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Synchronous dynamics of a delayed two-coupled oscillator

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Abstract. This paper presents a detailed analysis on the dynamics of a delayed two-coupled oscillator. Linear stability of the model is investigated by analyzing the associated characteristic transcendental equation. By means of the equivariant Hopf bifurcation theorem, we not only investigate the effect of time delay on the spatio-temporal patterns of periodic solutions emanating from the trivial equilibrium, but also derive the formula to determine the direction and stability of Hopf bifurcation. Moreover, we illustrate our results by numerical simulations.

Keywords: periodic solution, Hopf bifurcation, stability.

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1 Introduction

Synchronization phenomena are common in nature (see Nijmeijer and Rodriguez-Angeles [26] and references therein). An important avenue of study in synchronization focuses on coupled oscillators. One classical example is the Kuramoto model [22], which assumes full connectivity of the network. By using a combination of the Lyapunov functional method, matrix inequality techniques and properties of Kronecker product, Alofi *et al*. [1] investigated a so-called power-rate synchronization problem for the collective dynamics among genetic oscillators with unbounded time-varying delay. Wang *et al*. [27] investigated the synchronization of coupled Duffing-type oscillators. By means of the residue harmonic balance method, Xiao *et al*. [29] investigated the approximations to the periodic oscillations of the fractional order van der Pol equation.

The study of the dynamical behavior of oscillating systems is a central issue in physics and in mathematics. These systems provide basic and general results that found major applications not only in physics, but also in all the other branches of science. The harmonic oscillator is the simplest, and more fundamental theoretical model of oscillatory phenomena. Damped and forced oscillators provide, also, very fundamental results in physics and engineering.

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In this paper, we study the existence and stability of periodic orbits in a delayed two-coupled harmonic oscillator modelled by the following system of delay differential equations

$$\ddot{u}_i(t) + u_i(t) + \varepsilon \dot{u}_i(t) = \varepsilon f(u_{i+1}(t-\tau)), \tag{1.1}$$

where $f \in C^1(\mathbb{R};\mathbb{R})$ with $f(0) = 0$, $\tau \geq 0$ and $\varepsilon > 0$ are constants, and as well as in all subsequent expressions, the index i is taken to modulo 2, so that, for instance, $x_3 = x_1$. We also assume that each oscillator has no self-feedback and signal transmission is delayed due to the finite switching speed of oscillator. It can be seen that in system (1.1) the growth rate of one oscillator depends on the feedback from the other. Such a network has been found in a variety of neural structures and even in chemistry and electrical engineering. Despite the low number of units, two-oscillator networks with delay often display the same dynamical behaviors as large networks and, can thus be used as prototypes for us to understand the dynamics of large networks with delayed feedback (see, for example, [8,14–16,20,21]).

Here, we emphasize the importance of temporal delays in the coupling between cells, since in many chemical and biological oscillators (cells coupled via membrane transport of ions), the time needed for transport of processing of chemical components or signals may be of considerable length. Since we have symmetric coupling of identical oscillators, (1.1) has the reflection symmetry of interchange of two oscillators. Although model (1.1) is a little simple, it allows us to have a depth analysis and then to gain insight into possible mechanisms behind the observed behavior.

It is easy to see that every continuous $\psi = (\psi_1, \psi_2)^T : [-\tau, 0] \to \mathbb{R}^2$ uniquely determines a solution $u^\psi = (u_1^\psi, u_2^\psi)^T : [-\tau, \infty) \to \mathbb{R}^2$ of (1.1) with $u^\psi|_{[-\tau,0]} = \psi$. Clearly, if $\psi_1 = \psi_2$ then the uniquely determined solution satisfies $u_1^\psi = u_2^\psi$ in $[-\tau, \infty)$ and can be characterized by the scalar delay differential equation

$$\ddot{u}(t) + u(t) + \varepsilon \dot{u}(t) = \varepsilon f(u(t-\tau)), \tag{1.2}$$

Such solutions are said to be synchronous. Equation (1.2) has been used to model a variety of other biological and physical phenomena, and studied by many researchers (see, for example, [24,25]). More precisely, the local stability analysis has been discussed by a lot of investigators [3–6, 9, 10, 23] and complex dynamics including limit cycles and tori are also obtained by Campbell [7], Hou and Guo [19], Zhang and Guo [30, 31]. The existence of nonconstant periodic solutions of (1.2) has been proved in [2].

Our goal in this paper is to study the existence and stability of periodic orbits of (1.1). The plan for this paper is as follows. In Section 2, we consider the linear stability of the trivial solution (1.1). Section 3 is devoted to the spatio-temporal patterns of Hopf bifurcated periodic solutions when the the trivial solution lose its stability. In Section 4, we discuss the bifurcation direction and stability of periodic solutions emerging from from the trivial solution. In Section 5, we illustrate our results with some numerical simulations. Finally, some conclusions are made in Section 6.

2 Properties of bifurcated periodic solutions

Let $C([-\tau, 0], \mathbb{R}^2)$ denote the Banach space of continuous mapping from $[-\tau, 0]$ into $\mathbb{R}^2$ equipped with the supremum norm $\|\phi\| = \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)|$ for $\phi \in C([-\tau, 0], \mathbb{R}^2)$. In what follows, if $\sigma \in \mathbb{R}$, $A \geq 0$ and $x : [\sigma - 1, \sigma + A] \to \mathbb{R}^2$ is a continuous mapping, then $x_t \in C([-\tau, 0], \mathbb{R}^2)$, $t \in [\sigma, \sigma + A]$, is defined by $x_t(\theta) = x(t + \theta)$ for $-\tau \leq \theta \leq 0$. For

any two integers a and b, define $N(a) = \{a, a+1, \dots\}$, $N(a,b) = \{a, a+1, \dots, b\}$ when $a \le b$. $\mathbb{N} = N(0)$.

The linearization of (1.1) at the origin leads to

$$\ddot{u}_i(t) + u_i(t) + \varepsilon \dot{u}_i(t) = \varepsilon\alpha u_{i+1}(t-\tau), \qquad i \ (\mathrm{mod}\ 2). \tag{2.1}$$

where $\alpha = f'(0)$. It is well-known that the associated characteristic equation of (2.1) takes the form

$$\det \Delta(\tau, \lambda) = 0,$$

where the characteristic matrix $\Delta(\tau, \lambda)$ is given by

$$\Delta(\tau, \lambda) = (\lambda^2 + 1 + \varepsilon\lambda)\mathrm{Id} - \alpha\varepsilon M e^{-\lambda\tau}, \qquad \lambda \in \mathbb{C}$$

with Id denoting the identity matrix and

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

By an easy computation, we have

$$\det \Delta(\tau, \lambda) = (\lambda^2 + 1 + \varepsilon\lambda)^2 - (\alpha\varepsilon)^2 e^{-2\lambda\tau}.$$

Hence, by factoring the right side of the above equality, we can obtain

$$\det \Delta(\tau, \lambda) = [\lambda^2 + 1 + \varepsilon\lambda - \varepsilon\alpha \exp\{-\lambda\tau\}][\lambda^2 + 1 + \varepsilon\lambda + \varepsilon\alpha \exp\{-\lambda\tau\}]. \tag{2.2}$$

Thus, $\lambda \in \mathbb{C}$ is a zero of $\det \Delta(\tau, \lambda)$ if and only if there exists a $j \in \{0, 1\}$ such that

$$p_j(\tau, \lambda) \triangleq \lambda^2 + \varepsilon\lambda + 1 - (-1)^j \varepsilon\alpha \exp\{-\lambda\tau\} = 0. \tag{2.3}$$

We know that $\pm \mathrm{i}\omega$ ($\omega > 0$) are a pair of purely imaginary zeros of $p_j(\tau, \cdot)$ if and only if ω satisfies

$$\begin{cases} 1 - \omega^2 = (-1)^j \varepsilon\alpha \cos(\tau\omega), \\ -\omega = (-1)^j \alpha \sin(\tau\omega). \end{cases} \tag{2.4}$$

It follows from (2.4) that

$$\omega^4 + (\varepsilon^2 - 2)\omega^2 + 1 - \varepsilon^2\alpha^2 = 0. \tag{2.5}$$

The number of positive solutions to (2.5) may be zero, one, or two, which is determined by the signs of $(\varepsilon^2 + 4\alpha^2 - 4)$ and $(\varepsilon|\alpha| - 1)$. In fact, the curves $\varepsilon^2 + 4\alpha^2 = 4$ and $\varepsilon|\alpha| = 1$ divide the right half (ε, α)-plane into six regions:

$$\begin{aligned}
D_1 &= \left\{(\varepsilon, \alpha) \in \mathbb{R}^+ \times \mathbb{R} : \ \varepsilon^2 + 4\alpha^2 < 4\right\}, \\
D_2^+ &= \left\{(\varepsilon, \alpha) \in \mathbb{R}^+ \times \mathbb{R} : \ \varepsilon\alpha > 1\right\}, \\
D_2^- &= \left\{(\varepsilon, \alpha) \in \mathbb{R}^+ \times \mathbb{R} : \ \varepsilon\alpha < -1\right\}, \\
D_3^+ &= \left\{(\varepsilon, \alpha) \in \mathbb{R}^+ \times \mathbb{R} : \ \sqrt{1 - \frac{\varepsilon^2}{4}} < \alpha < \frac{1}{\varepsilon}, \ \varepsilon < \sqrt{2}\right\}, \\
D_3^- &= \left\{(\varepsilon, \alpha) \in \mathbb{R}^+ \times \mathbb{R} : \ \sqrt{1 - \frac{\varepsilon^2}{4}} < -\alpha < \frac{1}{\varepsilon}, \ \varepsilon < \sqrt{2}\right\}, \\
D_4 &= \left\{(\varepsilon, \alpha) \in \mathbb{R}^+ \times \mathbb{R} : \ \varepsilon^2 + 4\alpha^2 > 4, \ \varepsilon|\alpha| < 1, \ \varepsilon > \sqrt{2}\right\}.
\end{aligned}$$

