Electronic Journal of Qualitative Theory of Differential Equations
2011, No. **14**, 1-18; **http://www.math.u-szeged.hu/ejqtde/**

Upper and Lower Solutions for BVPs on the Half-line with Variable Coefficient and Derivative Depending Nonlinearity

Smaïl DJEBALI and Samira ZAHAR

Abstract

This paper is concerned with a second-order nonlinear boundary value problem with a derivative depending nonlinearity and posed on the positive half-line. The derivative operator is time dependent. Upon a priori estimates and under a Nagumo growth condition, the Schauder's fixed point theorem combined with the method of upper and lower solutions on unbounded domains are used to prove existence of solutions. A uniqueness theorem is also obtained and some examples of application illustrate the obtained results.

1 Introduction

In this work, we are concerned with the existence of solutions to the following boundary value problem

$$\begin{cases} x''(t) - k^2(t)x(t) + q(t)f(t, x(t), x'(t)) = 0, & t > 0, \\ x(0) = 0, \quad x(+\infty) = 0, \end{cases} \tag{1.1}$$

where $q \in C(0, +\infty) \cap L^1(0, \infty)$ while the nonlinearity $f : I \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and the coefficient $k : I \to (0, \infty)$ are continuous. Here $I = (0, +\infty)$ refers to the positive half-line.

Since BVPs on infinite intervals arise in many applications from physics, chemistry and biology, there has been so much work devoted to the investigation of positive solutions for such BVPs in the last couple of years (see e.g., [2, 3, 6, 16] and the references therein) where superlinear or sublinear nonlinearities are considered. The positivity of solutions is motivated by the fact that the unknown x may refer to a density, a temperature or the concentration of a product. For instance, the linear operator of derivation $-x'' + cx' + \lambda x$ $(c, \lambda > 0)$, which may be rewritten in reduced form as

2000 Mathematics Subject Classification. 34B15, 34B18, 34B40.
Key words. Lower and upper solution, infinity interval, variable coefficient, derivative depending nonlinearity, Nagumo condition, uniqueness

$-x'' + k^2 x$, stems from epidemiology and combustion theory and models the propagation of the wave front of a reaction-diffusion equation (see e.g., [6]). Methods used to investigate these problems range from the upper and lower solution techniques [15, 17] to the fixed point theory in weighted Banach spaces and the index fixed point theory on cones of some Banach spaces [1, 5, 16, 18].

When k is constant, the existence of solutions to problem (1.1) was established in [14] using the Tychonoff's fixed point theorem. It was also studied by Djebali *et al* in [7, 8, 9] where multiplicity results have been also given. In [17], B. Yan *et al* have used the upper and lower solution techniques to obtain some existence results when f is allowed to have a singularity at $x = 0$ and may change sign.

In the general case when the constant k is replaced by a bounded function $k = k(t)$, the problem was recently investigated by Ma and Zhu in [13]. The nonlinearity $f \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$ is assumed to satisfy a sublinear polynomial growth condition. The authors of [13] proved that if the parameter λ is less that some λ_0, then the following problem

$$\begin{cases} x''(t) - k^2(t)x(t) + \lambda q(t) f(t, x(t)) = 0, & t > 0, \\ x(0) = 0, \quad x(+\infty) = 0, \end{cases}$$

has a positive solution; a fixed point theorem in a cone of a Banach space has been employed. Their investigation relies heavily on estimates of the corresponding Green's function. In [11], the authors first applied fixed point index theory in cones of Banach spaces to prove existence results when $f = f(t, x)$ is positive and may exhibit a singularity at the origin with respect to the solution; then they used the Schauder's fixed point theorem together with the method of upper and lower solutions to prove existence of solutions when f is not necessarily positive.

Using an upper and lower solution method on infinity intervals, the aim of this paper is to investigate the more general problem where the nonlinearity $f = f(t, x, y)$ is derivative depending. In [12], Lian *et al.* used unbounded upper and lower solutions on noncompact intervals to prove an existence result for a class of BVPs. In [10], the authors considered problem (1.1) with $q = 1$ and used topological degree theory combined with the existence of C^1_B upper and lower solutions to prove existence of solutions on bounded intervals. Solutions are then extended to the positive half-line by means of sequential arguments. In the present paper, we complement these existence theorems via a direct approach.

This paper is organized as follows. Some preliminaries and definitions are given in Section 2. Then we will enunciate our assumptions in Section 3 and present a modified problem. In Section 4, bounded and unbounded upper and lower solutions will be established for problem (1.1) which allow us to prove correspondingly two existence results under a Nagumo type growth condition. The truncated problem is first studied. The proofs rely on suitable a priori estimates. A uniqueness result is also given in Section 5. Finally, we give two examples of application to illustrate our existence and uniqueness results.

