < 1\}$, ν is the unit outer normal vector to ∂B, $a(x)$ is a positive radially symmetric function in $\overline{B}$ (i.e. $a(x) = a(\|x\|)$) and $f \in C^1(\mathbb{R})$.

This kind of problem appears as a mathematical model in many distinct areas, for example: biological population growth process, selection-migration model or, more generally, any problem of the concentration of a diffusing substance in a heterogeneous medium whose diffusivity is $a^2(x)$, under the effect of the source or sink term $f(u)$.

Stable non-constant stationary solutions to (1.1) are sometimes simply referred to as *patterns*. On the existence and non-existence of patterns for scalar diffusion equation there is a vast literature that we can summarize as follows: [3, 9, 10, 15] in intervals; [5] in balls of $\mathbb{R}^n$; [1, 7, 13, 14] in surfaces of revolution or Riemannian manifold with or without boundary and [2, 6, 11] in bounded domains of $\mathbb{R}^n$. In particular, [2, 11] consider constant diffusivity (i.e. $a(x)$ =constant) and prove that there is no pattern if the domain is convex ($B \subset \mathbb{R}^n$, for instance) regardless of the function f. See also the references in these works.

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The technique used herein requires that the term of diffusivity $a(x)$ be analytic in B, see Theorem 1.1 below. This hypothesis allows us to conclude two properties of a: $a(\|x\|) = a(r)$ is analytic in $[0,1]$ (recall that a is radially symmetric) and $a'(0) = 0$ (throughout the text we use $'$ to denote the derivative in relation to r). Both play a key role in this work.

In order to present our main result, note that (1.1) is equivalent to the following one:

$$\begin{cases} u_t = (a^2(r)u')' + \dfrac{a^2(r)}{r}u' + f(u), \quad r \in (0,1), \\ u'(0) = u'(1) = 0 \end{cases} \tag{1.2}$$

since any radially symmetric solution of (1.1) must satisfy (1.2). Throughout the text, in many instances, we use (1.2) instead of (1.1).

Our main result can be stated as follows.

Theorem 1.1. *If for some $r_0 \in (0,1)$ it holds that*

$$\left(\frac{(ar)'}{r}\right)'(r_0) > 0 \tag{1.3}$$

and if $a(x)$ is analytic in B then there exists $f \in C^1(\mathbb{R})$ such that problem (1.1) admits a radially symmetric pattern.

In [5] do Nascimento considered the same problem and proved that if $a^2(r)$ satisfies $r^2a'' + ra' - a \leq 0$ on $(0,1)$ then every non-constant stationary solution of (1.1) is unstable, i.e, there are no patterns. This result extends those obtained by Yanagida [15] and Hale et al. [3] in a interval (namely, $a'' \leq 0$ and $(a^2)'' \leq 0$ respectively). The Theorem 1.1 shows that if a is analytic in B (i.e. $a(r)$ analytic in $[0,1]$) then the condition obtained by do Nascimento is also necessary for non-existence of patterns (see Remark 3.1 (1)). To see this, simply expand (1.3), namely

$$\left(\frac{(ar)'}{r}\right)' = a'' + \frac{a'}{r} - \frac{a}{r^2}.$$

Undoubtedly, this was the main motivation of the present study.

Our proof follows the steps proposed in [15] where the problem is considered in an interval and it is proved that if $a''(r_0) > 0$ for some r_0 in this interval then there exists f such that the corresponding problem possesses patterns. The same method has been adapted for problems on surfaces of revolution, see [1, 8, 13]. In particular, Punzo [13] considered the problem on surfaces of revolution without boundary and some of his ideas were adapted here due to the close relationship of symmetry present in both problems.

The paper is divided as follows: in the Preliminaries we proof three essential lemmas for our method while Section 3 is dedicated to the proof of Theorem 1.1.