More precisely, equation (2.5) has two positive solutions $\omega = \beta_\pm$ when $(\varepsilon, \alpha) \in D_3^+ \cup D_3^-$, has exactly one positive solution $\omega = \omega_+$ when $(\varepsilon, \alpha) \in D_2^+ \cup D_2^-$, and has no positive solution when $(\varepsilon, \alpha) \in D_1 \cup D_4$, where

$$\beta_\pm = \sqrt{1 - \frac{\varepsilon^2}{2} \pm \varepsilon\sqrt{\alpha^2 - 1 + \frac{\varepsilon^2}{4}}}.$$

Thus, the Hopf bifurcation values of $\tau_{j,k}^\pm$ are given as follows:

$$\tau_{j,k}^+ = \begin{cases} \frac{1}{\beta_+}(\arcsin\frac{\beta_+}{|\alpha|} + 2k\pi) & \text{if } (-1)^j\alpha \le -1, \\ \frac{1}{\beta_+}(\pi - \arcsin\frac{\beta_+}{|\alpha|} + 2k\pi) & \text{if } -1 \le (-1)^j\alpha < 0, \\ \frac{1}{\beta_+}(2\pi - \arcsin\frac{\beta_+}{|\alpha|} + 2k\pi) & \text{if } 0 < (-1)^j\alpha \le 1, \\ \frac{1}{\beta_+}(\pi + \arcsin\frac{\beta_+}{|\alpha|} + 2k\pi) & \text{if } (-1)^j\alpha > 1, \end{cases}$$

$$\tau_{j,k}^- = \begin{cases} \frac{1}{\beta_-}(\pi - \arcsin\frac{\beta_-}{|\alpha|} + 2k\pi) & \text{if } (-1)^j\alpha < 0, \\ \frac{1}{\beta_-}(2\pi - \arcsin\frac{\beta_-}{|\alpha|} + 2k\pi) & \text{if } (-1)^j\alpha > 0, \end{cases}$$

for $k \in \mathbb{N}_0$ and $j \in \{0, 1\}$. Thus, we have the following results about the zeros of $p_j(\tau, \lambda)$.

Lemma 2.1. *If $(\varepsilon, \alpha) \in D_1 \cup D_4$, then for each j and $\tau \ge 0$, $p_j(\tau, \cdot)$ has only zero points λ satisfying $\operatorname{Re}\lambda < 0$ and has no purely imaginary zero point.*

Proof. It follows from $(\varepsilon, \alpha) \in D_1 \cup D_4$ that $\varepsilon|\alpha| < 1$. We first notice the fact that there exist at most a finite number of zeros of $p_j(\tau, \lambda)$ in right half-plane for each $j \in \{0, 1\}$. Indeed, for any zero λ of $p_j(\tau, \lambda)$,

$$|\lambda^2 + \varepsilon\lambda + 1| = \varepsilon|\alpha| \exp\{-\tau \operatorname{Re}\lambda\}.$$

This implies that there is a real number η such that all zeros of $p_j(\tau, \lambda)$ satisfy $\operatorname{Re}\lambda < \eta$. Clearly, $p_j(\tau, \lambda)$ is an entire function. Hence, there can only be a finite number of zeros of $p_j(\tau, \lambda)$ in any compact set. Namely, there exist only a finite number of zeros in any vertical strip in the complex plane. We can regard λ as the continuous function of τ according to the implicit function theorem. Notice that

$$p_j(0, \lambda) = \lambda^2 + \varepsilon\lambda + 1 - (-1)^j\varepsilon\alpha = 0,$$

which has exactly two zero points with negative real parts. Recall the fact that all zeros of $p_j(\tau, \lambda)$ are simple and continuously depend on τ, then there exists a critical value τ_0 such that $p_j(\tau, \lambda)$ has only zero points with negative real parts if $\tau \in [0, \tau_0)$, and that as τ increases and passes through τ_0, the zero points with positive real parts may appear. Thus, $p_j(\tau_0, \lambda)$ has a pair of purely imaginary zero points $\pm i\omega$, where $\omega > 0$ is a solution to (2.5). In view of $(\varepsilon, \alpha) \in D_1 \cup D_4$, we see that $\tau_0 = \infty$. This completes the proof. □

Lemma 2.2. *Assume that $\varepsilon|\alpha| > 1$, i.e., $(\varepsilon, \alpha) \in D_2^+ \cup D_2^-$.*

(i) $p_j(\tau, \cdot)$ has a pair of simple imaginary zero $\pm i\beta_+$ at and only at $\tau = \tau_{j,k}^+ > 0$, $k \in \mathbb{N}$.

(ii) For each fixed pair $(j, k) \in \{0, 1\} \times \mathbb{N}_0$ such that $\tau_{j,k}^+ > 0$, there exist $\delta_{j,k}^1 > 0$ and C^1-mapping $\lambda_{j,k} : (\tau_{j,k}^+ - \delta_{j,k}^1, \tau_{j,k}^+ + \delta_{j,k}^1) \to \mathbb{C}$ such that $\lambda_{j,k}(\tau_{j,k}^+) = i\beta_+$ and $\lambda_{j,k}(\tau)$ is a zero of $p_j(\tau, \lambda)$ for all $\tau \in (\tau_{j,k}^+ - \delta_{j,k}^1, \tau_{j,k}^+ + \delta_{j,k}^1)$. Moreover, $\frac{d}{d\tau}\operatorname{Re}\{\lambda_{j,k}(\tau)\}|_{\tau=\tau_{j,k}^+} > 0$.

(iii) For each fixed $(\varepsilon,\alpha)\in D_2^+$, $p_0(\tau,\lambda)$ *has exactly one zero point with positive real parts when* $\tau\in[0,\tau_{0,0}^+)$, *and exactly* $2k+3$ *zero points with positive real parts* $\tau\in(\tau_{0,k}^+,\tau_{0,k+1}^+)$; $p_1(\tau,\lambda)$ *has only zero points with negative real parts when* $\tau\in[0,\tau_{1,0}^+)$, *and exactly* $2k+2$ *zero points with positive real parts* $\tau\in(\tau_{1,k}^+,\tau_{1,k+1}^+)$.

(iv) For each fixed $(\varepsilon,\alpha)\in D_2^-$, $p_1(\tau,\lambda)$ *has exactly one zero point with positive real parts when* $\tau\in[0,\tau_{1,0}^+)$, *and exactly* $2k+3$ *zero points with positive real parts* $\tau\in(\tau_{1,k}^+,\tau_{1,k+1}^+)$; $p_0(\tau,\lambda)$ *has only zero points with negative real parts when* $\tau\in[0,\tau_{0,0}^+)$, *and exactly* $2k+2$ *zero points with positive real parts* $\tau\in(\tau_{0,k}^+,\tau_{0,k+1}^+)$.

Proof. (i) Let $\lambda=\mathrm{i}v$ be a zero of $p_j(\tau,\lambda)$. Then, we get $v=\beta_+$, then $1-\beta_+^2=(-1)^j\varepsilon\alpha\cos(\tau\beta_+)$ and $\beta_+=(-1)^j\alpha\sin(\tau\beta_+)$. Namely, $\tau=\tau_{j,k}^+$ for some j and k.

(ii) The existence of $\delta_{j,k}^1$ and the mapping $\lambda_{j,k}$ follow from the implicit function theorem. We now differentiate the equality $p_j(\tau,\lambda)=0$ with respect to τ to get

$$\begin{aligned}\frac{d}{d\tau}\operatorname{Re}\{\lambda_{j,k}(\tau)\}\,|_{\tau_{j,k}^+}&=-\operatorname{Re}\left\{\frac{\lambda_{j,k}(\tau_{j,k}^+)(\lambda_{j,k}^2(\tau_{j,k}^+)-\varepsilon\lambda_{j,k}(\tau_{j,k}^+)+1)}{2\lambda_{j,k}(\tau_{j,k}^+)-\varepsilon+\tau_{j,k}^+(\lambda_{j,k}^2(\tau_{j,k}^+)-\varepsilon\lambda_{j,k}(\tau_{j,k}^+)+1)}\right\}\\&=\frac{\beta_+^2(\varepsilon^2+2\beta_+^2-2)}{\left[\tau_{j,k}^+(1-\beta_+^2)-\varepsilon\right]^2+\beta_+^2(2-\varepsilon\tau_{j,k}^+)^2}\\&=\frac{\varepsilon\beta_+^2\sqrt{\varepsilon^2+4\alpha^2-4}}{\left[\tau_{j,k}^+(1-\beta_+^2)-\varepsilon\right]^2+\beta_+^2(2-\varepsilon\tau_{j,k}^+)^2}>0.\end{aligned}$$

This completes the proof.

(iii) Using a similar argument as that in the proof of Lemma 2.1, we can regard λ as the continuous function of τ according to the implicit function theorem. If $\tau=0$ and $(\varepsilon,\alpha)\in D_2^+$ (respectively, $(\varepsilon,\alpha)\in D_2^-$), then $p_0(\tau,\lambda)$ (respectively, $p_1(\tau,\lambda)$) has exactly one zero point with positive real parts but $p_1(\tau,\lambda)$ (respectively, $p_0(\tau,\lambda)$) has only zero points with negative real parts. Recall the fact that all zeros of $p_j(\tau,\lambda)$ are simple and continuously depend on τ, then there exists a critical value $\tau_{j,0}$ such that the number of zero points of $p_j(\tau,\lambda)$ with positive real parts keeps the same if $\tau\in[0,\tau_0)$. It follows from conclusions (i) and (ii) that as τ increases and passes through τ_0, only one zero point of $p_j(\tau,\lambda)$, denoted by $\lambda^*(\tau)$, varies from a complex number with a negative real part to a purely imaginary number and then to a complex number with a positive real part. In fact, the proof of conclusion (i) yields that $\tau_0=\tau_{j,0}^+>0$.

