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Existence of Minimal and Maximal Solutions for a Quasilinear Elliptic Equation With Integral Boundary Conditions

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Abstract: This work is concerned with the construction of the minimal and maximal solutions for a quasilinear elliptic equation with integral boundary conditions, where the nonlinearity is a continuous function depending on the first derivative of the unknown function. We also give an example to illustrate our results.

Keywords: Integral boundary conditions; upper and lower solutions; monotone iterative technique; $p-$Laplacian; Nagumo–Wintner condition
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1 Introduction

This work is concerned with the construction of the minimal and the maximal solutions of the following nonlinear boundary value problem

$$\begin{cases} -\left(\varphi_p\left(u'\right)\right)' = f\left(x, u, u'\right), \ x \in (0,1), \\ u(0) - a_0 u'(0) = \int\limits_0^1 g_1(x)\, u(x)\, dx, \\ u(1) + a_1 u'(1) = \int\limits_0^1 g_2(x)\, u(x)\, dx, \end{cases} \tag{1}$$

where $\varphi_p(y) = |y|^{p-2} y$, $p > 1$, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ and $g_i : [0,1] \to \mathbb{R}_+$ are a continuous functions ($i = 1, 2$) and a_0 and a_1 are two positive real numbers.

Problems with integral boundary conditions arise naturally in thermal conduction problems [13], semiconductor problems [21], hydrodynamics problems [15], underground water flow [18] and medical sciences [see [14] and [23]].

It is well know that the method of upper and lower solutions coupled with monotone iterative technique has been used to prove existence of solutions of nonlinear boundary value problems by various authors (see [3], [6], [9], [16] and [17]).

The purpose of this work is to show that it can be applied successfully to problems with integral boundary conditions of type (1). Our results improve and generalize those obtained in [6], [7] and [9].

The plan of this paper is as follows: In section 2, we give some preliminary results that will be used throughout the paper. In section 3, we state and prove our main result. Finally in section 4, we give an example to illustrate our results.

2 Preliminary results

In this section, we give some preliminary results that will be used in the remainder of this paper.

We consider the following problem

$$\begin{cases} -\left(\varphi_p\left(u'\right)\right)' = F\left(x, u'\right) - \widehat{h}(x, u), \ x \in (0,1), \\ u(0) - a_2 u'(0) = a, \\ u(1) + a_3 u'(1) = b, \end{cases} \tag{2}$$

where $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\widehat{h} : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and strictly increasing in its second variable, a_2 and a_3 are a positive real numbers, $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

Lemma 1 *(Weak comparison principle).*

Let u_1, u_2 are such that $u_i \in C^1([0,1])$, $\varphi_p(u_i') \in C^1(0,1)$, $i = 1, 2$, and

$$\begin{cases} -\left(\varphi_p\left(u_1'\right)\right)' - F\left(x,u_1'\right) + \widehat{h}\left(x,u_1\right) \leq -\left(\varphi_p\left(u_2'\right)\right)' - F\left(x,u_2'\right) + \widehat{h}\left(x,u_2\right), \ x \in (0,1), \\ u_1(0) - a_2 u_1'(0) \leq u_2(0) - a_2 u_2'(0), \\ u_1(1) + a_3 u_1'(1) \leq u_2(1) + a_3 u_2'(1), \end{cases}$$

then $u_1(x) \leq u_2(x)$, *for all* $x \in [0,1]$.

Proof. Assume that there exists $x_0 \in [0,1]$ such that

$$u_2(x_0) - u_1(x_0) = \min_{0 \leq x_0 \leq 1} \left(u_2(x) - u_1(x)\right) < 0.$$

Then since $(u_2 - u_1) \in C^1([0,1])$, we have $(u_2 - u_1)'(x_0) = 0$.
If $x_0 = 0$, we obtain the contradiction

$$0 > u_2(0) - u_1(0) \geq a_2 \left(u_2'(0) - u_1'(0)\right) = 0.$$

A similar argument holds if $x_0 = 1$.
If $x_0 \in (0,1)$, we have

$$\varphi_p\left(u_2'(x_0)\right) = \varphi_p\left(u_1'(x_0)\right),$$

then since φ_p is strictly increasing, we obtain that

$$-\left(\varphi_p\left(u_2'\right)\right)'(x_0) + \left(\varphi_p\left(u_1'\right)\right)'(x_0) = \lim_{x \to x_0} \frac{-\varphi_p\left(u_2'\right)(x) + \varphi_p\left(u_1'\right)(x)}{x - x_0} \leq 0.$$

But on this point, we have

$$\begin{aligned} & -\left(\varphi_p\left(u_2'\right)\right)'(x_0) + \widehat{h}\left(x_0, u_2(x_0)\right) + \left(\varphi_p\left(u_1'\right)\right)'(x_0) - \widehat{h}\left(x_0, u_1(x_0)\right) \\ & \geq F\left(x_0, u_2'(x_0)\right) - F\left(x_0, u_1'(x_0)\right) = 0. \end{aligned}$$

Which means that,

$$-\left(\varphi_p\left(u_2'\right)\right)'(x_0) + \left(\varphi_p\left(u_1'\right)\right)'(x_0) \geq \widehat{h}\left(x_0, u_1(x_0)\right) - \widehat{h}\left(x_0, u_2(x_0)\right).$$

Since $u_1(x_0) > u_2(x_0)$ and the function h is strictly increasing in its second variable, we obtain that

$$-\left(\varphi_p\left(u_2'\right)\right)'(x_0) + \left(\varphi_p\left(u_1'\right)\right)'(x_0) > 0.$$

Which is a contradiction. ■

Definition 2.1: We say that α is a lower solution of (2) if

i) $\alpha \in C^1([0,1])$ and $\varphi_p(\alpha') \in C^1(0,1)$.