2 Preliminaries

We recall that by a stationary solution of problem (1.1) we mean a solution to the problem

$$\begin{cases} \operatorname{div}[a^2(x)\nabla u] + f(u) = 0, & x \in B, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial B, \end{cases} \tag{2.1}$$

and for the linearized problem ((2.1) in a neighborhood of U)

$$\begin{cases} \operatorname{div}[a^2(x)\nabla\phi] + f'(U)\phi + \lambda\phi = 0, & x \in B \\ \dfrac{\partial\phi}{\partial\nu} = 0, & x \in \partial B, \end{cases} \tag{2.2}$$

the sign of the principal eigenvalue λ_1 indicates the stability of U, i.e., if $\lambda_1 > 0$ then U is asymptotically stable and if $\lambda_1 < 0$ then U is unstable. If $\lambda_1 = 0$ then stability or instability can occur. This is so called linear stability and, roughly speaking, means that solutions of the corresponding parabolic equation (1.1) with the initial data near U will tend to U, as $t \to \infty$.

Lemma 2.1. *Let v be a radial solution of problem (1.1). Let there exist $w \in C^2((0,1)) \cap C^1([0,1])$ such that $w \geq 0$, w not identically zero on $[0,1]$,*

$$\mathcal{L}(w) \equiv \frac{(a^2rw')'}{r} + f'(v)w \leq 0 \quad \text{for } r \in (0,1) \text{ and } w'(1) > 0. \tag{2.3}$$

Then v is asymptotically stable.

Proof. Let λ_1 be the principal eigenvalue of the linearized problem

$$\begin{cases} \operatorname{div}[a^2(x)\nabla\phi] + f'(v)\phi + \lambda\phi = 0, & x \in B, \\ \dfrac{\partial\phi}{\partial\nu} = 0, & x \in \partial B, \end{cases} \tag{2.4}$$

and let ϕ_1 be the corresponding eigenfunction. Then $\phi_1 > 0$ in $\overline{B}$ and $\phi_1(x) = \phi_1(\|x\|)$ (see [5, Lemma 2.2]) so that

$$\begin{cases} \dfrac{(a^2r\phi_1')'}{r} + f'(v)\phi_1 + \lambda_1\phi_1 = 0, & r \in (0,1), \\ \phi_1'(0) = \phi_1'(1) = 0. \end{cases}$$

Hence,

$$\begin{aligned} 0 &\geq \int_0^1 \phi_1[(a^2rw')' + rf'(v)w]dr \\ &= \phi_1 a^2rw' \,|_0^1 - \int_0^1 \phi_1' a^2rw'dr + \int_0^1 \phi_1 rf'(v)wdr \\ &= \phi_1(1)a^2(1)w'(1) - \phi_1'a^2rw \,|_0^1 + \int_0^1 w(a^2r\phi_1')'dr + \int_0^1 \phi_1 rf'(v)wdr \\ &= \phi_1(1)a^2(1)w'(1) + \int_0^1 w[(a^2r\phi_1')' + rf'(v)\phi_1]dr \\ &> \int_0^1 -w\lambda_1\phi_1 rdr. \end{aligned}$$

It follows that $\lambda_1 > 0$ and v is asymptotically stable. □

Using (1.3) and the regularity of a, we can take $0 < R_1 < R_2 < R_3 < R_4 < 1$ in a neighborhood of r_0 such that

$$\left(\frac{(ar)'}{r}\right)'(r) > 0, \quad \text{for } r \in [R_1, R_4]. \tag{2.5}$$

It is not difficult to see that also occurs

$$\left(\frac{(a^2r)'}{r}\right)'(r) > 0, \quad \text{for } r \in [R_1, R_4]. \tag{2.6}$$