We can repeat the same analysis to conclude that there exists next critical value $\tau_{j,1}$ such that the number of zero points of $p_j(\tau,\lambda)$ with positive real parts keeps the same if $\tau\in(\tau_{j,0}^+,\tau_{j,1})$, and that as τ increases and passes through $\tau_{j,1}$, a new zero point of $p_j(\tau,\lambda)$ varies from a complex number with a negative real part to a purely imaginary number and then to a complex number with a positive real part. Similarly, it follows from the proof of conclusion (i) that $\tau_{j,1}=\tau_{j,1}^+$.

By induction, we can draw the conclusion that the number of zeros of $p_j(\tau,\lambda)$ with positive real parts increases as τ increases. This completes the proof. □

Lemma 2.3. *Assume that* $(\varepsilon,\alpha)\in D_3^+\cup D_3^-$.

(i) $p_j(\tau,\cdot)$ *has a pair of simple imaginary zeros* $\pm \mathrm{i}\beta_\pm$ *at and only at* $\tau=\tau_{j,k}^\pm>0$, $k\in\mathbb{N}$.

(ii) For each fixed pair $(j,k)\in\{0,1\}\times\mathbb{N}_0$ *such that* $\tau_{j,k}^+>0$, *there exist* $\delta_{j,k}^1>0$ *and* C^1*-mapping* $\lambda_{j,k}:(\tau_{j,k}^+-\delta_{j,k}^1,\tau_{j,k}^++\delta_{j,k}^1)\to\mathbb{C}$ *such that* $\lambda_{j,k}(\tau_{j,k}^+)=\mathrm{i}\beta_+$ *and* $\lambda_{j,k}(\tau)$ *is a zero of* $p_j(\tau,\lambda)$ *for all* $\tau\in(\tau_{j,k}^+-\delta_{j,k}^1,\tau_{j,k}^++\delta_{j,k}^1)$. *Moreover,* $\frac{d}{d\tau}\operatorname{Re}\{\lambda_{j,k}(\tau)\}|_{\tau=\tau_{j,k}^+}>0$.

(iii) For each fixed pair $(j,k)\in\{0,1\}\times\mathbb{N}_0$ *such that* $\tau_{j,k}^->0$, *there exist* $\delta_{j,k}^1>0$ *and* C^1*-mapping* $\lambda_{j,k}:(\tau_{j,k}^--\delta_{j,k}^1,\tau_{j,k}^-+\delta_{j,k}^1)\to\mathbb{C}$ *such that* $\lambda_{j,k}(\tau_{j,k}^-)=\mathrm{i}\beta_-$ *and* $\lambda_{j,k}(\tau)$ *is a zero of* $p_j(\tau,\lambda)$ *for all* $\tau\in(\tau_{j,k}^--\delta_{j,k}^1,\tau_{j,k}^-+\delta_{j,k}^1)$. *Moreover,* $\frac{d}{d\tau}\operatorname{Re}\{\lambda_{j,k}(\tau)\}|_{\tau=\tau_{j,k}^-}<0$.

(iv) For each fixed $j\in\{0,1\}$, *there exists a nonnegative integer* m_j *such that* $p_j(\tau,\lambda)$ *has exactly one pair of zeros with positive real parts when* $\tau_{j,k-1}^+<\tau<\tau_{j,k-1}^-$ *with* $\tau_{j,-1}^\pm=0$, *and all zeros of* $p_j(\tau,\lambda)$ *have negative real parts when* $\tau_{j,k-1}^-<\tau<\tau_{j,k}^+$ *with* $\tau_{j,-1}^-=0$, $k=0,1,2,\dots,m_j$, *and* $p_j(\tau,\lambda)$ *has only zeros with negative real parts when* $\tau>\tau_{0,m_j}^+$.

Proof. Using a similar argument as that in the proof of Lemma 2.2, we can prove conclusions (i)–(iii). We now prove conclusion (iv). First we notice the fact that

$$\tau_{0,0}^+=\frac{2\pi-\arcsin\frac{\beta_+}{\alpha}}{\beta_+}<\frac{2\pi-\arcsin\frac{\beta_-}{\alpha}}{\beta_-}=\tau_{0,0}^-,$$

when $0<\alpha\le 1$, and

$$\tau_{0,0}^+=\frac{\pi+\arcsin\frac{\beta_+}{\alpha}}{\beta_+}<\frac{2\pi-\arcsin\frac{\beta_-}{\alpha}}{\beta_-}=\tau_{0,0}^-,$$

when $\alpha>1$,

$$\tau_{0,0}^+=\frac{\pi-\arcsin\frac{\beta_+}{\alpha}}{\beta_+}<\frac{\pi-\arcsin\frac{\beta_-}{\alpha}}{\beta_-}=\tau_{0,0}^-,$$

when $-1<\alpha\le 0$, and

$$\tau_{0,0}^+=\frac{\arcsin\frac{\beta_+}{\alpha}}{\beta_+}<\frac{\pi-\arcsin\frac{\beta_-}{\alpha}}{\beta_-}=\tau_{0,0}^-,$$

when $\alpha<-1$. Then we have $\tau_{j,0}^+<\tau_{j,0}^-$, $j=0,1$. It follows from $\tau_{j,k+1}^\pm-\tau_{j,k}^\pm=\frac{2\pi}{\beta_\pm}$ and $\beta_+>\beta_-$ that

$$\tau_{j,k+1}^+-\tau_{j,k}^+<\tau_{j,k+1}^--\tau_{j,k}^-.$$

Thus, there exists an nonnegative integer m_j such that

$$\tau_{j,0}^+<\tau_{j,0}^-<\tau_{j,1}^+<\tau_{j,1}^-<\cdots<\tau_{j,m_j}^+<\tau_{j,m_j+1}^+<\tau_{j,m_j}^-.$$

□

Lemma 2.4.

(i) For any fixed $(\varepsilon,\alpha)\in D_1\cup D_4$ *and* $\tau\ge 0$, *all solutions* λ *to the characteristic equation* $\det\Delta(\tau,\lambda)=0$ *satisfy* $\operatorname{Re}\lambda<0$. *Furthermore, no Hopf bifurcation occurs at the origin.*

(ii) For any fixed $(\varepsilon,\alpha)\in D_2^+\cup D_2^-$ *and* $\tau\ge 0$, *the characteristic equation* $\det\Delta(\tau,\lambda)=0$ *has at least one solution* λ *satisfying* $\operatorname{Re}\lambda>0$. *Furthermore, system (1.1) undergoes Hopf bifurcation at the origin near* $\tau=\tau_{j,k}^+$, $j\in\{0,1\}$, $k\in\mathbb{N}_0$.

(iii) For any fixed $(\varepsilon,\alpha) \in D_3^+ \cup D_3^-$ *and* $\tau \geq 0$, *all solutions* λ *to the characteristic equation* $\det\Delta(\tau,\lambda)=0$ *satisfy* $\mathrm{Re}\lambda<0$ *when* $\tau \in [\cup_{k=0}^{m_0}(\tau_{0,k-1}^-,\tau_{0,k}^+)] \cap [\cup_{k=0}^{m_1}(\tau_{1,k-1}^-,\tau_{1,k}^+)]$. *Furthermore, system* (1.1) *undergoes Hopf bifurcation at the origin near* $\tau=\tau_{j,k}^{\pm}$, $j\in\{0,1\}$, $k\in\mathbb{N}_0$, *where* m_0 *and* m_1 *are given in Lemma 2.3.*

It follows from the above lemma that we have the following results on the linear stability of the equilibrium $x^*=0$ of system (1.1).

Theorem 2.5.

(i) If $(\varepsilon,\alpha)\in D_1\cup D_4$ *and* $\tau\geq 0$, *then the equilibrium* $x^*=0$ *of system* (1.1) *is stable for all* $\tau\geq 0$.

(ii) If $(\varepsilon,\alpha)\in D_2^+\cup D_2^-$, *then the equilibrium* $x^*=0$ *of system* (1.1) *is unstable for all* $\tau\geq 0$.

(iii) If $(\varepsilon,\alpha)\in D_3^+\cup D_3^-$, *then the equilibrium* $x^*=0$ *of system* (1.1) *is stable for all* $\tau\in [\cup_{k=0}^{m_0}(\tau_{0,k-1}^-,\tau_{0,k}^+)] \cap [\cup_{k=0}^{m_1}(\tau_{1,k-1}^-,\tau_{1,k}^+)]$, *where* m_0 *and* m_1 *are given in Lemma 2.3.*

3 Spatio-temporal patterns of periodic solutions

Throughout this section, we always assume that $(\varepsilon,\alpha)\in D_2^+\cup D_2^-\cup D_3^+\cup D_3^-$. Lemmas 2.2 and 2.3, together with the Hopf theorem (see, pp. 332 in [18]), imply that a Hopf bifurcation for (1.1) occurs at each $\tau=\tau_{j,k}^{\pm}>0$. Namely, in every neighborhood of $(x^*=0,\tau^*=\tau_{j,k}^{\pm})$ there is a unique branch of periodic solutions $x^{j,k}(t,\tau)$ with $x^{j,k}(t,\tau)\to 0$ as $\tau\to\tau_{j,k}^{\pm}$. The period $P^{j,k}(\gamma,\tau)$ of $x^{j,k}(t,\tau)$ satisfies that $P^{j,k}(\gamma,\tau)\to 2\pi/\beta_{\pm}$ as $\tau\to\tau_{j,k}^{\pm}$.