2 Auxiliary Lemmas

Let us first enunciate an assumption regarding the function k:

($\mathcal{H}_0$) the function $k : I \to [0,\infty)$ is bounded and continuous and

$$\exists\, d \in [\underline{k}, \overline{k}], \quad \forall\, \rho > 0, \lim_{t\to\infty} e^{-\rho t}\int_0^t e^{\rho s}[k^2(s) - d^2]ds \text{ exists}$$

$$\text{where } \overline{k} := \sup_{t\in[0,\infty)} k(t) \text{ and } \underline{k} := \inf_{t\in[0,\infty)} k(t) > 0.$$

In order to construct a Green's function of the corresponding linear problem, it is necessary to know a fundamental system of solutions. The following auxiliary results are brought from [13].

Lemma 2.1. *Assume that k is bounded and continuous. Then the Cauchy problem*

$$\begin{cases} x''(t) - k^2(t)x(t) = 0, & t > 0, \\ x(0) = 0, \quad x'(0) = 1 \end{cases} \tag{2.1}$$

has a unique solution ϕ_1 defined on $[0,+\infty)$. Moreover ϕ_1 is nondecreasing and unbounded.

Lemma 2.2. *(See also [1], Thm. 7) Assume that k is bounded and continuous. Then the problem*

$$\begin{cases} x''(t) - k^2(t)x(t) = 0, & t > 0, \\ x(0) = 1, \quad \lim_{t\to+\infty} x(t) = 0 \end{cases} \tag{2.2}$$

has a unique solution ϕ_2 defined on $[0,+\infty)$ with

$$0 < \phi_2 \leq 1,\ \phi_2' < 0.$$

If further ($\mathcal{H}_0$) holds, then

$$\lim_{t\to\infty} \frac{\phi_2'(t)}{\phi_2(t)} = -d.$$

Lemma 2.3. *Assume that ($\mathcal{H}_0$) holds. Then there exists $M > 0$ such that $\sup_{t\in[0,\infty)} \phi_1(t)\phi_2(t) < M$.*

Lemma 2.4. *Assume ($\mathcal{H}_0$) holds. Then for any function $y \in L^1[0,\infty)$, the problem*

$$\begin{cases} x''(t) - k^2(t)x(t) + y(t) = 0, & t > 0, \\ x(0) = 0, \quad x(+\infty) = 0 \end{cases}$$

is equivalent to the integral equation

$$x(t) = \int_0^\infty G(t,s)y(s)ds, \quad t > 0,$$

where

$$G(t,s) = \begin{cases} \phi_1(t)\phi_2(s), & s \geq t, \\ \phi_1(s)\phi_2(t), & t \geq s. \end{cases} \tag{2.3}$$

Lemma 2.5. *For all* $(t,s) \in [0,\infty) \times [0,\infty)$, $G(t,s) < \frac{1}{2\underline{k}}$.

Moreover, we have

Lemma 2.6. *For any* $t \in (0,\infty)$, *we have*

$$0 \leq \phi_1'(t)\phi_2(t) \leq 1$$

and

$$-1 \leq \phi_1(t)\phi_2'(t) \leq 0.$$

Proof. Since $\{\phi_1, \phi_2\}$ is the fundamental system, we have that $\phi_1(t)\phi_2'(t) - \phi_1'(t)\phi_2(t) = -1$ which means that $\phi_1'(t)\phi_2(t) + [-\phi_1(t)\phi_2'(t)] = 1$. Then our claim follows from the sign and the monotonicity of ϕ_1, ϕ_2. □

Lemma 2.7. *We have*

$$k_0 := \int_0^\infty k^2(s)\phi_2(s)ds < \infty.$$

Proof. By Lemma 2.2, it is clear that the function $U = \frac{\phi_2'}{\phi_2}$ satisfies the Ricatti equation $U' + U^2 = k^2$ both with the terminal condition $U(+\infty) = -d$. Hence if, by contradiction, $U(0) = -\infty$, then $U'(0) = -\infty$, which is impossible. Thus $-\infty < U(0) < 0$ and as a consequence $-\infty < \phi_2'(0) < 0$ which implies that $0 < k_0 = -\phi_2'(0) < \infty$, as claimed. □

Now we define what we mean by lower and upper solutions.

Definition 2.1.