Now, consider the linear ordinary differential equation

$$z'' + \frac{p(r)}{r}z' + \frac{q(r)}{r^2}z = 0, \tag{2.7}$$

where

$$p(r) := \frac{a^2 + 2(a^2)'r}{a^2} \quad \text{and} \quad q(r) := \frac{r^2}{a^2}\left[\left(\frac{(a^2r)'}{r}\right)' - K\right]$$

with $K > 0$ a parameter to be chosen later. Since $a(r)$ is analytic in $[0,1]$ we can infer that the functions $p(r)$ and $q(r)$ are analytic in $[0,1]$ and

$$p_0 := \lim_{r\to 0} p(r) = 1 \quad \text{and} \quad q_0 := \lim_{r\to 0} q(r) = -1.$$

It follows that $\bar{r} = 0$ is a regular singular point for the differential equation (2.7) and its indicial equation is $\mu^2 - 1 = 0$. Thus, for $r \in [0, R_2)$, the problem (2.7) has a solution of the form $\tilde{z}(r) = r\eta(r)$ where η is an analytic function in $[0, R_2)$ and $\eta(0) \neq 0$ (for this matter we cite [4]).

The above steps – inspired by [13] where a problem on surfaces of revolution without boundary was considered – are to ensure that $z_1 := \tilde{z}/\eta(0)$ is a solution of the initial value problem

$$\begin{cases} \left(\frac{(a^2rz)'}{r}\right)' - Kz = 0, & r \in [0, R_2), \\ z(0) = 0, \quad z'(0) = 1. \end{cases} \tag{2.8}$$

Also consider $z_2 = z_2(r)$ a solution of the initial value problem

$$\begin{cases} \left(\frac{(a^2rz)'}{r}\right)' - Kz = 0, & r \in (R_3, 1], \\ z(1) = 0, \quad z'(1) = -1. \end{cases} \tag{2.9}$$

We can find $\overline{K} > 0$ such that

$$\left(\frac{(a^2r)'}{r}\right)' - K < 0, \quad \text{for } r \in (0,1) \tag{2.10}$$

for every $K \geq \overline{K}$. Indeed, it suffices to note that

$$\left(\frac{(a^2r)'}{r}\right)' = (a^2)'' + \frac{(a^2)'}{r} - \frac{a^2}{r^2}$$

and $a'(0) = 0$.

We shall write $z_i(r) = z_i(r, K)$ $(i = 1, 2)$ to indicate the dependence of the solution on the parameter K.

Lemma 2.2. *The solution z_1 of problem (2.8) and the solution z_2 of problem (2.9) have the following properties:*

(1) $z_1 > 0$ *in* $(0, R_2)$*;*

(2) $z_1(\cdot, K)$ *is increasing in* $[0, R_2)$ *for any* $K \geq \overline{K}$*;*

(3) $z_1(r, \cdot)$ *is increasing on* $(\overline{B}, \infty)$ *for any* $r \in (0, R_2)$*;*

(4) $\lim_{K\to\infty} z_1(r, K) = \infty$ *for any* $r \in (0, R_2)$*;*

(5) $z_2 > 0$ *in* $(R_3, 1)$*;*

(6) $z_2(\cdot, K)$ *is decreasing in* $(R_3, 1]$ *for any* $K \geq \overline{K}$*;*

(7) $z_2(r, \cdot)$ *is increasing on* $(\overline{B}, \infty)$ *for any* $r \in (R_3, 1)$*;*

(8) $\lim_{K\to\infty} z_2(r, K) = \infty$ *for any* $r \in (R_3, 1)$*.*

Proof. (1) Assuming otherwise, we could take $r_1 \in (0, R_2)$ such that $z_1(r_1) = 0$ and $z_1(r) > 0$ for all $r \in (0, r_1)$. For some $s_2 \in (0, r_1)$, $z_1(r_2) = \max_{[0,r_1]} z_1 > 0$, i.e., $z_1'(r_2) = 0$ and $z_1''(r_2) \leq 0$.