In what follows, we aim to analyze the spatio-temporal patterns of these bifurcated periodic solutions. It is well-known that the symmetry of a system is important in determining the patterns of oscillation that it can support. To explore the possible (spatial) symmetry of the system (1.1), we need to introduce two compact Lie groups. One is the cycle group $\mathbb{S}^1$, the other is $\mathbb{Z}_2$, the cyclic group of order 2 (the order of a finite group is the number of the elements it contains). Clearly, we have

Lemma 3.1. *Denote by* ρ *the generator of the cyclic subgroup* $\mathbb{Z}_2$. *Define the action of* $\mathbb{Z}_2$ *on* $\mathbb{R}^2$ *by*

$$\rho\cdot(x_1,x_2)=(x_2,x_1)\quad \textit{for all } (x_1,x_2)\in\mathbb{R}^2.$$

Then system (1.1) *is* $\mathbb{Z}_2$*-equivariant.*

Proof. Define a mapping F: $C([-\tau,0],\mathbb{R}^2)\to\mathbb{R}^2$ by

$$(F(\phi))_i=-\phi_i(0)+\varepsilon\dot{\phi}_i(0)+\varepsilon f(\phi_{i+1}(-\tau))$$

for $\phi\in C([-\tau,0],\mathbb{R}^2)$ and i (mod 2). Then

$$\begin{aligned}(F(\rho\cdot\phi))_i=&-(\rho\cdot\phi)_i(0)+\varepsilon(\rho\cdot\dot{\phi})_i(0)+\varepsilon f((\rho\cdot\phi)_{i+1}(-\tau))\\ =&-\phi_{i+1}(0)+\varepsilon\dot{\phi}_{i+1}(0)+\varepsilon f(\phi_i(-\tau))\\ =&(\rho\cdot F(\phi))_i\end{aligned}$$

for $\phi\in C([-\tau,0],\mathbb{R}^2)$ and i (mod 2). Namely, F is $\mathbb{Z}_2$-equivariant. This completes the proof. □

Let $\omega_0 = \beta_\pm$, $\omega = \frac{2\pi}{\omega_0}$ and P_ω the Banach space of continuous ω-periodic mappings $x : \mathbb{R} \to \mathbb{R}^2$. $\mathbb{Z}_2 \times S^1$ acts on P_ω by

$$(\delta, e^{i\theta}) \cdot x(t) = \delta \cdot x(t+\theta), \qquad e^{i\theta} \in \mathbb{S}^1,\ x \in P_\omega,\ \delta \in \mathbb{Z}_2,$$

Let SP_ω denote the subspace of P_ω consisting of all ω-periodic solutions of (1.1) with $\tau = \tau_{j,k}$. Then

$$SP_\omega = \left\{ x_1\epsilon^1 + x_2\epsilon^2 :\ x_1, x_2 \in \mathbb{R} \right\},$$

where ϵ^1 and ϵ^2 are 2-dimensional vector functions defined on $\mathbb{R}$ with the m-th components defined by $\epsilon^1_m(t) = \cos(\omega_0 t + (m-1)j\pi)$ and $\epsilon^2_m(t) = \sin(\omega_0 t + (m-1)j\pi)$ for $t \in \mathbb{R}$ respectively. Note that for all $t \in \mathbb{R}$ and $m \in \{1, 2\}$,

$$\begin{aligned}
(\rho \cdot \epsilon^1(t))_m &= \epsilon^1_{m+1}(t) = \cos(\omega_0 t + mj\pi) \\
&= \cos(\omega_0 t + (m-1)j\pi + j\pi) \\
&= \epsilon^1_m\left(t + \frac{j\pi}{\omega_0}\right), \\
(\rho \cdot \epsilon^2(t))_m &= \epsilon^2_{m+1}(t) = \sin(\omega_0 t + mj\pi) \\
&= \sin(\omega_0 t + (m-1)j\pi + j\pi) \\
&= \epsilon^2_m\left(t + \frac{j\pi}{\omega_0}\right).
\end{aligned}$$

Then we have

$$\rho \cdot \epsilon^1 = \epsilon^1\left(t + \frac{j}{2}\omega\right), \qquad \rho \cdot \epsilon^2 = \epsilon^2\left(t + \frac{j}{2}\omega\right). \tag{3.1}$$

It has been verified in [28] that, under usual non-resonance and transversality conditions, for every subgroup $\Sigma \leq \mathbb{Z}_2 \times \mathbb{S}^1$ such that the Σ-fixed-point subspace of SP_ω (i.e., $\mathrm{Fix}(\Sigma, SP_\omega) = \{x \in SP_\omega : \gamma x = x \text{ for all } \gamma \in \Sigma\}$) is of dimension 2, symmetric delay differential equations has a bifurcation of periodic solutions whose spatial-temporal symmetry can be completely characterized by Σ.

Here, we consider the following subgroup of $\mathbb{Z}_3 \times \mathbb{S}^1$ to describe the symmetry of periodic solution of system (1.1) (see [13] for more details):

$$\Sigma = \langle (\rho, e^{-\mathrm{i}\frac{j}{2}\omega}) \rangle.$$

The two equations in (3.1) imply that the Σ-fixed-point set of SP_ω is itself, i.e., $\mathrm{Fix}(\Sigma, SP_\omega) = SP_\omega$. Thus, the general symmetric local Hopf bifurcation theorem (Theorem 2.1 in [28]) enables us we obtain the following result on the existence of smooth local Hopf bifurcations of wave solutions.

Theorem 3.2. *Assume that $(\varepsilon, \alpha) \in D_2^+ \cup D_2^- \cup D_3^+ \cup D_3^-$. Then near each $\tau^\pm_{j,k} > 0$, there exists a branch of small-amplitude periodic solutions of (1.1) emerging from the trivial solution $x = 0$. More precisely, there exist $\varepsilon^{j,k} > 0$ and $\delta^{j,k} > 0$ such that for each $\theta \in [0, 2\pi]$, $\alpha \in (0, \varepsilon^{j,k}_\pm)$, system (1.1) with $\tau = \tau^\pm_{j,k} + \tau^{j,k}(\alpha, \theta)$ has a periodic solution $x^{j,k} = x^{j,k}(t; \alpha, \theta)$ with period $\omega^{j,k} = \omega^{j,k}(\alpha, \theta)$ such that*

$$x^{j,k}_i(t) = x^{j,k}_{i+1}\left(t - \frac{j\omega^{j,k}}{2}\right), \qquad i = 0, 1 \tag{3.2}$$

$$x_i^{j,k}(t;\alpha,\theta) = \alpha\left[\cos\theta\epsilon_i^1(t) + \sin\theta\epsilon_i^2(t)\right] + o(|\alpha|)$$
$$= \alpha\cos\left(\omega_0 t + (i-1)j\pi - \theta\right) + o(|\alpha|)$$

as $\alpha \to 0$. *The mapping* $(x^{j,k}, \tau^{j,k}, \omega^{j,k}) : (0, \varepsilon^{j,k}) \times [0, 2\pi] \to C(\mathbb{R}, \mathbb{R}^3) \times \mathbb{R} \times \mathbb{R}$ *is continuously differentiable and*

$$\omega^{j,k}(0,\theta) = \frac{2\pi}{\omega_0}, \qquad \tau^{j,k}(0,\theta) = 0.$$

Furthermore, if $|\tau - \tau_{j,k}| < \delta^{j,k}$ *and* $|\omega - \frac{2\pi}{\beta_{j,k}}| < \delta^{j,k}$ *then every* ω*-periodic solution of* (1.1) *satisfying* $x_i(t) = x_{i+1}(t - j\omega^{j,k})$, *and* $\sup_{t\in\mathbb{R}} |x(t)| < \delta^{j,k}$ *must be given by* $x^{j,k}(t;\alpha,\theta)$ *for some* $\alpha \in (0, \varepsilon^{j,k})$ *and* $\theta \in [0, 2\pi)$.

We call the above periodic solutions *discrete waves*. They are also called *synchronous oscillations* (if $j = 0$) or *phase-locked oscillations* (if $j \neq 0$) as each neuron oscillates just like others except not necessarily in phase with each other.

4 Properties of bifurcated periodic solutions

Theorem 3.2 means that in every neighborhood of $(x^* = 0, \tau^* = \tau_{j,k}^{\pm})$ there is a unique branch of periodic solutions with the spatio-temporal pattern (3.2). In order to be able to analyze the Hopf bifurcation in more detail, we compute the reduced system on the center manifold associated with the pair of conjugate complex, purely imaginary solutions $\Lambda = \{i\omega_0, -i\omega_0\}$ of the characteristic equation, where $\omega_0 = \beta_{\pm}$. By this reduction we can determine the Hopf bifurcation direction, i.e., to answer the question of whether the bifurcating branch of periodic solution exists locally for all $\tau > \tau_{j,k}^{\pm}$ (supercritical bifurcation) or $\tau < \tau_{j,k}^{\pm}$ (subcritical bifurcation). Throughout this section, we always assume that the function f satisfies

(P1). $f \in C^2(\mathbb{R}, \mathbb{R})$, $uf(u) \neq 0$ when $u \neq 0$.