(a) We say that α *is a lower solution of problem* (1.1) *if* $\alpha \in \mathcal{C}^1[0,\infty) \cap \mathcal{C}^2(0,\infty)$ *and*

$$\begin{cases} \alpha''(t) - k^2(t)\alpha(t) + q(t)f(t,\alpha(t),\alpha'(t)) \geq 0, \\ \alpha(0) \leq 0, \quad \alpha(+\infty) \leq 0. \end{cases}$$

(b) A function β *is an upper solution of problem* (1.1) *if* $\beta \in \mathcal{C}^1[0,\infty) \cap \mathcal{C}^2(0,\infty)$ *and*

$$\begin{cases} \beta''(t) - k^2(t)\beta(t) + q(t)f(t,\beta(t),\beta'(t)) \leq 0, \\ \beta(0) \geq 0, \quad \beta(+\infty) \geq 0. \end{cases}$$

3 General Assumptions and a Modified Problem

We first posit some assumptions:

($\mathcal{H}_1$) *There exist $\alpha \leq \beta$ lower and upper solutions of problem* (1.1) *respectively.*

($\mathcal{H}_2$) $\alpha_0 := \sup_{t\in[0,\infty)} \{|\alpha(t)|\phi_2^{-1}(t)\} < \infty$ and $\beta_0 := \sup_{t\in[0,\infty)} \{|\beta(t)|\phi_2^{-1}(t)\} < \infty$, where $\phi_2^{-1}(t) := \frac{1}{\phi_2(t)}$.

($\mathcal{H}_3$) *There exist continuous functions $\psi : I \to [0,\infty)$ and $h : \mathbb{R} \to [1,\infty)$ such that*

$$\int_0^\infty \psi(s)q(s)ds < \infty, \tag{3.1}$$

$$\int_0^\infty \frac{ds}{h(s)} = +\infty \tag{3.2}$$

and

$$|f(t,x,y)| \leq \psi(t)h(y),\ \forall\,(t,x,y) \in D_\alpha^\beta \times \mathbb{R}, \tag{3.3}$$

where D_α^β is defined by

$$D_\alpha^\beta := \{(t,x) \in (0,\infty)\times\mathbb{R} : \alpha(t) \leq x \leq \beta(t)\}.$$

($\mathcal{H}_4$) $\alpha_1 := \sup_{t\in\mathbb{R}^+} \alpha'(t) < \infty$, $\beta_1 := \inf_{t\in\mathbb{R}^+} \beta'(t) > -\infty$, and for any $y \in \mathbb{R}$ and $t \in (0,\infty)$, we have

$$\begin{cases} y < \beta'(t) & \Longrightarrow \quad f(t,\beta(t),y) \leq f(t,\beta(t),\beta'(t)) \\ y > \alpha'(t) & \Longrightarrow \quad f(t,\alpha(t),y) \geq f(t,\alpha(t),\alpha'(t)). \end{cases}$$

($\mathcal{H}_4$)$'$

$$\alpha_1 := \sup_{t\in\mathbb{R}^+} |\alpha'(t)| < \infty \quad \text{and} \quad \beta_1 := \sup_{t\in\mathbb{R}^+} |\beta'(t)| < \infty.$$

Now, define the Banach space

$$X = \{x \in \mathcal{C}^1[0,\infty) : \lim_{t\to+\infty} x(t) \text{ and } \lim_{t\to+\infty} x'(t) \text{ exist}\}$$

equipped with the norm $\|x\| = \max\left\{\sup_{t\in[0,\infty)} |x(t)|,\ \sup_{t\in[0,\infty)} |x'(t)|\right\}$.

The following compactness criterion will be needed (see [4], p. 62).

Lemma 3.1. *Let $M \subseteq X$. Then M is relatively compact in X if the following conditions hold*

(a) M is uniformly bounded in X,

(b) the functions belonging to M and the functions belonging to $\{u: u(t) = x'(t),\, x \in M\}$ are locally equicontinuous on $[0, +\infty)$,

(c) the functions belonging to M and the functions belonging to $\{u: u(t) = x'(t),\, x \in M\}$ are equiconvergent at $+\infty$.

Given two continuous functions α and β such that $\alpha \le \beta$, we define the truncated function $\widetilde{f}$ by

$$\widetilde{f}(t,x,y) = \begin{cases} f_R(t,\beta(t),y) + \frac{\beta(t)-x}{1+|\beta(t)-x|}, & \beta(t) < x, \\ f_R(t,x,y), & \alpha(t) \le x \le \beta(t), \\ f_R(t,\alpha(t),y) + \frac{x-\alpha(t)}{1+|\alpha(t)-x|}, & x < \alpha(t). \end{cases}$$

where

$$f_R(t,x,y) = \begin{cases} f(t,x,-R), & y < -R, \\ f(t,x,y), & |y| \le R, \\ f(t,x,R), & R < y \end{cases}$$

and the real number R is such that $R > \max\{|\alpha_1|, |\beta_1|\}$. Finally, consider the modified problem

$$\begin{cases} x''(t) - k^2(t)x(t) + q(t)\widetilde{f}(t,x(t),x'(t)) = 0, & t \in (0,\infty) \\ x(0) = 0, \quad x(+\infty) = 0. \end{cases} \tag{3.4}$$