ii) $\begin{cases} -\left(\varphi_p\left(\alpha'\right)\right)' \leq F\left(x,\alpha'\right) - \widehat{h}\left(x,\alpha\right), \ x \in (0,1), \\ \alpha(0) - a_2\alpha'(0) \leq a, \ \alpha(1) + a_3\alpha'(1) \leq b. \end{cases}$

Definition 2.2: We say that β is an upper solution of (2) if

i) $\beta \in C^1([0,1])$ and $\varphi_p(\beta') \in C^1(0,1)$.

ii) $\begin{cases} -(\varphi_p(\beta'))' \geq F(x,\beta') - \widehat{h}(x,\beta), \ x \in (0,1), \\ \beta(0) - a_2\beta'(0) \geq a, \ \beta(1) + a_3\beta'(1) \geq b. \end{cases}$

Now, if moreover F is a bounded function, then we have the following result.

Theorem 2 *Suppose that α and β are lower and upper solutions of problem (2) such that $\alpha(x) \leq \beta(x)$ for all $0 \leq x \leq 1$. Then the problem (2) admits a unique solution $u \in C^1([0,1])$ with $\varphi_p(u') \in C^1(0,1)$ such that*

$$\alpha(x) \leq u(x) \leq \beta(x), \ \text{for all } 0 \leq x \leq 1.$$

Proof. Using a proof similar to that of Theorem 1 in [25], we can prove that the problem (2) admits at least one solution and by Lemma 1, it follows that this problem admits a unique solution. ■

Now, we consider the following problem

$$\begin{cases} -(\varphi_p(u'))' = f(x,u,u'), \ x \in (0,1), \\ u(0) - a_0u'(0) = \int_0^1 g_1(x)u(x)\,dx, \\ u(1) + a_1u'(1) = \int_0^1 g_2(x)u(x)\,dx, \end{cases} \tag{3}$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, $g_i : [0,1] \to \mathbb{R}_+$ are a continuous functions $(i = 0,1)$ and a_0 and a_1 are two positive real numbers.

Definition 2.3: We say that u is a solution of (3) if

i) $u \in C^1([0,1])$ and $\varphi_p(u') \in C^1(0,1)$.

ii) u satisfies (3).

Definition 2.4: We say that $\underline{u}$ is a lower solution of (3) if

i) $\underline{u} \in C^1([0,1])$ and $\varphi_p(\underline{u}') \in C^1(0,1)$.

ii) $\begin{cases} -(\varphi_p(\underline{u}'))' \leq f(x,\underline{u},\underline{u}'), \ x \in (0,1), \\ \underline{u}(0) - a_0\underline{u}'(0) \leq \int_0^1 g_1(x)\underline{u}(x)\,dx, \\ \underline{u}(1) + a_1\underline{u}'(1) \leq \int_0^1 g_2(x)\underline{u}(x)\,dx. \end{cases}$

Definition 2.5: We say that $\overline{u}$ is an upper solution of (3) if

i) $\overline{u} \in C^1([0,1])$ and $\varphi_p(\overline{u}') \in C^1(0,1)$.

ii) $$\begin{cases} -(\varphi_p(\overline{u}'))' \geq f(x, \overline{u}, \overline{u}'), \ x \in (0,1), \\ \overline{u}(0) - a_0 \overline{u}'(0) \geq \int\limits_0^1 g_1(x) \overline{u}(x) \, dx, \\ \overline{u}(1) + a_1 \overline{u}'(1) \geq \int\limits_0^1 g_2(x) \overline{u}(x) \, dx. \end{cases}$$

Now, we define the Nagumo–Wintner condition.

Definition 2.6: We say that $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies a Nagumo–Wintner condition relative to the pair $\underline{u}$ and $\overline{u}$, if there exist $C \geq 0$ and a functions $Q \in L^p([0,1])$ and $\Psi : [0, +\infty) \to (0, +\infty)$ continuous, such that

$$|f(x,u,v)| \leq \Psi(|v|) \left(Q(x) + C |v|^{\frac{1}{p-1}} \right), \tag{4}$$

for all $(x,u,v) \in D$, where

$$D = \left\{ (x,u,v) \in [0,1] \times \mathbb{R}^2 : \underline{u}(x) \leq u(x) \leq \overline{u}(x) \right\}$$

and

$$\int\limits_0^{+\infty} \frac{s^{\frac{1}{p}}}{\Psi\left(|s|^{\frac{1}{p-1}} \right)} ds = +\infty. \tag{5}$$

We have the following result

Lemma 3 *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfying Nagumo–Wintner conditions (4) and (5) in D. Then there exists a constant $K > 0$, such that every solution of problem (3) verifying $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$, for all $x \in [0,1]$, satisfies $\|u'\|_0 \leq K$.*

Proof.

Since $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$, for all $x \in [0,1]$, we have

$$\underline{u}(1) - \overline{u}(0) \leq u(1) - u(0) \leq \overline{u}(1) - \underline{u}(0).$$

Let

$$\eta := \max\left\{ |\underline{u}(1) - \overline{u}(0)|, |\overline{u}(1) - \underline{u}(0)| \right\}.$$

By the mean value theorem, there exists $x_0 \in (0,1)$ such that

$$u(1) - u(0) = u'(x_0),$$

and then,

$$|u'(x_0)| \leq \eta.$$

We put by definition

$$L := |u'(x_0)| \text{ and } \widetilde{\delta} := 2\max\left(\|\underline{u}\|_0, \|\overline{u}\|_0\right).$$

Take $K > (\eta, \|\underline{u}'\|_0, \|\overline{u}'\|_0)$ such that

$$\int_{\varphi_p(\eta)}^{\varphi_p(K)} \frac{s^{\frac{1}{p}}}{\Psi\left(|s|^{\frac{1}{p-1}}\right)} ds > \|Q\|_p \widetilde{\delta}^{\frac{p-1}{p}} + C\widetilde{\delta}^{\frac{p-1}{p}}. \tag{6}$$

Now, we are going to prove that $|u'(x)| \leq K$, for all $x \in [0,1]$.