To simplify the presentation, we first note that with the transformation

$$(w_1, w_2, w_3, w_4) = (u_1, \dot{u}_1, u_2, \dot{u}_2),$$

we can rewrite (1.1) as the following system of delay differential equations

$$\begin{aligned}
\dot{w}_1(t) &= w_2(t),\\
\dot{w}_2(t) &= -w_1(t) - \varepsilon w_2(t) + \varepsilon f(w_3(t-\tau)),\\
\dot{w}_3(t) &= w_4(t),\\
\dot{w}_4(t) &= -w_3(t) - \varepsilon w_4(t) + \varepsilon f(w_1(t-\tau)).
\end{aligned} \tag{4.1}$$

Recall that the characteristic matrix $\Delta^*(\tau,\lambda)$ of the linearization of (4.1) is given by

$$\Delta^*(\tau,\lambda) = \begin{bmatrix} \lambda & -1 & 0 & 0\\ 1 & \lambda+\varepsilon & -\alpha\varepsilon e^{-\lambda\tau} & 0\\ 0 & 0 & \lambda & -1\\ -\alpha\varepsilon e^{-\lambda\tau} & 0 & 1 & \lambda+\varepsilon \end{bmatrix}, \quad \lambda \in \mathbb{C},$$

then

$$\det\Delta^*(\tau_{j,k}^{\pm}, \pm i\omega_0) = 0$$

for all $j \in \{0,1\}$ and $k \in \mathbb{N}_0$. In particular,

$$\Delta^*(\tau_{j,k}^{\pm}, \mathrm{i}\omega_0)v_j = 0, \qquad \Delta^*(\tau_{j,k}^{\pm}, -\mathrm{i}\omega_0)\overline{v_j} = 0. \tag{4.2}$$

where $v_j = (1, \mathrm{i}\omega_0, (-1)^j, (-1)^j\mathrm{i}\omega_0)^T$. According to Theorem 3.2, near each $\tau = \tau_{j,k}^{\pm}$, there exists a branch of small-amplitude periodic solutions of (4.1) bifurcated from the trivial solution $u = 0$. The spatio-temporal pattern of the bifurcated periodic solution takes the form

$$u_i^{j,k}(t) = u_{i+1}^{j,k}\left(t - \frac{j\omega}{2}\right), \qquad i \ (\mathrm{mod}\ 2),$$

where ω represents its period and is sufficiently near to $2\pi/\omega_0$. Our purpose is to compute the reduced system of (4.1) on the center manifold associated with the pair of conjugate complex, purely imaginary solutions $\Lambda = \{\mathrm{i}\omega_0, -\mathrm{i}\omega_0\}$ of the characteristic equation.

Let us give the Taylor expansion of the right hand side of (4.1). Then we can rewrite (4.1) as

$$\dot{x}(t) = L_\tau x_t + G(x_t, \tau) \tag{4.3}$$

with

$$L_\tau\varphi = (\varphi_2(0), -\varphi_1(0) - \varepsilon\varphi_2(0) + \varepsilon\alpha\varphi_3(-\tau), \varphi_4(0), -\varphi_3(0) - \varepsilon\varphi_4(0) + \varepsilon\alpha\varphi_1(-\tau))^T$$

and

$$\begin{aligned} G(\varphi,\tau) = {} & \frac{\varepsilon f''(0)}{2}(0, \varphi_3^2(-\tau), 0, \varphi_1^2(-\tau))^T \\ & + \frac{\varepsilon f'''(0)}{6}(0, \varphi_3^3(-\tau), 0, \varphi_1^3(-\tau))^T + o(|(0, \varphi_3^3(-\tau), 0, \varphi_1^3(-\tau))^T|) \end{aligned}$$

for all $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^T \in C([-\tau_{j,k}^{\pm}, 0], \mathbb{R}^4)$. By the Riesz representation theorem, there exists an 4×4 matrix-valued function $\eta(\cdot, \tau) : [-\tau, 0] \to \mathbb{R}^4$ whose components each have bounded variation and are such that

$$L_\tau\varphi = \int_{-\tau}^{0} d\eta(\theta, \mu)\varphi(\theta) \quad \text{for } \varphi \in C([-\tau, 0], \mathbb{R}^4).$$

Next, we define for $\varphi \in C^1([-\tau, 0], \mathbb{R}^4)$,

$$A_\tau\varphi = \begin{cases} d\varphi/d\theta, & \text{if} \quad \theta \in [-\tau, 0), \\ \int_{-\tau}^{0} d\eta(\xi, \tau)\varphi(\xi) = L_\tau\varphi, & \text{if} \quad \theta = 0. \end{cases} \tag{4.4}$$

Let $\varphi_j(\theta)$ be the eigenvector for $A_{\tau_{j,k}^{\pm}}$ associated with $i\omega_0$; namely,

$$A_{\tau_{j,k}^{\pm}}\varphi_j(\theta) = \mathrm{i}\omega_0\varphi_j(\theta). \tag{4.5}$$

In view of (4.2), we can choose $\varphi_j(\theta) = v_j e^{i\omega_0\theta}$ for $\theta \in [-\tau_{j,k}^{\pm}, 0]$. So, the center space at $\tau = \tau_{j,k}^{\pm}$ and in complex coordinates is $X = \mathrm{span}\{\varphi_j, \overline{\varphi}_j\}$. Hence, $\Phi = (\varphi_j, \overline{\varphi}_j)$ is a basis for the center space X. The adjoint operator $A^*_{\tau_{j,k}^{\pm}}$ is defined by

$$A^*_{\tau_{j,k}^{\pm}}\psi = \begin{cases} -d\psi/d\xi, & \text{if} \quad \xi \in (0, \tau_{j,k}^{\pm}], \\ \int_{-\tau_{j,k}^{\pm}}^{0} \psi(-t)d\eta(t, \tau_{j,k}^{\pm}), & \text{if} \quad \xi = 0. \end{cases}$$

Note that the domains of $A_{\tau_{j,k}^{\pm}}$ and$A^*_{\tau_{j,k}^{\pm}}$ are $C^1([-\tau_{j,k}^{\pm},0],\mathbb{R}^4)$ and $C^1([0,\tau_{j,k}^{\pm}],\mathbb{R}^{4*})$, respectively, where for convenience in computation we shall allow functions with range $\mathbb{C}^4$ instead of $\mathbb{R}^4$. It follows from (4.5) that $\pm i\omega_0$ are also eigenvalues for $A^*_{\tau_{j,k}^{\pm}}$, and there is a nonzero row-vector function $\psi_j^*(\xi)$, $\xi\in[0,\tau_{j,k}^{\pm}]$ such that

$$A^*_{\tau_{j,k}^{\pm}}\psi_j=-i\omega_0\psi_j.$$

Then, $\Psi=(\psi_j,\overline{\psi}_j)^T$ is a basis for the adjoint space X^*. In order to construct coordinates to describe the center manifold $\mathcal{C}_{\tau_{j,k}^{\pm}}$ near to the origin, we need an inner product as follows:

$$\langle\psi,\varphi\rangle=\overline{\psi}(0)\varphi(0)-\int_{\theta=-\tau_{j,k}^{\pm}}^{0}\int_{\xi=0}^{\theta}\overline{\psi}(\xi-\theta)d\eta(\theta,\tau_{j,k}^{\pm})\varphi(\xi)d\xi \tag{4.6}$$

for $\psi\in C([0,\tau_{j,k}^{\pm}],\mathbb{R}^{4*})$ and $\varphi\in C([-\tau_{j,k}^{\pm},0],\mathbb{R}^4)$. Then, as usual,

$$\langle\psi,A_{\tau_{j,k}^{\pm}}\varphi\rangle=\langle A^*_{\tau_{j,k}^{\pm}}\psi,\varphi\rangle$$

for $(\varphi,\psi)\in \mathrm{Dom}(A_{\tau_{j,k}^{\pm}})\times\mathrm{Dom}(A^*_{\tau_{j,k}^{\pm}})$. We normalize ψ_j by the condition $\langle\psi_j,\varphi_j\rangle=1$ and $\langle\psi_j,\overline{\varphi}_j\rangle=0$. By direct computation, we obtain that

$$\psi_j(\xi)=u_je^{i\omega_0\xi},$$

where $u_j=\overline{D}_j(-i\omega_0+\varepsilon,1,(-1)^{j+1}(i\omega_0-\varepsilon),(-1)^j)$ and

$$D_j=\frac{1}{2}\left[2i\omega_0-\varepsilon+(-1)^j\varepsilon\alpha\tau_{j,k}^{\pm}e^{-i\omega_0\tau_{j,k}^{\pm}}\right]^{-1}.$$

Let $Q=\{\varphi\in C^1([-\tau_{j,k}^{\pm},0],\mathbb{R}^4))\,|\,(\Psi,\varphi)=0\}$, then $C([-\tau_{j,k}^{\pm},0],\mathbb{R}^4)=X\bigoplus Q$. So Eq. (4.3) can be written in the following abstract form

$$\frac{dU_t}{dt}=\mathcal{A}_\tau U_t+X_0G(U_t,\tau), \tag{4.7}$$

where

$$X_0(\theta)=\begin{cases}0, & \theta\in[-\tau,0),\\ \mathrm{Id}_4, & \theta=0.\end{cases}$$

Then by using the decomposition

$$U_t=2\,\mathrm{Re}\{z(t)\varphi_j\}+y_t,\qquad z(t)\in\mathbb{C},\qquad y_t\in Q^1:=Q\bigcap C^1([-\tau_{j,k}^{\pm},0],\mathbb{R}^4),$$

we decompose (4.3) as

$$\begin{aligned}\dot{z}&=i\omega_0z+\overline{u}_jG^*(2\,\mathrm{Re}\{z\varphi_j\}+y,\tau),\\ \dot{y}&=A_{\tau_{j,k}^{\pm}}y+[X_0-\Phi\overline{\Psi(0)}]G^*(2\,\mathrm{Re}\{z\varphi_j\}+y,\tau),\end{aligned} \tag{4.8}$$

where $z\in\mathbb{C}$, $y\in Q^1$, and $G^*(x_t,\tau)=L_\tau x_t-L_{\tau_{j,k}^{\pm}}x_t+G(x_t,\tau)$.