4 Existence Results

4.1 A priori Estimates

Proposition 4.1. *Assume that either $(\mathcal{H}_1)$ and $(\mathcal{H}_4)$, or $(\mathcal{H}_1)$ and $(\mathcal{H}_4)'$ hold. Then all possible solutions of problem* (3.4) *satisfy*

$$\alpha(t) \le x(t) \le \beta(t), \quad \forall\, t \in I.$$

Proof. We prove that $x(t) \le \beta(t)$, $\forall\, t \in I$. Suppose, on the contrary that $\sup_{t\in[0,\infty)}(x-\beta)(t) > 0$. Since $(x-\beta)(+\infty) = -\beta(+\infty) \le 0$ and $(x-\beta)(0) = -\beta(0) \le 0$, then there exists $t_0 \in (0,\infty)$ such that $x(t_0) - \beta(t_0) = \sup(x-\beta)(t_0) > 0$; hence $(x''-\beta'')(t_0) \le 0$ and $x'(t_0) - \beta'(t_0) = 0$. Moreover, by definition of an upper solution, we have the successive estimates:

$$\begin{aligned} (x''-\beta'')(t_0) &= k^2(t_0)x(t_0) - q(t_0)\widetilde{f}(t_0,x(t_0),x'(t_0)) - \beta''(t_0) \\ &\ge k^2(t_0)x(t_0) - q(t_0)\widetilde{f}(t_0,x(t_0),x'(t_0)) - k^2(t_0)\beta(t_0) \\ &\quad + q(t_0)f(t_0,\beta(t_0),\beta'(t_0)) \end{aligned}$$

Hence

$$\begin{aligned}(x''-\beta'')(t_0) &\geq k^2(t_0)(x-\beta)(t_0)-q(t_0)\widetilde{f}(t_0,x(t_0),\beta'(t_0))\\ &\quad +q(t_0)f(t_0,\beta(t_0),\beta'(t_0))\\ &= k^2(t_0)(x-\beta)(t_0)-q(t_0)f_R(t_0,\beta(t_0),\beta'(t_0))\\ &\quad -q(t_0)\tfrac{(\beta-x)(t_0)}{1+|(\beta-x)(t_0)|}+q(t_0)f(t_0,\beta(t_0),\beta'(t_0))\\ &> -q(t_0)\left[f_R(t_0,\beta(t_0),\beta'(t_0))-f(t_0,\beta(t_0),\beta'(t_0))\right].\end{aligned}$$

To check that the last right-hand term is nonnegative, we distinguish between two cases:

(a) In case $(\mathcal{H}_4)$ holds, consider the sub-cases:

(a1) $\beta'(t_0)<-R$ implies that $|\beta_1|>R$ which does not hold true.

(a2) If $-R\leq\beta'(t_0)\leq R$, then $f_R(t_0,\beta(t_0),\beta'(t_0))=f(t_0,\beta(t_0),\beta'(t_0))$.

(a3) If $\beta'(t_0)>R$, then $f_R(t_0,\beta(t_0),\beta'(t_0))=f(t_0,\beta(t_0),R)\leq f(t_0,\beta(t_0),\beta'(t_0))$ follows from the first part of $(\mathcal{H}_4)$.

(b) If $(\mathcal{H}_4)'$ holds then $-R\leq\beta'(t_0)\leq R$. Consequently $f_R(t_0,\beta(t_0),\beta'(t_0))=f(t_0,\beta(t_0),\beta'(t_0))$. Our claim is then proved leading to a contradiction.

Similarly, we can prove that $x(t)\geq\alpha(t)$ for every $t\in[0,\infty)$. □

Remark 4.1. *Assumption* $(\mathcal{H}_4)$ *is essential in Proposition* 4.1. *Such an hypothesis is missing to complete the proof of Theorem* 3.1 *in [12].*

4.2 The Truncated Problem

Theorem 4.1. *Under Assumptions* $(\mathcal{H}_0)$, $(\mathcal{H}_1)$, *and* $(\mathcal{H}_3)$, *the truncated problem* (3.4) *has at least one solution in* X.