Suppose, on the contrary that there exists $x_1 \in [0,1]$ such that $|u'(x_1)| > K$.

Then by the continuity of u', we can choose $x_2 \in [0,1]$ verifying one of the following situations:

i) $u'(x_0) = L$, $u'(x_2) = K$ and $L \leq u'(x) \leq K$, for all $x \in (x_0, x_2)$.

ii) $u'(x_2) = K$, $u'(x_0) = L$ and $L \leq u'(x) \leq K$, for all $x \in (x_2, x_0)$.

iii) $u'(x_0) = -L$, $u'(x_2) = -K$ and $-K \leq u'(x) \leq -L$, for all $x \in (x_0, x_2)$.

iv) $u'(x_2) = -K$, $u'(x_0) = -L$ and $-K \leq u'(x) \leq -L$, for all $x \in (x_2, x_0)$.

Assume that the case i) holds. The others can be handled in a similar way.

Since u is a solution of the problem (3) and by the Nagumo–Wintner condition (4), we have

$$(\varphi_p(u'))'(x) \leq \Psi(u'(x))\left(Q(x) + C.(u'(x))^{\frac{1}{p-1}}\right), \text{ for all } x \in (x_0, x_2). \tag{7}$$

Since $L \leq \eta$ and φ_p is increasing, we have

$$\int_{\varphi_p(\eta)}^{\varphi_p(K)} \frac{s^{\frac{1}{p}}}{\Psi\left(s^{\frac{1}{p-1}}\right)} ds \leq \int_{\varphi_p(L)}^{\varphi_p(K)} \frac{s^{\frac{1}{p}}}{\Psi\left(s^{\frac{1}{p-1}}\right)} ds. \tag{8}$$

Now if we put $s = \varphi_p(u'(x))$, we obtain that

$$\int_{\varphi_p(L)}^{\varphi_p(K)} \frac{s^{\frac{1}{p}}}{\Psi\left(s^{\frac{1}{p-1}}\right)} ds = \int_{x_0}^{x_2} \frac{(\varphi_p(u'(x)))^{\frac{1}{p}}}{\Psi(u'(x))} (\varphi_p(u'(x)))' dx.$$

Then by (7) and (8), it follows that

$$\begin{aligned}
\int_{\varphi_p(\eta)}^{\varphi_p(K)} \frac{s^{\frac{1}{p}}}{\Psi\left(s^{\frac{1}{p-1}}\right)} ds &\leq \int_{x_0}^{x_2} \frac{\left(\varphi_p\left(u'\left(x\right)\right)\right)^{\frac{1}{p}}}{\Psi\left(u'\left(x\right)\right)} \left(\varphi_p\left(u'\left(x\right)\right)\right)' dx \\
&\leq \int_{x_0}^{x_2} \left(\varphi_p\left(u'\left(x\right)\right)\right)^{\frac{1}{p}} \left[Q\left(x\right) + C.\left(u'\left(x\right)\right)^{\frac{1}{p-1}}\right] dx \\
&= \int_{x_0}^{x_2} \left(u'\left(x\right)\right)^{\frac{p-1}{p}} \left[Q\left(x\right) + C.\left(u'\left(x\right)\right)^{\frac{1}{p-1}}\right] dx \\
&= \int_{x_0}^{x_2} \left(u'\left(x\right)\right)^{\frac{p-1}{p}} Q\left(x\right) dx + C \int_{x_0}^{x_2} \left(u'\left(x\right)\right)^{\frac{1}{p}} dx \\
&\leq \begin{array}{l} \left\|Q\right\|_p \left(\int_{x_0}^{x_2} \left(\left(u'\left(x\right)\right)^{\frac{p-1}{p}}\right)^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} + \\ +C.\left(\left(\int_{x_0}^{x_2} \left(\left(u'\left(x\right)\right)^{\frac{1}{p}}\right)^p dx\right)^{\frac{1}{p}}\right).\left(\int_{x_0}^{x_2} 1^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \end{array} \\
&= \left\|Q\right\|_p \left(u\left(x_2\right) - u\left(x_0\right)\right)^{\frac{p-1}{p}} + C.\left(u\left(x_2\right) - u\left(x_0\right)\right)^{\frac{1}{p}}.\left(x_2 - x_0\right)^{\frac{p-1}{p}} \\
&\leq \left\|Q\right\|_p \widetilde{\delta}^{\frac{p-1}{p}} + C\widetilde{\delta}^{\frac{p-1}{p}}.
\end{aligned}$$

Which a contradiction with (6). ■

3 Main result

In this section, we state and prove our main result.

On the nonlinearity f, we shall impose the following condition:

H) There exists a continuous function $h : \mathbb{R} \to \mathbb{R}$ strictly increasing such that $s \mapsto f\left(x, s, z\right) + h\left(s\right)$ is increasing for all $x \in [0, 1]$ and all $z \in \mathbb{R}$.

The main result of this work is:

Theorem 4 *Let $\underline{u}$ and $\overline{u}$ be a lower and upper solution solution respectively for problem (3) and such that $\underline{u} \leq \overline{u}$ in $[0, 1]$. Assume that H) is satisfied and the Nagumo–Wintner conditions relative to $\underline{u}$ and $\overline{u}$ holds. Then the problem (3) has a maximal solution u^* and a minimal solution u_* such that for every solution u of (3) with $\underline{u} \leq u \leq \overline{u}$ in $[0, 1]$, we have $\underline{u} \leq u_* \leq u \leq u^* \leq \overline{u}$ in $[0, 1]$.*

For the proof of this theorem, we need a preliminary lemma.