It follows that

$$\begin{aligned}
\left\{\left[\frac{(a^2 r z_1)'}{r}\right]' - K z_1\right\}(r_2) &= \left[\left(\frac{(a^2 r)'}{r}\right) z_1' + \left(\frac{(a^2 r)'}{r}\right)' z_1 + (a^2)' z_1' + a^2 z_1''\right](r_2) - K z_1(r_2) \\
&= (a^2 z_1'')(r_2) + z_1(r_2)\left[\left(\frac{(a^2 r)'}{r}\right)'(r_2) - K\right] \\
&< 0,
\end{aligned}$$

what contradicts the definition of z_1.

(2) Again, suppose by contradiction that exists $r_1 \in (0, R_2)$ such that $z_1'(s) > 0$ for all $r \in (0, r_1)$ and $z_1'(r_1) = 0$. Thus $z_1''(r_1) \leq 0$.

On the other hand,

$$z_1''(r_1) = -\left(\frac{z_1}{a^2}\right)(r_1)\left[\left(\frac{(a^2 r)'}{r}\right)'(r_1) - K\right] > 0,$$

since (2.6) occurs and $z_1(r_1) > 0$. It follows that z_1 is increasing in $(0, R_2)$.

(3) Take $K_1 > K_2 \geq \overline{K}$. It is not difficult to see that

$$\begin{cases}
\left[\dfrac{(a^2 r z_1(r, K_1))'}{r}\right]' - K_2 z_1(r, K_1) \geq 0, & r \in (0, R_2), \\
z_1(0, K_1) = 0, \quad z_1'(0, K_1) = 1.
\end{cases}$$

Now, as $z_1(r, K_2)$ satisfies

$$\begin{cases}
\left[\dfrac{(a^2 r z_1(s, K_2))'}{r}\right]' - K_2 z_1(r, K_2) = 0, & r \in (0, R_2), \\
z_1(0, K_2) = 0, \quad z_1'(0, K_2) = 1
\end{cases}$$

following the procedure used to prove Theorem 13 in Chapter 1 of [12], we can prove that $z_1(r, K_2) \leq z_1(r, K_1)$, for all $r \in (0, R_2)$.

(4) Fix any $K_1 > \overline{K}$. By integrating the equation (2.8), and remembering that z_1 is a solution, we get for any $K \geq K_1$,

$$(a^2 \eta z_1(\eta, K))' = \eta \int_0^\eta K z_1(t, K) dt + \eta c_1.$$

Integrating again

$$a^2 r z_1(r, K) = K \int_0^r \eta \int_0^\eta z_1(t, K) dt d\eta + c_1 \int_0^r \eta d\eta + c_2,$$

where c_1 and c_2 are constants independent of K. As $a > 0$ and using the item *(3)* of this lemma, we obtain

$$z_1(r, K) \geq \frac{1}{a^2 r} \left[K \int_0^r \eta \int_0^\eta z_1(t, K_1) dt d\eta + c_1 \frac{r^2}{2} + c_2 \right], \qquad \forall r \in (0, R_2).$$

The claim follows by letting $K \to \infty$. The proofs for *(5)–(8)* are analogous. □

For our next lemma we define the function $z : [0, 1] \to \mathbb{R}$,

$$z(r) := \begin{cases} z_1(r), & \text{if } r \in [0, R_2), \\ z_3(r), & \text{if } r \in [R_2, R_3], \\ z_2(r), & \text{if } r \in (R_3, 1], \end{cases} \tag{2.11}$$

where z_3 is a positive smooth function such that z is smooth at the points $r = R_2$ and $r = R_3$. Thus, z is smooth in $[0, 1]$, $z > 0$ in $(0, 1)$ and $z(0) = z(1) = 0$.