Proof. Since solving problem (3.4) amounts to proving existence of a fixed point for T, let us consider the operator $T:X\longrightarrow X$ defined by

$$(Tx)(t)=\int_0^\infty G(t,s)q(s)\widetilde{f}(s,x(s),x'(s))ds. \tag{4.1}$$

(a) $T:X\longrightarrow X$ **is well defined.** Let $x\in X$. From (3.1) and (3.3), we get

$$\begin{aligned}(Tx)(t) &\leq \textstyle\int_0^\infty G(t,s)q(s)\widetilde{f}(s,x(s),x'(s))ds\\ &\leq \textstyle\int_0^\infty G(t,s)q(s)\left(\psi(s)h(x'(s))+1\right)ds\\ &\leq \tfrac{1}{2\underline{k}}\textstyle\int_0^\infty q(s)\left(H_0\psi(s)+1\right)ds<\infty,\end{aligned}$$

where $H_0 = H_0(x) = \max_{0\le t\le ||x'||} h(t)$. From the monotonicity of ϕ_1 and ϕ_1' together with Lemma 2.6, we obtain that

$$\begin{aligned}
|(Tx)'(t)| &= \left|\int_0^\infty G_t(t,s)q(s)\widetilde{f}(s,x(s),x'(s))ds\right| \\
&= \left|\int_0^t \phi_1(s)\phi_2'(t)q(s)\widetilde{f}(s,x(s),x'(s))ds + \int_t^\infty \phi_1'(t)\phi_2(s)q(s)\widetilde{f}(s,x(s),x'(s))ds\right| \\
&\le \int_0^t \left|\phi_1(t)\phi_2'(t)|q(s)|\widetilde{f}(s,x(s),x'(s))|ds + \int_t^\infty \phi_1'(s)\phi_2(s)q(s)|\widetilde{f}(s,x(s),x'(s))\right| ds \\
&\le \int_0^t q(s)\left|\widetilde{f}(s,x(s),x'(s))|ds + \int_t^\infty q(s)|\widetilde{f}(s,x(s),x'(s))\right| ds.
\end{aligned}$$

Hence

$$\begin{aligned}
|(Tx)'(t)| &\le \int_0^\infty q(s)\left|\widetilde{f}(s,x(s),x'(s))\right| ds \\
&\le \int_0^\infty q(s)\left(H_0\psi(s)+1\right)ds \\
&< \infty.
\end{aligned}$$

Lemma 2.4 implies that $\lim_{t\to\infty} Tx(t) = 0$. Moreover

$$\begin{aligned}
\lim_{t\to\infty}(Tx)'(t) &= \lim_{t\to\infty}\left[\int_0^t \phi_1(s)\phi_2'(t)q(s)\widetilde{f}(s,x(s),x'(s))ds \right.\\
&\quad \left. + \int_t^\infty \phi_1'(t)\phi_2(s)q(s)\widetilde{f}(s,x(s),x'(s))ds\right].
\end{aligned}$$

For $s \le t$, we have

$$\lim_{t\to\infty}\phi_1(s)\phi_2'(t) = \lim_{t\to\infty}\phi_1(s)\phi_2(t)\frac{\phi_2'(t)}{\phi_2(t)} = \lim_{t\to\infty} G(t,s)\lim_{t\to\infty}\frac{\phi_2'(t)}{\phi_2(t)} = 0.$$

Hence for any $\varepsilon > 0$, there exists $N > 0$ such that for $t \ge N$, we have

$$\phi_1(s)\phi_2'(t) \le \frac{\varepsilon}{\int_0^\infty q(s)(H_0\psi(s)+1)ds} := \varepsilon$$

and then

$$\int_0^t \phi_1(s)\phi_2'(t)q(s)\widetilde{f}(s,x(s),x'(s))ds \le \varepsilon\int_0^t q(s)(H_0\psi(s)+1)ds \le \varepsilon.$$

For $s \ge t$, we have

$$\phi_1'(t)\phi_2(s) \le \phi_1'(t)\phi_2(t) \le 1.$$

Therefore

$$\lim_{t\to\infty}\left|\int_t^\infty \phi_1'(t)\phi_2(s)q(s)\widetilde{f}(s,x(s),x'(s))ds\right| \le \lim_{t\to\infty}\left|\int_t^\infty q(s)(H_0\psi(s)+1)ds\right| = 0.$$

It follows that $\lim_{t\to\infty}(Tx)'(t) = 0$.