Let $\underline{w}$, $\overline{w} \in C^1\left([0, 1]\right)$ be fixed such that

i) $\varphi_p\left(\underline{w'}\right)$, $\varphi_p\left(\overline{w}'\right) \in C^1\left(0, 1\right)$.

ii) $\underline{u} \leq \underline{w} \leq \overline{w} \leq \overline{u}$ in $[0,1]$.

Let $\delta(v) := \max(-K, \min(v, K))$, for all $v \in \mathbb{R}$, where K is the constant defined in the proof of lemma 3. Then the function δ is continuous and bounded. In fact, we have $\delta(v) = v$ for all v such that $|v| \leq K$; and $|\delta(v)| \leq K$ for all $v \in \mathbb{R}$.

We consider the following problems:

$$\begin{cases} -(\varphi_p(u'))' + h(u) = f(x, \overline{w}, \delta(u')) + h(\overline{w}), \ x \in (0,1), \\ u(0) - a_0 u'(0) = \int\limits_0^1 g_0(s)\, \overline{w}(s)\, ds, \\ u(1) + a_1 u'(1) = \int\limits_0^1 g_1(s)\, \overline{w}(s)\, ds, \end{cases} \tag{9}$$

and

$$\begin{cases} -(\varphi_p(u'))' + h(u) = g(x, \underline{w}, \delta(u')) + h(\underline{w}), \ x \in (0,1), \\ u(0) - a_0 u'(0) = \int\limits_0^1 g_0(s)\, \underline{w}(s)\, ds, \\ u(1) + a_1 u'(1) = \int\limits_0^1 g_1(s)\, \underline{w}(s)\, ds. \end{cases} \tag{10}$$

Lemma 5 *Let $\underline{w}$ and $\overline{w}$ be a lower and upper solution respectively for problem (3). Assume that $H)$ is satisfied and the Nagumo–Wintner conditions relative to $\underline{u}$ and $\overline{u}$ holds. Then there exists a unique solution $\widetilde{u}$ and $\widehat{u}$ of (9) and (10) such that $\underline{u} \leq \underline{w} \leq \widetilde{u} \leq \widehat{u} \leq \overline{w} \leq \overline{u}$.*

Proof.

The proof will be given in several steps.

Step 1: $\underline{w}$ is a lower solution of (9).

Proof: Let $x \in (0,1)$, we have

$$\begin{aligned} -(\varphi_p(\underline{w}'))' + h(\underline{w}) &\leq f(x, \underline{w}, \underline{w}') + h(\underline{w}) \\ &\leq f(x, \overline{w}, \underline{w}') + h(\overline{w}). \end{aligned}$$

This means that,

$$\forall\ x \in (0,1), \ -(\varphi_p(\underline{w}'))' + h(\underline{w}) \leq f(x, \overline{w}, \underline{w}') + h(\overline{w}).$$

Now since $\underline{w}$ is a lower solution of (3) and $\underline{u} \leq \underline{w} \leq \overline{u}$ in $[0,1]$, then by using a proof similar to that of lemma 3, we prove that $\|\underline{w}'\|_0 \leq K$. Hence $\delta(\underline{w}') = \underline{w}'$ and we obtain that

$$\forall x \in (0,1), \ -(\varphi_p(\underline{w}'))' + h(\underline{w}) \leq f(x, \overline{w}, \delta(\underline{w}')) + h(\overline{w}). \tag{11}$$

On the other hand, we have

$$\begin{aligned}\underline{w}(0)-a_0\underline{w}'(0) &\le \int_0^1 g_0(s)\,\underline{w}(s)\,ds\\ &\le \int_0^1 g_0(s)\,\overline{w}(s)\,ds.\end{aligned}$$

That is,

$$\underline{w}(0)-a_0\underline{w}'(0)\le \int_0^1 g_0(s)\,\overline{w}(s)\,ds. \tag{12}$$

Similarly, we have

$$\underline{w}(1)+a_1\underline{w}'(1)\le \int_0^1 g_1(s)\,\overline{w}(s)\,ds. \tag{13}$$

Then by (11), (12) and (13), it follows that $\underline{w}$ is a lower solution of (9).
Step 2: $\overline{w}$ is an upper solution of (9).
Proof: Let $x\in(0,1)$, we have

$$-\left(\varphi_p\left(\overline{w}'\right)\right)'+h\left(\overline{w}\right)\ge f\left(x,\overline{w},\overline{w}'\right)+h\left(\overline{w}\right).$$

Now by using a proof similar to that of lemma 3, we prove that $\left\|\overline{w}'\right\|_0\le K$. Hence $\delta\left(\overline{w}'\right)=\overline{w}'$ and we obtain that

$$\forall\, x\in(0,1)\,,\;\; -\left(\varphi_p\left(\overline{w}'\right)\right)'+h\left(\overline{w}\right)\ge f\left(x,\overline{w},\delta\left(\overline{w}'\right)\right)+h\left(\overline{w}\right). \tag{14}$$

Also, we have

$$\overline{w}(0)-a_0\overline{w}'(0)\ge \int_0^1 g_0(s)\,\overline{w}(s)\,ds, \tag{15}$$

and

$$\overline{w}(1)+a_1\overline{w}'(1)\ge \int_0^1 g_1(s)\,\overline{w}(s)\,ds. \tag{16}$$

Then by (14), (15) and (16), it follows that $\overline{w}$ is an upper solution of (9).