As the formulas to be developed for the bifurcation direction and stability are all relative to $\tau=\tau_{j,k}^{\pm}$ only, we set $\tau=\tau_{j,k}^{\pm}$ in (4.8) and obtain a center manifold $y=W(z,\bar{z})$ with the range in Q. The flow of (4.8) on the center manifold can be written as

$$U_t=\Phi\cdot(z(t),\bar{z}(t))^T+W(z(t),\bar{z}(t)),$$

where

$$\dot{z}(t) = \mathrm{i}\omega_0 z(t) + g^{(j)}(z,\bar{z}), \tag{4.9}$$

with

$$\begin{aligned} g^{(j)}(z,\bar{z}) &= \overline{u}_j G^*(2\mathrm{Re}\{z\varphi_j\} + W(z,\bar{z}),\tau) \\ &= g_{20}^{(j)}\frac{z^2}{2} + g_{11}^{(j)} z\bar{z} + g_{02}^{(j)}\frac{\bar{z}^2}{2} + g_{21}^{(j)}\frac{z^2\bar{z}}{2} + \cdots. \end{aligned}$$

Hence we have

$$\begin{aligned} g_{20}^{(j)} &= [1+(-1)^j] D_j \varepsilon f''(0) e^{-2i\omega_0\tau_{j,k}^{\pm}}, \\ g_{11}^{(j)} &= [1+(-1)^j] D_j \varepsilon f''(0), \\ g_{02}^{(j)} &= [1+(-1)^j] D_j \varepsilon f''(0) e^{-2i\omega_0\tau_{j,k}^{\pm}}, \end{aligned}$$

and

$$\begin{aligned} g_{21}^{(j)} &= (-1)^j D_j \varepsilon f'''(0) e^{-i\omega_0\tau_{j,k}^{\pm}} \\ &\quad + \frac{1}{2} D_j \varepsilon f''(0) \left[W_{20,3}(-\tau_{j,k}^{\pm}) e^{i\omega_0\tau_{j,k}^{\pm}} + 2W_{11,3}(-\tau_{j,k}^{\pm}) e^{-i\omega_0\tau_{j,k}^{\pm}} \right] \\ &\quad + \frac{(-1)^j}{2} D_j \varepsilon f''(0) \left[W_{20,1}(-\tau_{j,k}^{\pm}) e^{i\omega_0\tau_{j,k}^{\pm}} + 2W_{11,1}(-\tau_{j,k}^{\pm}) e^{-i\omega_0\tau_{j,k}^{\pm}} \right]. \end{aligned} \tag{4.10}$$

So in order to compute $g_{21}^{(j)}$, we need to compute $W_{11} = (W_{11,1}, W_{11,2}, W_{11,3}, W_{11,4})$ and $W_{20} = (W_{20,1}, W_{20,2}, W_{20,3}, W_{20,4})$.

Since $W(z(t),\bar{z}(t))$ satisfies

$$\begin{aligned} \dot{W} &= \dot{x}_t - \varphi_j \dot{z}(t) - \bar{\varphi}_j \dot{\bar{z}}(t) \\ &= \mathcal{A}_{\tau_{j,k}^{\pm}} x_t + X_0 G(x_t, \tau_{j,k}^{\pm}) - \varphi_j \dot{z}(t) - \bar{\varphi}_j \dot{\bar{z}}(t) \\ &= \mathcal{A}_{\tau_{j,k}^{\pm}} W + X_0 G(x_t, \tau_{j,k}^{\pm}) - \varphi_j g(z,\bar{z}) - \bar{\varphi}_j \bar{g}(z,\bar{z}) \\ &= \mathcal{A}_{\tau_{j,k}^{\pm}} W + H_{20}\frac{z^2}{2} + H_{11} z\bar{z} + H_{02}\frac{\bar{z}^2}{2} + \cdots \end{aligned} \tag{4.11}$$

then by using the chain rule

$$\dot{W} = \frac{\partial W(z,\bar{z})}{\partial z}\dot{z} + \frac{\partial W(z,\bar{z})}{\partial \bar{z}}\dot{\bar{z}},$$

we have

$$\begin{cases} (2\mathrm{i}\omega_0 - \mathcal{A}_{\tau_{n,\lambda}}) W_{20} = H_{20} \\ -\mathcal{A}_{\tau_{n,\lambda}} W_{11} = H_{11}. \end{cases} \tag{4.12}$$

Note that

$$-\varphi_j(\theta) g(z,\bar{z}) - \bar{\varphi}_j(\theta)\bar{g}(z,\bar{z}) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta) z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots$$

for $-\tau_{j,k}^{\pm} \le \theta < 0$, then we have

$$\begin{aligned} H_{20}(\theta) &= -\varphi_j(\theta) g_{20}^{(j)} - \bar{\varphi}_j(\theta)\bar{g}_{02}, \\ H_{11}(\theta) &= -\varphi_j(\theta) g_{11}^{(j)} - \bar{\varphi}_j(\theta)\bar{g}_{11} \end{aligned} \tag{4.13}$$

for $-\tau_{j,k}^{\pm} \le \theta < 0$. It follows that

$$W_{20}(\theta) = \frac{\mathrm{i}g_{20}^{(j)}}{\omega_0}\varphi_j(\theta) + \frac{\mathrm{i}\bar{g}_{02}}{3\omega_0}\bar{\varphi}_j(\theta) + Ee^{2\mathrm{i}\omega_0\theta}, \tag{4.14}$$

and

$$W_{11}(\theta) = -\frac{\mathrm{i}g_{11}^{(j)}}{\omega_0}\varphi_j(\theta) + \frac{\mathrm{i}\bar{g}_{11}}{\omega_0}\bar{\varphi}_j(\theta) + F. \tag{4.15}$$

Note that

$$H_{20}(0) = -[\varphi_j(0)g_{20}^{(j)} + \bar{\varphi}_j(0)\bar{g}_{02}] + \varepsilon f''(0)e^{-2\mathrm{i}\omega_0\tau_{j,k}^{\pm}}(0,1,0,1)^T.$$

It follows from (4.11) and (4.12) and the definition of $\mathcal{A}_{\tau_{n,\lambda}}$ that

$$(2\mathrm{i}\omega_0 - \mathcal{A}_{\tau_{j,k}^{\pm}})Ee^{2\mathrm{i}\omega_0\theta}|_{\theta=0} = f''(0)e^{-2\mathrm{i}\omega_0\tau_{j,k}^{\pm}}(0,1,0,1)^T.$$

Namely,

$$\Delta^*(\tau_{j,k}^{\pm}, 2\mathrm{i}\omega_0)E = -f''(0)e^{-2\mathrm{i}\omega_0\tau_{j,k}^{\pm}}(0,1,0,1)^T. \tag{4.16}$$

As we know, $2\mathrm{i}\omega_\lambda$ is not the eigenvalue of $\mathcal{A}_{\tau_{n,\lambda}}$ and hence

$$E = -f''(0)e^{-2\mathrm{i}\omega_0\tau_{j,k}^{\pm}}[\Delta^*(\tau_{j,k}^{\pm}, 2\mathrm{i}\omega_0)]^{-1}(0,1,0,1)^T. \tag{4.17}$$

Similarly, 0 is not the eigenvalue of $\mathcal{A}_{\tau_{n,\lambda}}$ and hence

$$F = -f''(0)[\Delta^*(\tau_{j,k}^{\pm}, 0)]^{-1}(0,1,0,1)^T. \tag{4.18}$$

It is well known that the following quantities determine the direction and stability of bifurcating periodic orbits (see [11, 12, 17]):

$$\begin{aligned}
C_1(0) &= \frac{\mathrm{i}}{2\omega_0}\left(g_{11}^{(j)}g_{20}^{(j)} - 2|g_{11}^{(j)}|^2 - \frac{|g_{02}^{(j)}|^2}{3}\right) + \frac{g_{21}^{(j)}}{2},\\
\mu_2 &= -\frac{\mathrm{Re}(C_1(0))}{\mathrm{Re}(\lambda'(\tau_{j,k}^{\pm}))},\\
\beta_2 &= 2\,\mathrm{Re}(C_1(0)),\\
T_2 &= -\frac{\mathrm{Im}(C_1(0)) + \mu_2\,\mathrm{Im}(\lambda'(\tau_{j,k}^{\pm}))}{\tau_{j,k}^{\pm}}.
\end{aligned}$$

We have the following results:

(i) μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0$ (respectively, $\mu_2 < 0$), then the bifurcating periodic solutions exist for $\tau > \tau_{j,k}^{\pm}$ (respectively, $\tau < \tau_{j,k}^{\pm}$) and the bifurcation is called forward (respectively, backward);

(ii) β_2 determines the stability of bifurcating periodic solutions: the bifurcating periodic solutions are orbitally asymptotically stable (respectively, unstable) on the center manifold if $\beta_2 < 0$ (respectively, $\beta_2 > 0$);

(iii) T_2 determines the period of the bifurcating periodic solutions: the period increases (respectively, decreases) if $T_2 > 0$ (respectively, $T_2 < 0$).

If we further assume the following:

(H). $f'(0)=\alpha$, $f''(0)=0$ and $f'''(0)\neq 0$,

then $E=F=0$, $g_{20}^{(j)}=g_{11}^{(j)}=g_{02}^{(j)}=0$, and

$$g_{21}^{(j)}=(-1)^j D_j \varepsilon f'''(0) e^{-\mathrm{i}\omega_0\tau_{j,k}^{\pm}}.$$

Namely, $\operatorname{sign}\operatorname{Re}\{g_{21}^{(j)}\}=\operatorname{sign}\{f'(0)f'''(0)\}$. Therefore, we have the following corollary.