(b) $T : X \longrightarrow X$ **is continuous.** Let $(x_n)_{n\in\mathbb{N}}$ be a sequence converging to some limit x in X; then there exists $r > 0$ such that $||x|| \leq r$ and $||x_n|| \leq r$. Let $H_r = \max_{0\leq t\leq r} h(t)$. We have

$$\begin{aligned}
&\int_0^\infty q(s)|\widetilde{f}(s, x_n(s), x_n'(s)) - \widetilde{f}(s, x(s), x'(s))|ds \\
&\leq 2\int_0^\infty q(s)(H_r\psi(s) + 1)ds < \infty
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&||Tx_n - Tx|| \\
= \;& \max\left\{ \sup_{t\in[0,\infty)} \left| \int_0^\infty G(t,s)q(s)\left[\widetilde{f}(s, x_n(s), x_n'(s)) - \widetilde{f}(s, x(s), x'(s))\right] ds \right|, \right. \\
& \left. \sup_{t\in[0,\infty)} \left| \int_0^\infty G_t(t,s)q(s)\left[\widetilde{f}(s, x_n(s), x_n'(s)) - \widetilde{f}(s, x(s), x'(s))\right] ds \right| \right\} \\
\leq \;& \max\left\{ \tfrac{1}{2\underline{k}} \int_0^\infty q(s)\left|\widetilde{f}(s, x_n(s), x_n'(s)) - \widetilde{f}(s, x(s), x'(s))\right| ds, \right. \\
& \sup_{t\in[0,\infty)} \left[\int_0^t |\phi_1(t)\phi_2'(t)|q(s)\left|\widetilde{f}(s, x_n(s), x_n'(s)) - \widetilde{f}(s, x(s), x'(s))\right| ds \right. \\
& \left.\left. + \int_t^\infty \phi_1'(s)\phi_2(s)q(s)\left|\widetilde{f}(s, x_n(s), x_n'(s)) - \widetilde{f}(s, x(s), x'(s))\right| ds \right] \right\} \\
\leq \;& \max\left\{ \tfrac{1}{2\underline{k}} \int_0^\infty q(s)\left|\widetilde{f}(s, x_n(s), x_n'(s)) - \widetilde{f}(s, x(s), x'(s))\right| ds, \right. \\
& \left. \int_0^\infty q(s)\left|\widetilde{f}(s, x_n(s), x_n'(s)) - \widetilde{f}(s, x(s), x'(s))\right| ds \right\} \\
\leq \;& \max\left\{1, \tfrac{1}{2\underline{k}}\right\} \int_0^\infty q(s)\left|\widetilde{f}(s, x_n(s), x_n'(s)) - \widetilde{f}(s, x(s), x'(s))\right| ds.
\end{aligned}$$

From continuity of f, (4.2) and the Lebesgue dominated convergence theorem, the last term goes to 0 as $n \to \infty$.

(c) $T : X \longrightarrow X$ **is compact.** Let B be any bounded subset of X and let $x \in B$. Then there exists $r > 0$ such that $||x|| \leq r$. First, notice that as above we have

$$\|Tx\| \leq \max\left\{1, \frac{1}{2\underline{k}}\right\} \int_0^\infty q(s)\left(H_r\psi(s) + 1\right) ds < \infty.$$

Now, given $T > 0$ and $t_0, t_1 \in [0, T]$, we have the estimates

$$\begin{aligned} |(Tx)(t_0) - (Tx)(t_1)| &\leq \int_0^\infty |G(t_0,s) - G(t_1,s)|\, q(s)(H_r\psi(s)+1)ds \\ &\leq \int_0^T |G(t_0,s) - G(t_1,s)|\, q(s)(H_r\psi(s)+1)ds \\ &\quad + \int_T^\infty |\phi_1(t_0)\phi_2(s) - \phi_1(t_1)\phi_2(s)|\, q(s)(H_r\psi(s)+1)ds \\ &\leq \int_0^T |G(t_0,s) - G(t_1,s)|\, q(s)(H_r\psi(s)+1)ds \\ &\quad + |\phi_1(t_0) - \phi_1(t_1)| \int_0^\infty q(s)(H_r\psi(s)+1)ds. \end{aligned}$$

By (3.1), the continuity of the Green's function and the Lebesgue dominated convergence theorem, we get

$$\lim_{|t_1 - t_0| \to 0} \int_0^T |G(t_0,s) - G(t_1,s)|\, q(s)(H_r\psi(s)+1)ds = 0.$$

In addition, (3.1) and the continuity of ϕ_1 imply that

$$\lim_{|t_1 - t_0| \to 0} |\phi_1(t_0) - \phi_1(t_1)| \int_0^\infty q(s)(H_r\psi(s)+1)ds = 0.$$