By Steps 1 and 2 and since the functions $(x,u')\mapsto f\left(x,\overline{w},\delta\left(u'\right)\right)+h\left(\overline{w}\right)$ is a bounded continuous function and $u\mapsto h(u)$ is continuous and strictly increasing, then by theorem 2, it follows the existence of a unique solution $\widetilde{u}$ of (9) such that $\underline{w}\le\widetilde{u}\le\overline{w}$.

Similarly, we can prove the existence and uniqueness of a solution to (10), which we call $\widehat{u}$ such that $\underline{w}\le\widehat{u}\le\overline{w}$.

Finally, by using a proof similar to that of lemma 1, we prove that $\widehat{u} \leq \widetilde{u}$ in $[0,1]$. ■

Proof. of Theorem 4

The proof will be given in several steps.

We take $\overline{u}_0 = \overline{u}$, $\underline{u}_0 = \underline{u}$ and define the sequences $(\overline{u}_n)_{n\geq 1}$, $(\underline{u}_n)_{n\geq 1}$ by

$$\begin{cases} -\left(\varphi_p\left(\overline{u}_{n+1}'\right)\right)' + h\left(\overline{u}_{n+1}\right) = f\left(x, \overline{u}_n, \delta\left(\overline{u}_{n+1}'\right)\right) + h\left(\overline{u}_n\right),\ x \in (0,1), \\ \overline{u}_{n+1}(0) - a_0\overline{u}_{n+1}'(0) = \int\limits_0^1 g_0(s)\,\overline{u}_n(s)\,ds, \\ \overline{u}_{n+1}(1) + a_1\overline{u}_{n+1}'(1) = \int\limits_0^1 g_1(s)\,\overline{u}_n(s)\,ds, \end{cases} \tag{P_{n+1}}$$

and

$$\begin{cases} -\left(\varphi_p\left(\underline{u}_{n+1}'\right)\right)' + h\left(\underline{u}_{n+1}\right) = g\left(x, \underline{u}_n, \delta\left(\underline{u}_{n+1}'\right)\right) + h\left(\underline{u}_n\right),\ x \in (0,1), \\ \underline{u}_{n+1}(0) - a_0\underline{u}_{n+1}'(0) = \int\limits_0^1 g_0(s)\,\underline{u}_n(s)\,ds, \\ \underline{u}_{n+1}(1) + a_1\underline{u}_{n+1}'(1) = \int\limits_0^1 g_1(s)\,\underline{u}_n(s)\,ds. \end{cases} \tag{Q_{n+1}}$$

Step 1: For all $n \in \mathbb{N}$, we have

$$\underline{u} \leq \underline{u}_n \leq \underline{u}_{n+1} \leq \overline{u}_{n+1} \leq \overline{u}_n \leq \overline{u} \text{ in } [0,1].$$

Proof:

i) For $n = 0$, we have

$$\begin{cases} -\left(\varphi_p\left(\overline{u}_1'\right)\right)' + h\left(\overline{u}_1\right) = f\left(x, \overline{u}, \delta\left(\overline{u}_1'\right)\right) + h\left(\overline{u}\right),\ x \in (0,1), \\ \overline{u}_1(0) - a_0\overline{u}_1'(0) = \int\limits_0^1 g_0(s)\,\overline{u}(s)\,ds, \\ \overline{u}_1(1) + a_1\overline{u}_1'(1) = \int\limits_0^1 g_1(s)\,\overline{u}(s)\,ds, \end{cases} \tag{P_1}$$

$$\begin{cases} -\left(\varphi_p\left(\underline{u}_1'\right)\right)' + h\left(\underline{u}_1\right) = g\left(x, \underline{u}, \delta\left(\underline{u}_1'\right)\right) + h\left(\underline{u}\right),\ x \in (0,1), \\ \underline{u}_1'(0) - a_0\underline{u}_1(0) = \int\limits_0^1 g_0(s)\,\underline{u}(s)\,ds, \\ \underline{u}_1'(1) + a_1\underline{u}_1(1) = \int\limits_0^1 g_1(s)\,\underline{u}(s)\,ds. \end{cases} \tag{Q_1}$$

Since $\underline{u}$ and $\overline{u}$ are lower and upper solutions of problem (3), then by lemma 5, it follows that

$$\underline{u} = \underline{u}_0 \leq \underline{u}_1 \leq \overline{u}_1 \leq \overline{u}_0 = \overline{u} \text{ in } [0,1].$$

ii) Assume for fixed $n > 1$, we have

$$\underline{u} \leq \underline{u}_{n-1} \leq \underline{u}_n \leq \overline{u}_n \leq \overline{u}_{n-1} \leq \overline{u} \text{ in } [0,1],$$

and we show that

$$\underline{u} \leq \underline{u}_n \leq \underline{u}_{n+1} \leq \overline{u}_{n+1} \leq \overline{u}_n \leq \overline{u} \text{ in } [0,1].$$

Let $x \in (0,1)$, we have

$$-\left(\varphi_p\left(\overline{u}_n'\right)\right)' + h\left(\overline{u}_n\right) = f\left(x, \overline{u}_{n-1}, \delta\left(\overline{u}_n'\right)\right) + h\left(\overline{u}_{n-1}\right). \tag{17}$$

Since $\overline{u}_{n-1} \geq \overline{u}_n$ and using the hypothesis H), we obtain

$$f\left(x, \overline{u}_{n-1}, \delta\left(\overline{u}_n'\right)\right) + h\left(\overline{u}_{n-1}\right) \geq f\left(x, \overline{u}_n, \delta\left(\overline{u}_n'\right)\right) + h\left(\overline{u}_n\right). \tag{18}$$