Lemma 2.3. *Let the function z be defined by* (2.11)*. Then there exists $f \in C^1(\mathbb{R})$ such that the function*

$$Z(r) := \int_0^r z(t) dt \quad \text{for } r \in [0, 1], \tag{2.12}$$

is a stationary non-constant solution of problem (1.2) *(i.e. a radial stationary non-constant solution of* (1.1)*).*

Proof. The function $u = Z(r)$ is increasing in $(0, 1)$, since $z > 0$ in $(0, 1)$. Hence we can define the inverse function $X(u) = Z^{-1}(u)$. Put

$$f(u) := \begin{cases} -Ku - a^2(0), & \text{if } u \leq 0 \\ -\dfrac{\frac{d}{du}\left\{X(u) a^2(X(u)) z(X(u))\right\}}{X(u) \frac{d}{du}\left\{X(u)\right\}}, & \text{if } 0 < u < Z(1) \\ -Ku + KZ(1) + a^2(1), & \text{if } u \geq Z(1). \end{cases} \tag{2.13}$$

The rest of the proof follows by the same arguments as in the proof of Lemma 3.5 in [8]. □

3 Proof of the main theorem

This section is devoted to prove the Theorem 1.1. Let z be the function defined by (2.11) and $m_1, m_2 > 0$ constants to be chosen later. Define

$$w(r) := \begin{cases} a(r) z(r) - m_1 z(R_1)(r - R_2)^3, & \text{if } r \in [0, R_2) \\ a(r) z(r), & \text{if } r \in [R_2, R_3] \\ a(r) z(r) + m_2 z(R_4)(r - R_3)^3, & \text{if } r \in (R_3, 1]. \end{cases} \tag{3.1}$$

We have that $w = w(r, K, m_1, m_2)$ (see (2.8), (2.9) and (2.11)), $w(\cdot, K, m_1, m_2) \in C^2((0,1)) \cap C^1([0,1])$ and is positive in $[0,1]$. In order to use Lemma 2.1, we prove that if Z is a stationary solution of (1.1) defined by (2.12), then there exist $m_1 > 0$, $m_2 > 0$ and $K > 0$ such that

$$\mathcal{L}(w) \equiv \frac{(a^2 r w')'}{r} + f'(v)w \leq 0 \quad \text{for } r \in (0,1) \text{ and } w'(1) > 0. \tag{3.2}$$

Note that this proves the Theorem 1.1 with f given by (2.13).

First we divide the interval $(0,1)$ as follows

$$(0,\epsilon] \cup (\epsilon, R_1) \cup [R_1, R_2) \cup [R_2, R_3] \cup (R_3, R_4] \cup (R_4, 1),$$

where $\epsilon > 0$ is so small so that:

$$0 < r^2 a^2(r) a''(r) + r a^2(r) a'(r) + a^3(r) \leq 2a^3(0) \quad \text{in } (0,\epsilon); \tag{3.3a}$$

$$\frac{a^2(0)}{2} \leq (a^2(r) r)' \leq 2a^2(0) \quad \text{in } (0,\epsilon); \tag{3.3b}$$

$$0 < \frac{z(r)}{r} \leq 2 \quad \text{in } (0,\epsilon). \tag{3.3c}$$

Before looking at each sub-interval of $(0,1)$, we note that

$$f'(Z(r)) = -K, \qquad \forall\, r \in (0, R_2) \cup (R_3, 1). \tag{3.4}$$

Indeed, in this case $0 < Z(r) < Z(1)$ and then we calculate (here $'$ denotes d/du)

$$f'(u) = -(a^2 z)''(X(u))X'(u) - \frac{(a^2 z)'(X(u))X'(u)}{X(u)} + \frac{(a^2 z)(X(u))X'(u)}{X^2(u)}, \qquad 0 < u < Z(1).$$

Now, as $X(u) = Z^{-1}(u)$ and $X'(u) = 1/z(X(u))$, we use the equations (2.8) (if $r \in (0, R_2)$) or (2.9) (if $r \in (R_3, 1)$) to conclude (3.4).