Corollary 4.1. *Under the assumption (H), system (1.1) undergoes a Hopf bifurcation at $\tau=\tau_{j,k}^{\pm}$, $j\in\{0,1\}$, $k\in\mathbb{N}_0$. The direction of Hopf bifurcation is determined by $\operatorname{sign}\{f'(0)f'''(0)\}$. More precisely, the Hopf bifurcation at $\tau=\tau_{j,k}^{+}$ is subcritical (respectively, supercritical) if $f'(0)f'''(0)>0$ (respectively, <0), while the Hopf bifurcation at $\tau=\tau_{j,k}^{-}$ is supercritical (respectively, subcritical) if $f'(0)f'''(0)>0$ (respectively, <0).*

5 Numerical simulations

In this section, we will give some numerical simulations to illustrate our theoretical results. We consider the following system

$$\begin{cases}\ddot{u}_0(t)=-u_0(t)-\varepsilon\dot{u}_0(t)+\varepsilon\tanh(u_1(t-\tau))\\ \ddot{u}_1(t)=-u_1(t)-\varepsilon\dot{u}_1(t)+\varepsilon\tanh(u_0(t-\tau)).\end{cases}\tag{5.1}$$

It is easy to check that $\alpha=1$, $\beta_+=1$ and $\tau_{j,k}^{+}=(1.5+2k-j)\pi$, $j\in\{0,1\}$, $k\in\mathbb{N}_0$. We first consider system (5.1) with $\varepsilon=3(\sqrt{2}-1)$. Note that $(3(\sqrt{2}-1),1)\in D_2^{+}$. It follows from Lemmas 2.2 and 2.4 that the trivial equilibrium is unstable and system (5.1) undergoes Hopf bifurcation at the origin near $\tau=\tau_{j,k}^{+}$, $j\in\{0,1\}$, $k\in\mathbb{N}_0$. If $\tau=1.5\pi$, as shown in Figure 5.1, the trivial equilibrium is unstable and the trajectory starting from sufficiently close to the trivial equilibrium will be away from a neighborhood of trivial equilibrium.

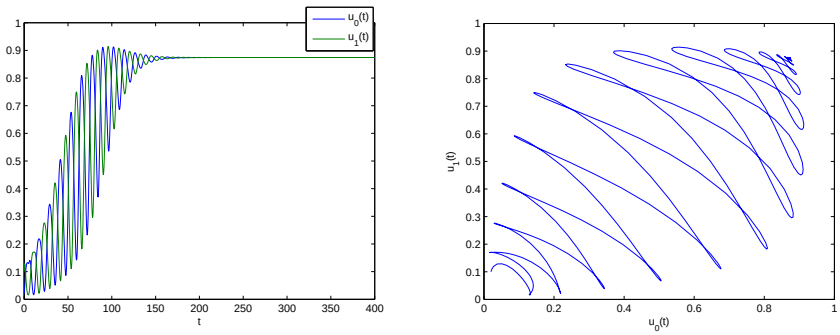
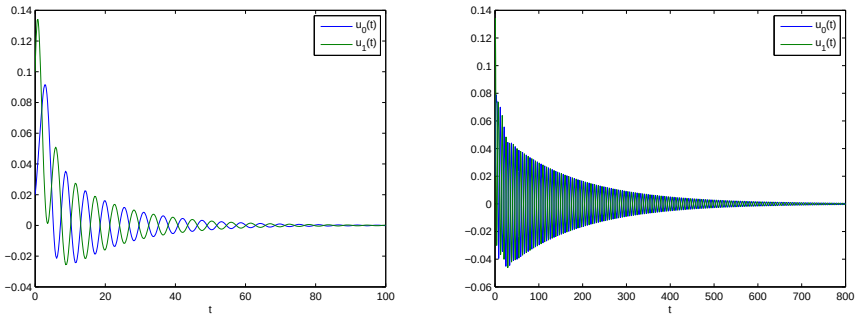


Figure 5.1: Simulations of system (5.1) with $\varepsilon=3(\sqrt{2}-1)$ and $\tau=1.5\pi$ illustrate that the trivial equilibrium is unstable.

Consider system (5.1) with $\varepsilon=1.5(\sqrt{2}-1)$. Note that $(1.5(\sqrt{2}-1),1)\in D_3^{+}$, $\beta_-=0.7836$, and $\tau_{j,k}^{-}=1.2762(1.7134+2k-j)\pi$, $j\in\{0,1\}$, $k\in\mathbb{N}_0$. It follows from Lemma 2.3 that system (5.1) undergoes Hopf bifurcation at the origin near $\tau=\tau_{j,k}^{\pm}$, $k\in\mathbb{N}_0$, $j\in\{0,1\}$. Note that

$$\begin{aligned}&\tau_{1,0}^{+}=0.5\pi, \quad \tau_{1,0}^{-}\approx 0.9104\pi, \quad \tau_{0,0}^{+}=1.5\pi, \quad \tau_{0,0}^{-}\approx 2.1866\pi,\\ &\tau_{1,1}^{+}=2.5\pi, \quad \tau_{1,1}^{-}\approx 3.4628\pi, \quad \tau_{0,1}^{+}=3.5\pi, \quad \tau_{0,1}^{-}\approx 4.7390\pi.\end{aligned}$$

Theorem 2.5 means that the trivial equilibrium of system (5.1) with $\varepsilon = 1.5(\sqrt{2}-1)$ is stable when $\tau \in [0, \tau^+_{1,0}) \cup (\tau^-_{1,0}, \tau^+_{0,0}) \cup (\tau^-_{0,0}, \tau^+_{1,1}) \cup (\tau^-_{1,1}, \tau^+_{0,1})$, and is unstable when $\tau \in (\tau^+_{1,0}, \tau^-_{1,0}) \cup (\tau^+_{0,0}, \tau^-_{0,0}) \cup (\tau^+_{1,1}, \tau^-_{1,1}) \cup (\tau^+_{0,1}, \tau^-_{0,1})$. It follows from $0 < 0.3\pi < \tau^+_{1,0}$, $\tau^-_{1,0}\pi < \pi < \tau^+_{0,0}$, and $\tau^+_{1,1} < 3\pi < \tau^-_{1,1}$ that the trivial equilibrium is stable when either $\tau = 0.3\pi$ or $\tau = \pi$ (see Figure 5.2), but is unstable when $\tau = 3\pi$ (see Figure 5.3). As τ increases and crosses the critical values $\tau^+_{1,0}$ and $\tau^+_{1,1}$ (respectively, $\tau^+_{0,0}$ and $\tau^+_{0,1}$), the trivial equilibrium loses its stability and a synchronous (phased-locked) periodic solution bifurcating from the trivial equilibrium, as depicted in Figures 5.4 and 5.5.



(i) (ii)

Figure 5.2: Simulations of system (5.1) with $\varepsilon = 1.5(\sqrt{2}-1)$ and (i) $\tau = 0.3\pi$, (ii) $\tau = \pi$ illustrate that trivial equilibrium is stable.

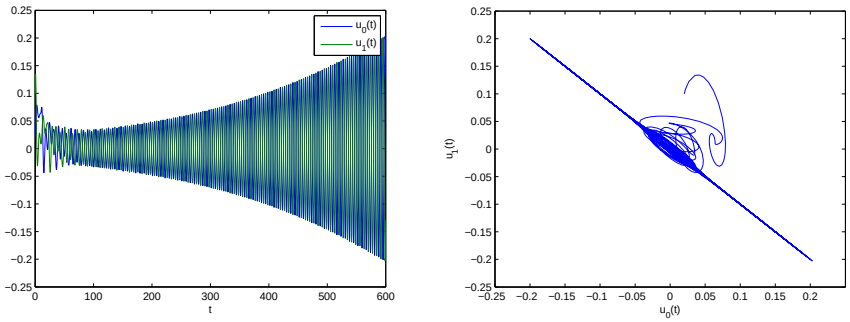


Figure 5.3: Simulations of system (5.1) with $\varepsilon = 1.5(\sqrt{2}-1)$ and $\tau = 3\pi$ illustrate that trivial equilibrium is unstable.

In what follows, we take $f(x) = -\tanh(x)$ and consider the following system

$$\begin{cases} \ddot{u}_0(t) = -u_0(t) - \varepsilon \dot{u}_0(t) - \varepsilon \tanh(u_1(t-\tau)), \\ \ddot{u}_1(t) = -u_1(t) - \varepsilon \dot{u}_1(t) - \varepsilon \tanh(u_0(t-\tau)). \end{cases} \tag{5.2}$$

Now, we have $\alpha = f'(0) = -1$, $\beta_+ = 1$ and $\tau^+_{j,k} = (\frac{1}{2} + 2k + j)\pi$, $j \in \{0,1\}$, $k \in \mathbb{N}_0$. Consider system (5.2) with $\varepsilon = 3(\sqrt{2}-1)$. Note that $(3(\sqrt{2}-1), -1) \in D_2^-$. It follows from Lemmas 2.2 and 2.4 that the trivial equilibrium is unstable and system (5.2) undergoes Hopf bifurcation at the origin near $\tau = \tau^+_{j,k}$, $j \in \{0,1\}$, $k \in \mathbb{N}_0$. Figure 5.6 presents that the trivial equilibrium of system (5.2) with $\varepsilon = 3(\sqrt{2}-1)$ and $\tau = \pi$ is unstable, the trajectory starting from sufficiently close to the trivial equilibrium will be away from a neighborhood of the trivial equilibrium,

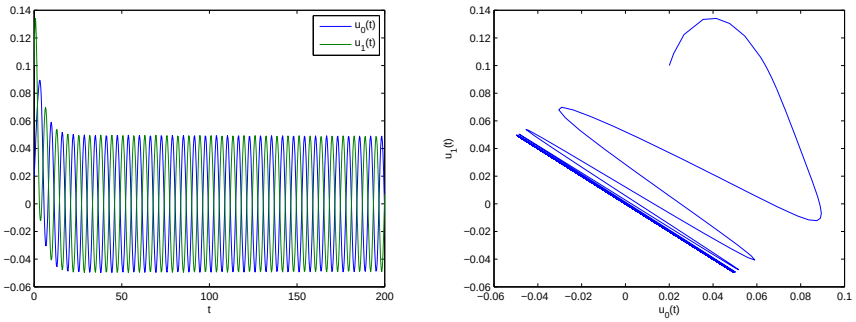


Figure 5.4: Simulations of system (5.1) with $\varepsilon = 1.5(\sqrt{2}-1)$ and $\tau = 0.5\pi$ illustrate that trivial equilibrium is unstable.