Then by (17) and (18), it follows that

$$\forall\ x \in (0,1), -\left(\varphi_p\left(\overline{u}_n'\right)\right)' \geq f\left(x, \overline{u}_n, \delta\left(\overline{u}_n'\right)\right).$$

Now by using a proof similar to that of lemma 3, we can prove that $\left\|\overline{u}_n'\right\|_0 \leq K$. Hence $\delta\left(\overline{u}_n'\right) = \overline{u}_n'$ and we obtain that

$$\forall\ x \in (0,1), -\left(\varphi_p\left(\overline{u}_n'\right)\right)' \geq f\left(x, \overline{u}_n, \overline{u}_n'\right). \tag{19}$$

On the other hand, we have

$$\begin{aligned}
\overline{u}_n(0) - a_0 \overline{u}_n'(0) &= \int_0^1 g_0(s)\, \overline{u}_{n-1}(s)\, ds \\
&\geq \int_0^1 g_0(s)\, \overline{u}_n(s)\, ds.
\end{aligned}$$

That is,

$$\overline{u}_n(0) - a_0 \overline{u}_n'(0) \geq \int_0^1 g_0(s)\, \overline{u}_n(s)\, ds, \tag{20}$$

and

$$\begin{aligned}
\overline{u}_n(1) + a_1 \overline{u}_n'(1) &= \int_0^1 g_1(s)\, \overline{u}_{n-1}(s)\, ds \\
&\geq \int_0^1 g_1(s)\, \overline{u}_n(s)\, ds.
\end{aligned}$$

That is,

$$\overline{u}_n(1) + a_1 \overline{u}_n'(1) \geq \int_0^1 g_1(s)\, \overline{u}_n(s)\, ds \tag{21}$$

Then by (19), (20) and (21), it follows that $\overline{u}_n$ is an upper solution of (3). Similarly, we can prove that $\underline{u}_n$ is a lower solution of (3). Then by lemma 5, there exists a unique solution $\overline{u}_{n+1}$ and $\underline{u}_{n+1}$ of (P_{n+1}) and (Q_{n+1}) such that

$$\underline{u} \leq \overline{u}_{n+1} \leq \overline{u}_n \leq \underline{u}_n \leq \underline{u}_{n+1} \leq \overline{u} \text{ in } [0,1].$$

Hence, we have

$$\forall n \in \mathbb{N}, \underline{u} \leq \overline{u}_{n+1} \leq \overline{u}_n \leq \underline{u}_n \leq \underline{u}_{n+1} \leq \overline{u} \text{ in } [0,1].$$

Step 2: The sequence $(\overline{u}_n)_{n\in\mathbb{N}}$ converge to a maximal solution of (3).

Proof: By Step 1 and since $\|\overline{u}_n'\|_0 \leq K$, for all $n \in \mathbb{N}$, it follows that the sequence $(\overline{u}_n)_{n\in\mathbb{N}}$ is uniformly bounded in $C^1([0,1])$.

Now let $\varepsilon_1 > 0$ and t, $s \in [0,1]$ such that $t < s$, then for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\left|\varphi_p\left(\overline{u}_{n+1}'(s)\right) - \varphi_p\left(\overline{u}_{n+1}'(t)\right)\right| &= \left|\int_t^s \left(f\left(\tau, \overline{u}_n(\tau), \delta\left(\overline{u}_{n+1}'(\tau)\right)\right) + h\left(\overline{u}_n(\tau)\right) - h\left(\overline{u}_{n+1}(\tau)\right)\right) d\tau\right| \\
&\leq \int_t^s \left|\left(f\left(\tau, \overline{u}_n(\tau), \delta\left(\overline{u}_{n+1}'(\tau)\right)\right) + h\left(\overline{u}_n(\tau)\right) - h\left(\overline{u}_{n+1}(\tau)\right)\right)\right| d\tau \\
&\leq \left(M_1(f) + 2M_2(h)\right)|s-t|,
\end{aligned}$$

where

$$M_1(f) := \max\left\{|f(x,s,z)| : x \in [0,1],\ \underline{u} \leq u \leq \overline{u} \text{ and } |z| \leq K\right\},$$

and

$$M_2(h) := \max\left\{|h(u)| : \underline{u} \leq u \leq \overline{u}\right\}.$$

If we put $K_1 := M_1(f) + 2M_2(h)$, one has

$$\left|\varphi_p\left(\overline{u}_{n+1}'(s)\right) - \varphi_p\left(\overline{u}_{n+1}'(t)\right)\right| \leq K_1 |s-t|.$$

Then if we choose $|s-t| \leq \dfrac{\varepsilon_1}{K_1+1}$, we obtain

$$\left|\varphi_p\left(\overline{u}_{n+1}'(s)\right) - \varphi_p\left(\overline{u}_{n+1}'(t)\right)\right| < \varepsilon_1.$$

Therefore the sequence $\left(\varphi_p\left(\overline{u}_n'\right)\right)_{n\in\mathbb{N}}$ is equicontinuous on $[0,1]$.

Now since the mapping φ_p^{-1} is an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, we deduce from

$$\left|\overline{u}_n'(s) - \overline{u}_n'(t)\right| = \left|\varphi_p^{-1}\left(\varphi_p\left(\overline{u}_n'(s)\right)\right) - \varphi_p^{-1}\left(\varphi_p\left(\overline{u}_n'(t)\right)\right)\right|$$

that the sequence $(\overline{u}'_n)_{n\in\mathbb{N}}$ is equicontinuous on $[0,1]$.

Hence by the Arzéla-Ascoli theorem, there exists a subsequence $\left(\overline{u}_{n_j}\right)$ of $(\overline{u}_n)_{n\in\mathbb{N}}$ which converges in $C^1([0,1])$.