A simple but laborious calculation shows that

$$\frac{(a^2 r (az)')'}{r} = a\left[(a^2 z)'' + \left(\frac{a^2 z}{r}\right)' - az\left(\frac{(ar)'}{r}\right)'\right]$$

and then

$$\begin{aligned}\mathcal{L}(az) &= \frac{(a^2 r (az)')'}{r} + f'(Z)az \\ &= a\left[(a^2 z)'' + \left(\frac{a^2 z}{r}\right)' + f'(Z)z\right] - a^2 z\left(\frac{(ar)'}{r}\right)' \\ &= -a^2 z\left(\frac{(ar)'}{r}\right)'.\end{aligned} \tag{3.5}$$

In order to conclude that the term between brackets above is zero, simply derive the equation

$$(a^2 Z')' + \frac{a^2}{r} Z' + f(Z) = 0$$

and recall that $Z' = z$.

Moreover, for any $r \in (0, R_2)$,

$$\begin{aligned}
\mathcal{L}(w) &= \frac{(a^2rw')'}{r} + f'(Z)w = \frac{(a^2rw')'}{r} - Kw \quad \text{(by (3.4))}\\
&= \frac{(a^2r(az)' - 3a^2rm_1z(R_1)(r-R_2)^2)'}{r} - Kaz + Km_1z(R_1)(r-R_2)^3\\
&= \underbrace{\frac{(a^2r(az)')'}{r} - Kaz}_{\mathcal{L}(az)} - \frac{3(a^2r)'m_1z(R_1)(r-R_2)^2}{r}\\
&\quad - 6a^2m_1z(R_1)(r-R_2) + Km_1z(R_1)(r-R_2)^3\\
&\overset{(*)}{=} -a^2z\left(\frac{(ar)'}{r}\right)' + m_1z(R_1)(R_2-r)\left(6a^2 + \frac{3(a^2r)'(r-R_2)}{r} - K(r-R_2)^2\right)\\
&= \frac{1}{r}\left[(-r^2a^2a'' - ra^2a' + a^3)\frac{z}{r} - 3m_1z(R_1)(a^2r)'(r-R_2)^2\right]\\
&\quad + m_1z(R_1)(R_2-r)\left(6a^2 - K(r-R_2)^2\right).
\end{aligned} \tag{3.6}$$

In $(*)$ we use (3.4) and (3.5).
We denote

$$\tilde{a} := \max_{[0,1]}\left\{a^2(r)\right\} > 0. \tag{3.7}$$

Now, we have 6 steps.
Step 1: By (3.3a)–(3.3c), for any $r \in (0, \epsilon)$

$$\begin{aligned}
\mathcal{L}(w) &\leq \frac{1}{r}\left[4a^3(0) - 3\frac{a^2(0)}{2}m_1z(R_1)(\epsilon - R_2)^2\right] + m_1z(R_1)(R_2-r)(6\tilde{a} - K(R_2-\epsilon)^2)\\
&\leq 0
\end{aligned}$$

if

$$z(R_1) \geq \frac{8a(0)}{3m_1(\epsilon - R_2)^2} \quad \text{and} \quad K \geq \frac{6\tilde{a}}{(R_2-\epsilon)^2}. \tag{3.8}$$

Recall that $z(R_1) = z_1(R_1)$ and (3.8) occur for K sufficiently large due to Lemma 2.2 *(4)*.
Step 2: For $r \in [\epsilon, R_1)$ we consider

$$C_1 := \max_{[\epsilon,R_1]}\left\{\left|-a^2a'' - \frac{a^2a'}{r} + \frac{a^3}{r^2}\right|\right\} > 0 \quad \text{and} \quad C_2 := \max_{[\epsilon,R_1]}\left\{\frac{|(a^2r)'|}{r}\right\} > 0.$$