Hence the right-hand term goes to 0 as $|t_1 - t_0| \to 0$. Moreover, for $t_0 \leq t_1$, the following estimates hold true

$$\begin{aligned} |(Tx)'(t_0) - (Tx)'(t_1)| &\leq \int_0^\infty |G_t(t_0,s) - G_t(t_1,s)| q(s)(H_r\psi(s)+1)ds \\ &= \int_0^{t_0} |G_t(t_0,s) - G_t(t_1,s)| q(s)(H_r\psi(s)+1)ds \\ &\quad + \int_{t_0}^{t_1} |G_t(t_0,s) - G_t(t_1,s)| q(s)(H_r\psi(s)+1)ds \\ &\quad + \int_{t_1}^{T} |G_t(t_0,s) - G_t(t_1,s)| q(s)(H_r\psi(s)+1)ds \\ &\quad + \int_{T}^{\infty} |G_t(t_0,s) - G_t(t_1,s)| q(s)(H_r\psi(s)+1)ds \\ &\leq \int_0^{t_0} \phi_1(s)|\phi_2'(t_0) - \phi_2'(t_1)| q(s)(H_r\psi(s)+1)ds \\ &\quad + \int_{t_0}^{t_1} |\phi_1'(t_0)\phi_2(s) - \phi_1(s)\phi_2'(t_1)| q(s)(H_r\psi(s)+1)ds \\ &\quad + \int_{t_1}^{T} \phi_2(s)|\phi_1'(t_0) - \phi_1'(t_1)| q(s)(H_r\psi(s)+1)ds \\ &\quad + \int_{T}^{\infty} \phi_2(s)|\phi_1'(t_0) - \phi_1'(t_1)| q(s)(H_r\psi(s)+1)ds. \end{aligned}$$

Consequently,

$$\begin{aligned}
|(Tx)'(t_0)-(Tx)'(t_1)| &\leq \phi_1(t_0)|\phi_2'(t_0)-\phi_2'(t_1)|\int_0^{t_0} q(s)(H_r\psi(s)+1)ds \\
&+(|\phi_1'(t_1)\phi_2(t_1)|+|\phi_1(t_1)\phi_2'(t_1)|)\int_{t_0}^{t_1} q(s)(H_r\psi(s)+1)ds \\
&+|\phi_1'(t_0)-\phi_1'(t_1)|\int_{t_1}^{T} q(s)(H_r\psi(s)+1)ds \\
&+|\phi_1'(t_0)-\phi_1'(t_1)|\int_{T}^{\infty} q(s)(H_r\psi(s)+1)ds,
\end{aligned}$$

each of the four terms above tends to 0 as $|t_1-t_0|$ tends to 0, proving that Tx is almost equicontinuous.

To prove equiconvergence, we first notice that $\lim_{t\to\infty} Tx(t)=0$. Moreover from $\lim\limits_{t\to\infty}\phi_2(t)=0$ and $\int_0^\infty q(s)(H_r\psi(s)+1)ds<\infty$, for any $\varepsilon>0$, there exists $N>0$ such that for $t\geq N$, the following estimates hold true:

$$\begin{aligned}
0 &\leq \sup_{x\in B}|Tx(t)-0| \leq \sup_{x\in B}\int_0^\infty G(t,s)q(s)|\widetilde{f}(s,x(s),x'(s))|ds \\
&\leq \int_0^t \phi_1(s)\phi_2(t)q(s)(H_r\psi(s)+1)ds+\int_t^\infty \phi_1(t)\phi_2(s)q(s)(H_r\psi(s)+1)ds \\
&= \int_0^N \phi_1(s)\phi_2(t)q(s)(H_r\psi(s)+1)ds+\int_N^t \phi_1(s)\phi_2(t)q(s)(H_r\psi(s)+1)ds \\
&\quad +\int_t^\infty \phi_1(t)\phi_2(s)q(s)(H_r\psi(s)+1)ds \\
&\leq \int_0^N \phi_1(s)\phi_2(t)q(s)(H_r\psi(s)+1)ds+\int_N^\infty \phi_1(t)\phi_2(t)q(s)(H_r\psi(s)+1)ds \\
&\quad +\int_N^\infty \phi_1(t)\phi_2(t)q(s)(H_r\psi(s)+1)ds \\
&\leq \phi_1(N)\phi_2(t)\int_0^N q(s)(H_r\psi(s)+1)ds+M\int_N^\infty q(s)(H_r\psi(s)+1)ds \\
&\quad +M\int_N^\infty q(s)(H_r\psi(s)+1)ds.
\end{aligned}$$