Let

$$u := \lim_{n_j\to+\infty} \overline{u}_{n_j}.$$

Then

$$u' = \lim_{n_j\to+\infty} \overline{u}'_{n_j}.$$

But by Step 1 the sequence $(\overline{u}_n)_{n\in\mathbb{N}}$ is decreasing and bounded from below, then the pointwise limit of this sequence exists and it is denoted by u^*. Hence, we have $u = u^*$ and moreover, the whole sequence converges in $C^1([0,1])$ to u^*.

Let $x \in (0,1)$, we have

$$-\varphi_p\left(\overline{u}'_{n+1}(x)\right) = \varphi_p\left(\overline{u}'_{n+1}(0)\right) + \int_0^x \left(f\left(\tau, \overline{u}_n(\tau), \delta\left(\overline{u}'_{n+1}(\tau)\right)\right) + h\left(\overline{u}_n(\tau)\right) - h\left(\overline{u}_{n+1}(\tau)\right)\right) d\tau.$$

Now, as n tends to $+\infty$, we obtain that

$$f\left(\tau, \overline{u}_n(\tau), \delta\left(\overline{u}'_{n+1}(\tau)\right)\right) + h\left(\overline{u}_n(\tau)\right) - h\left(\overline{u}_{n+1}(\tau)\right) \to f\left(\tau, u^*(\tau), \delta\left(u^{*\prime}(\tau)\right)\right).$$

Also, we have

$$\exists\ K_4 > 0,\ \forall\ n \in \mathbb{N},\ \forall\ \tau \in [0,1],\ \left|f\left(\tau, \overline{u}_n(\tau), \delta\left(\overline{u}'_{n+1}(\tau)\right)\right) + h\left(\overline{u}_n(\tau)\right) - h\left(\overline{u}_{n+1}(\tau)\right)\right| \le K_4.$$

Hence, the dominated convergence theorem of Lebesgue implies that

$$-\varphi_p\left(u^{*\prime}(x)\right) = \varphi_p\left(u^{*\prime}(0)\right) + \int_0^x \left(f\left(\tau, u^*(\tau), \delta\left(u^{*\prime}(\tau)\right)\right) + h\left(u^*(\tau)\right) - h\left(u^*(\tau)\right)\right) d\tau.$$

Thus, we obtain

$$\forall\ x \in (0,1),\ -\left(\varphi_p\left(u^{*\prime}\right)\right)' = f\left(x, u^*, \delta\left(u^{*\prime}\right)\right). \tag{22}$$

Also, by the dominated convergence theorem of Lebesgue, we have

$$u^*(0) - a_0 u^{*\prime}(0) = \int_0^1 g_0(s)\, u^*(s)\, ds, \tag{23}$$

and

$$u^*(1) + a_1 u^{*\prime}(1) = \int_0^1 g_1(s)\, u^*(s)\, ds. \tag{24}$$

By (22), (23) and (24), it follows that u^* is a solution of the following problem

$$\begin{cases} -\left(\varphi_p\left(u^{*\prime}\right)\right)' = f\left(x, u^*, \delta\left(u^{*\prime}\right)\right), \ x \in (0,1), \\ u^*(0) - a_0 u^{*\prime}(0) = \int\limits_0^1 g_0(s)\, u^*(s)\, ds, \\ u^*(1) + a_1 u^{*\prime}(1) = \int\limits_0^1 g_1(s)\, u^*(s)\, ds. \end{cases} \tag{25}$$

Now using a proof similar to that of lemma 3, we prove that $\|u^*\| \leq K$. Hence $\delta\left(u^{*\prime}\right) = u^{*\prime}$ and consequently u^* is a solution of (3).

Now, we prove that if u is another solution of (3) such that $\underline{u} \leq u \leq \overline{u}$ in $[0,1]$, then $u \leq u^*$ in $[0,1]$.

Since u is a lower solution of (3), then by Step 1, we have

$$\forall\ n \in \mathbb{N},\ u \leq \overline{u}_n.$$

Letting $n \to +\infty$, we obtain that

$$u \leq \lim_{n\to+\infty} \overline{u}_n = u^*.$$

Which means that u^* is a maximal solution of problem (3).

Step 3: The sequence $\left(\underline{u}_n\right)_{n\in\mathbb{N}}$ converges to a minimal solution of (3).

Proof: The proof is similar to that of Step 2, so it is omitted.

The proof of our result is complete. ■

4 Application

In this section, we apply the previous result to the following problem

$$\begin{cases} -\left(\varphi_p\left(u'\right)\right)' = \lambda_1 u^{k_1} - u^{k_2} + \lambda_2 u^{k_1} \left|u'\right|^{\frac{1}{p-1}} \text{ in } (0,1), \\ u(0) - a_0 u'(0) = 0,\ u(1) + a_1 u'(1) = \int\limits_0^1 g_1(s)\, u(s)\, ds, \end{cases} \tag{26}$$

where $0 < k_1 < p-1$, $k_2 > k_1$, λ_1, and are a positive real parameters and $\int\limits_0^1 g_1(s)\, ds \leq 1$.

To study this problem, we need first consider the following problem:

$$\begin{cases} -\left(\varphi_p\left(u'\right)\right)' = u^{k_3} \text{ in } (0,1), \\ u(0) = 0,\ u(1) = 0, \end{cases} \tag{27}$$

where $k_3 > 0$ and $k_3 \neq p-1$.

Theorem 6 *The problem (27) admits a unique positive solution* $\Phi_{k_3,p}$.

Proof.