Hence, by (3.6)

$$\begin{aligned}
\mathcal{L}(w) &\leq z(R_1)\left[\left|-a^2a'' - \frac{a^2a'}{r} + \frac{a^3}{r^2}\right| + 3m_1\frac{|(a^2r)'|}{r}(R_2-R_1)^2 + 6\tilde{a}m_1(R_2-\epsilon) - Km_1(R_2-R_1)^3\right]\\
&\leq z(R_1)\left[C_1 + 3m_1C_2(R_2-R_1)^2 + 6\tilde{a}m_1(R_2-\epsilon) - Km_1(R_2-R_1)^3\right]\\
&\leq 0
\end{aligned}$$

if

$$K \geq \frac{1}{(R_2-R_1)^3}\left[\frac{C_1}{m_1} + 3C_2(R_2-R_1)^2 + 6\tilde{a}(R_2-\epsilon)\right].$$

Step 3: For $r \in [R_1, R_2)$ we consider

$$C_3 := \min_{[R_1,R_2]}\left\{a^2\left(\frac{(ar)'}{r}\right)'\right\} > 0 \quad \text{(see (2.5))}$$

and

$$C_4 := \max_{[R_1,R_2]} \left\{ \frac{|(a^2r)'|}{r} \right\} > 0.$$

Again by (3.6),

$$\begin{aligned}\mathcal{L}(w) &\le z(R_1)\left[-a^2\left(\frac{(ar)'}{r}\right)' + 3m_1\frac{|(a^2r)'|}{r}(R_2-R_1)^2 + 6\tilde{a}m_1(R_2-R_1)\right]\\ &\le z(R_1)\left[-C_3 + 3m_1C_4(R_2-R_1)^2 + 6\tilde{a}m_1(R_2-R_1)\right]\\ &\le 0\end{aligned}$$

if

$$m_1 \le \frac{C_3}{3C_4(R_2-R_1)^2 + 6\tilde{a}(R_2-R_1)}.$$

Step 4: For $r \in [R_2, R_3]$ we use (2.5) and (3.5) in order to conclude that

$$\mathcal{L}(w) = \mathcal{L}(az) = -a^2(r)z(r)\left(\frac{(a(r)r)'}{r}\right)' < 0.$$

Now, the steps 5 ($r \in (R_3, R_4]$) and 6 ($r \in (R_4, 1)$) are similar to steps 2 and 3 respectively, i.e., $\mathcal{L}(w) < 0$ for $r \in (R_3, 1)$ when K is large enough and m_2 is small enough.

Finally, to complete the proof, we have that $w'(r) = a'(r)z(r) + a(r)z'(r) + 3m_2z(R_4)(r-R_3)^2$. Hence

$$w'(1) = -a(1) + 3m_2z_2(R_4)(1-R_3)^2 > 0$$

if

$$z_2(R_4) > \frac{a(1)}{3m_2(1-R_3)^2} \tag{3.9}$$

and (3.9) occur for K sufficiently large (see Lemma 2.2 *(8)*)

The Theorem 1.1 is proved.

Remark 3.1.

1. Note that our condition (1.3) is equivalent to $r_0^2a''(r_0) + r_0a'(r_0) - a(r_0) > 0$ for some $r_0 \in (0,1)$. Thus, since a be analytic, our result shows that condition found by do Nascimento in [5, Theorem 5.2] is also necessary to non-existence of patterns to (1.1).

2. Our results are easily extended to balls in $\mathbb{R}^n$ with $n > 2$. In this case (1.2) it would be replaced by

$$\begin{cases} u_t = (a^2(r)u')' + \dfrac{(n-1)a^2(r)}{r}u' + f(u), & r \in (0,1)\\ u'(0) = u'(1) = 0. \end{cases}$$

3. Theorem 1.1 allows to create many examples of existence of patterns to (1.1). Two simple examples are $a_1(r) = r^3 + 1/2$ and (1.3) occurs when $r_0 = 1/2$ or $a_2(r) = r\ln(r+1) + 1/2$ and $r_0 = 4/5$, for instance. In both cases there is f such that problem (1.1) admits radially symmetric patterns.

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