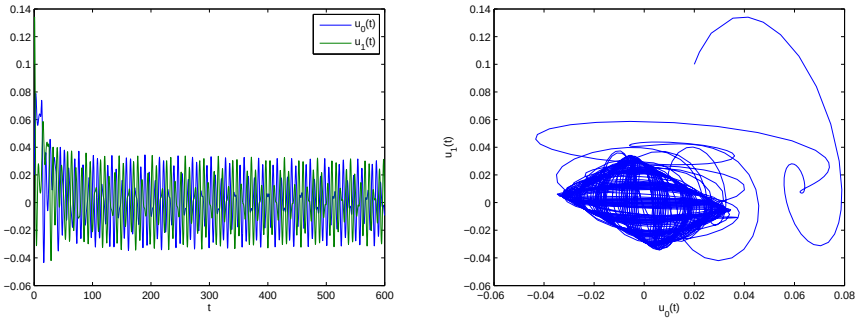


Figure 5.5: Simulations of system (5.1) with $\varepsilon = 1.5(\sqrt{2}-1)$ and $\tau = 3.5\pi$ illustrate that a periodic solution appears via Hopf bifurcation.

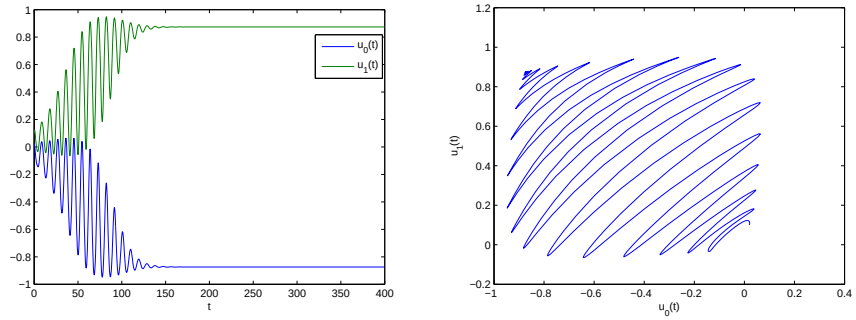


Figure 5.6: Simulations of system (5.2) with $\varepsilon = 3(\sqrt{2}-1)$ and $\tau = \pi$ illustrate that the trivial equilibrium is unstable.

solutions of system (5.2) form some interesting spatial-temporal patterns. In Figure 5.7, we see a periodic solution emerging from the origin.

Finally, consider system (5.2) with $\varepsilon = 1.5(\sqrt{2}-1)$. Note that $(1.5(\sqrt{2}-1), -1) \in D_3^-$, $\beta_- = 0.7836$, and $\tau_{j,k}^- = 1.2762(0.7134 + 2k + j)\pi$, $j \in \{0,1\}$, $k \in \mathbb{N}_0$. It follows from Lemma 2.3 that system (5.2) undergoes Hopf bifurcation at the origin near $\tau = \tau_{j,k}^{\pm}$, $k \in \mathbb{N}_0$, $j \in \{0,1\}$. Note that

$$\tau_{0,0}^+ = 0.5\pi, \quad \tau_{0,0}^- \approx 0.9104\pi, \quad \tau_{1,0}^+ = 1.5\pi, \quad \tau_{1,0}^- \approx 2.1866\pi,$$
$$\tau_{0,1}^+ = 2.5\pi, \quad \tau_{0,1}^- \approx 3.4628\pi, \quad \tau_{1,1}^+ = 3.5\pi, \quad \tau_{1,1}^- \approx 4.7390\pi.$$

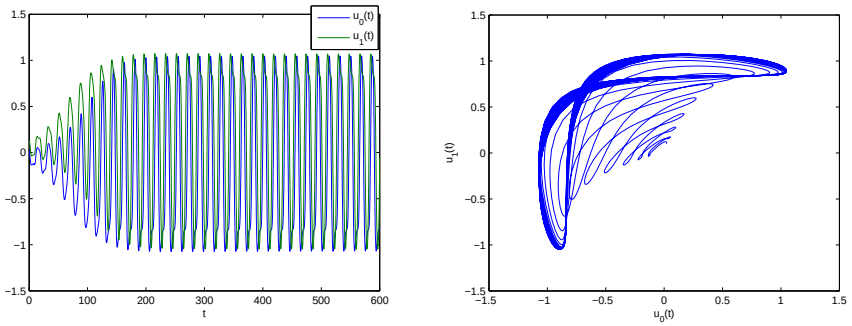


Figure 5.7: Simulations of system (5.2) with $\varepsilon = 3(\sqrt{2}-1)$ and $\tau = 2.5\pi$.

Theorem 2.5 means that the trivial equilibrium of system (5.2) with $\varepsilon = 1.5(\sqrt{2}-1)$ is stable when $\tau \in [0, \tau^+_{0,0}) \cup (\tau^-_{0,0}, \tau^+_{1,0}) \cup (\tau^-_{1,0}, \tau^+_{0,1}) \cup (\tau^-_{0,1}, \tau^+_{1,1})$, and is unstable when $\tau \in (\tau^+_{0,0}, \tau^-_{0,0}) \cup (\tau^+_{1,0}, \tau^-_{1,0}) \cup (\tau^+_{0,1}, \tau^-_{0,1}) \cup (\tau^+_{1,1}, \tau^-_{1,1})$. As shown in Figures 5.8 and 5.9, the trivial equilibrium is stable when $\tau = 1.3\pi$ and is unstable when $\tau = 2\pi$. A Hopf bifurcation occurs when $\tau = 3.5\pi$, the origin loses its stability and a synchronous periodic solution is bifurcating from the origin, as depicted in Figure 5.10.

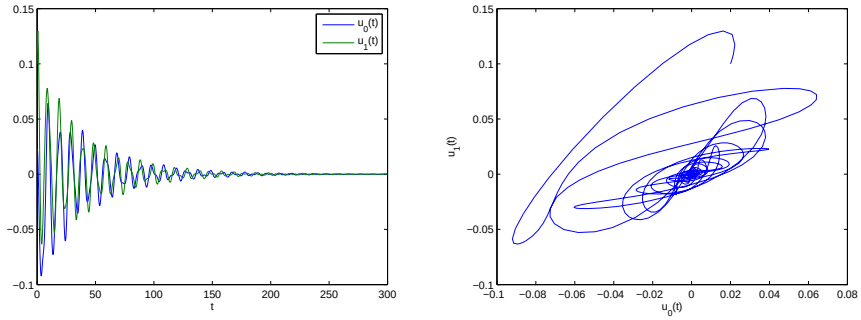


Figure 5.8: Simulations of system (5.2) with $\varepsilon = 1.5(\sqrt{2}-1)$ and $\tau = 1.3\pi$ illustrate that trivial equilibrium is stable.

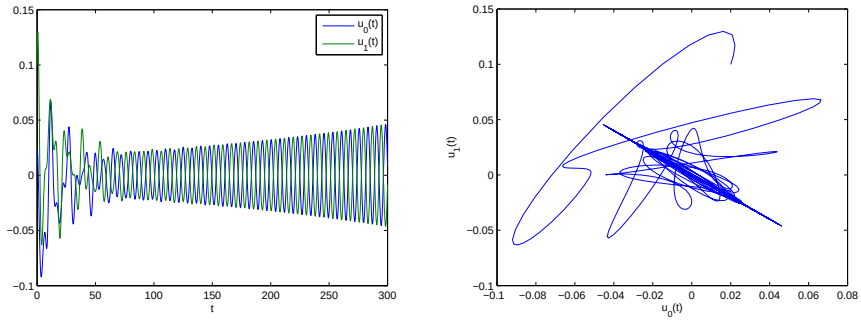


Figure 5.9: Simulations of system (5.2) with $\varepsilon = 1.5(\sqrt{2}-1)$ and $\tau = 2\pi$ illustrate that trivial equilibrium is unstable.

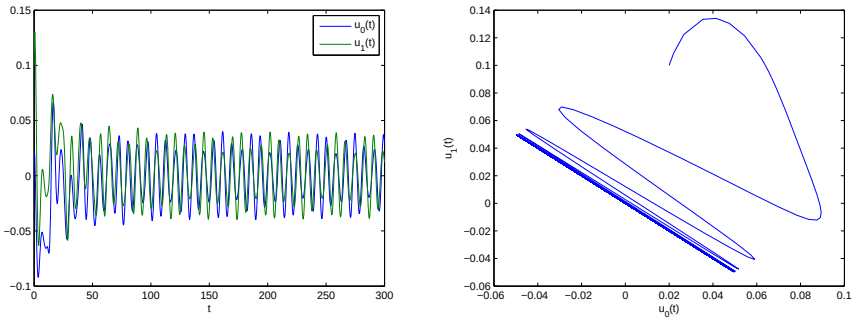


Figure 5.10: Simulations of system (5.2) with $\varepsilon = 1.5(\sqrt{2}-1)$ and $\tau = 3.5\pi$ illustrate that a periodic solution appears via Hopf bifurcation.

6 Conclusion

The goal of this paper is to study the existence and stability of periodic orbits of delay differential equations. To achieve this, a novel model based on a delayed two-coupled harmonic oscillator is proposed. The local Hopf bifurcations and the spatio-temporal patterns of Hopf bifurcating periodic orbits are also investigated. Numerical simulations are adopted to validate the theoretical results. By using different suitable parameters and coefficient numbers, the simulation results reveal that the bifurcating periodic solutions are orbitally asymptotically stable.

Acknowledgments

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