Multiplying the differential equation in (27) by u' and integrating the resulting equation over $[0,x]$, we obtain

$$|u'(x)|^p = |u'(0)|^p - \frac{p}{(p-1)(k_3+1)} u^{k_3+1}(x),\ x \in [0,1]. \tag{28}$$

We note that u is symmetric about $x = \dfrac{1}{2}$ and $u'(x) > 0$, for all $x \in \left[0, \dfrac{1}{2}\right)$.

If we put by definition $\rho = \max\limits_{x\in[0,1]} u(x)$, then $u\left(\dfrac{1}{2}\right) = \rho$ and $\rho > 0$.

Now substituting $x = \dfrac{1}{2}$ in (28), we obtain

$$u'(x) = \left[\frac{p}{(p-1)(k_3+1)} \left(\rho^{k_3+1} - u^{k_3+1}(x)\right)\right]^{\frac{1}{p}},\ \text{ for all } x \in \left[0, \frac{1}{2}\right],$$

and thus,

$$\frac{u'(x)}{\left[\dfrac{p}{(p-1)(k_3+1)} \left(\rho^{k_3+1} - u^{k_3+1}(x)\right)\right]^{\frac{1}{p}}} = 1,\ \text{ for all } x \in \left[0, \frac{1}{2}\right).$$

Integrating the last equality on $(0,x)$, where $x \in \left[0, \frac{1}{2}\right)$, we obtain

$$\int_0^{u(x)} \frac{dv}{\left[\dfrac{p}{(p-1)(k_3+1)} \left(\rho^{k_3+1} - v^{k_3+1}\right)\right]^{\frac{1}{p}}} = x. \tag{29}$$

Letting $x \to \dfrac{1}{2}$ in (29), we obtain

$$G(\rho) := \int_0^{\rho} \frac{dv}{\left[\dfrac{p}{(p-1)(k_3+1)} \left(\rho^{k_3+1} - v^{k_3+1}\right)\right]^{\frac{1}{p}}} = \frac{1}{2}.$$

Thus, if the problem (27) admits a positive solution u, with $\max\limits_{x\in[0,1]} u(x) = u\left(\dfrac{1}{2}\right) = \rho$, then we have $G(\rho) = \dfrac{1}{2}$.

Conversely, if $G(\rho) = \dfrac{1}{2}$. Defining u via equation (29), we can prove that problem (27) admits a unique positive solution u, with $\max\limits_{x\in[0,1]} u(x) = u\left(\dfrac{1}{2}\right) = \rho$.

Hence the problem (27) admits a unique positive solution u, with $\max_{x\in[0,1]} u(x) = u\left(\frac{1}{2}\right) = \rho$ if and only if $G(\rho) = \frac{1}{2}$.

Some easy computations shows that

$$G(\rho) = \left[\frac{(p-1)(k_3+1)}{p}\right]^{\frac{1}{p}} \frac{B\left(\frac{p-1}{p}, \frac{1}{k_3+1}\right)}{k_3+1} \rho^{\frac{p-1-k_3}{p}},$$

where $B(k,l)$ is the Euler beta function defined by

$$B(k,l) = \int_0^1 (1-t)^{k-1} t^{l-1} dt, \; k > 0 \text{ and } l > 0.$$

It is not difficult to see that the equation $G(\rho) = \frac{1}{2}$ admits a unique solution.

Hence the problem (27) admits a unique positive solution. ■

Theorem 7 *Assume that* $\lambda_1 > 1$, *then the problem (26) admits a maximal solution* u^* *and a minimal solution* u_*.

Proof. We put $(\underline{u}, \overline{u}) = (\varepsilon\Phi_{k_1,p}, L)$ where ε and L are a positive constants.

First, since $\Phi'_{k_1,p}(0) > 0$, $\Phi'_{k_1,p}(1) < 0$ and $\int_0^1 g_1(s)\,ds \leq 1$ and, it is easy to check that

$$\begin{cases} \varepsilon\Phi_{k_1,p}(0) - a_0\varepsilon\Phi'_{k_1,p}(0) \leq 0, \\ \varepsilon\Phi_{k_1,p}(1) + a_1\varepsilon\Phi'_{k_1,p}(1) \leq \int_0^1 g_1(s)\,\varepsilon\Phi_{k_1,p}(s)\,ds, \\ L \geq \int_0^1 g_1(s)\,L ds. \end{cases}$$

Now $\underline{u}$ and $\overline{u}$-are lower and upper solutions of (26), if we have

$$\begin{cases} \varepsilon^{p-1}\Phi^{k_1}_{k_1,p}(x) \leq \lambda_1\varepsilon^{k_1}\Phi^{k_1}_{k_1,p}(x) - \varepsilon^{k_2}\Phi^{k_2}_{k_1,p}(x) + \lambda_2\varepsilon^{k_1}\Phi^{k_1}_{k_1,p}(x)\left|\varepsilon\Phi'_{k_1,p}(x)\right|^{\frac{1}{p-1}} \text{ in } (0,1), \\ 0 \geq \lambda_1 L^{k_1} - L^{k_2}. \end{cases}$$

That is

$$\begin{cases} \varepsilon^{p-1-k_1} \leq \lambda_1 - \varepsilon_1^{k_2-k_1}\Phi^{k_2-k_1}_{k_1,p}(x) + \lambda_2\left|\varepsilon\Phi'_{k_1,p}(x)\right|^{\frac{1}{p-1}} \text{ in } (0,1), \\ L^{k_2} \geq \lambda_1 L^{k_1}. \end{cases}$$

Since $k_2 > k_1$, if we choose $\lambda_1 > 1$, ε sufficiently small and L sufficiently large, we obtain that $\underline{u}$ and $\overline{u}$ is a lower and upper solutions of (26).

This implies that the problem (26) admits a maximal solution u^* and a minimal solution u_*. ■

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