Hence

$$\sup_{x\in B}|Tx(t)|\leq\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

Furthermore, for any $\varepsilon>0$, there exists $N>0$ such that for $t\geq N$

$$\begin{aligned}
|\phi_1(s)\phi_2'(t)| &= |G(t,s)(\tfrac{\phi_2'(t)}{\phi_2(t)}+d-d)| \\
&\leq G(t,s)|\tfrac{\phi_2'(t)}{\phi_2(t)}+d|+dG(t,s) \\
&\leq \tfrac{\varepsilon}{2\int_0^\infty q(s)(H_r\psi(s)+1)ds}
\end{aligned}$$

and

$$\int_t^\infty \phi_1'(t)\phi_2(s)q(s)(H_r\psi(s)+1)ds \le \int_t^\infty q(s)(H_r\psi(s)+1)ds \le \frac{\varepsilon}{2}.$$

As a consequence, for $t \ge N$, we obtain the estimates

$$\begin{aligned}
\sup_{x\in B}|(Tx)'(t) - \lim_{t\to\infty}(Tx)'(t)| &= \sup_{x\in B}\left|\int_0^\infty G_t(t,s)q(s)\widetilde{f}(s,x(s),x'(s))ds\right| \\
&\le \sup_{x\in B}\Big[\left|\int_0^t \phi_1(s)\phi_2'(t)q(s)\widetilde{f}(s,x(s),x'(s))ds\right| \\
&\quad + \left|\int_t^\infty \phi_1'(t)\phi_2(s)q(s)\widetilde{f}(s,x(s),x'(s))ds\right|\Big] \\
&\le \int_0^t |\phi_1(s)\phi_2'(t)|\, q(s)(H_r\psi(s)+1)ds \\
&\quad + \int_t^\infty \phi_1'(t)\phi_2(s)q(s)(H_r\psi(s)+1)ds \\
&\le \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus we have proved equiconvergence of T ending the proof that T is completely continuous. Finally, by the Leray-Schauder fixed point theorem, we deduce that T has at least a fixed point x, solution of problem (3.4). □

4.3 The Original Problem

Theorem 4.2. *Assume that either Assumptions* $(\mathcal{H}_0)-(\mathcal{H}_4)$ *or* $(\mathcal{H}_0)-(\mathcal{H}_3)$ *and* $(\mathcal{H}_4)'$ *hold. Then problem* (1.1) *has at least one solution* x *having the representation*

$$x(t) = \int_0^\infty G(t,s)q(t)f(s,x(s),x'(s))ds$$

with

$$\alpha(t) \le x(t) \le \beta(t), \quad t \in [0,\infty),$$

where $G(t,s)$ is the Green's function defined in (2.3).

Proof. From (3.2), we can find two real numbers $R > \max\{|\alpha_1|,|\beta_1|\}$ and $\eta > 0$ such that

$$\int_\eta^R \frac{ds}{h(s)} \ge k_0 \max\{\alpha_0,\beta_0\} + \int_0^\infty \psi(s)q(s)ds \tag{4.3}$$

and

$$\eta \ge \max\left\{\sup_{t\in[\gamma,\infty)} \frac{\beta(t)-\alpha(0)}{t}, \sup_{t\in[\gamma,\infty)} \frac{\beta(0)-\alpha(t)}{t}\right\},$$

for some $\gamma > 0$. Note that $\alpha(t) \le \alpha_0\phi_2(t) < \alpha_0$ and $\beta(t) \le \beta_0\phi_2(t) < \beta_0$.

(a) By Theorem 4.1, problem (3.4) has at least one solution in X. In addition, proposition 4.1 implies that any solution x of problem (3.4) satisfies the bounds

$$\alpha(t) \leq x(t) \leq \beta(t).$$

Hence $\widetilde{f}(t, x(t), x'(t)) = f_R(t, x(t), x'(t)),\ \forall\, t \in (0, \infty)$.

(b) It remains to prove that $|x'(t)| \leq R$, for every $t \in [0, \infty)$.

Case 1. Assume that $|x'(t)| > \eta, \forall\, t \in [0, \infty)$ and that $x'(t) > \eta, \forall\, t \in [0, \infty)$. Then for $t \geq \gamma$, we have

$$\frac{\beta(t) - \alpha(0)}{t} \geq \frac{x(t) - x(0)}{t} = \frac{1}{t}\int_0^t x'(s)ds > \eta \geq \frac{\beta(t) - \alpha(0)}{t},$$

which is a contradiction. Hence there exists $t_0 \in [0, \infty)$ such that $|x'(t_0)| \leq \eta$.

Case 2. If $|x'(t)| \leq \eta, \forall\, t \in [0, \infty)$, then one may take $R = \max\{|\alpha_1|, |\beta_1|, \eta\}$.

Case 3. There exists an interval $[t_0, t_1] \subset [0, \infty)$ such that either

$$|x'(t_0)| = \eta \text{ and } x'(t) > \eta,\ \forall\, t \in (t_0, t_1]$$

or

$$|x'(t_1)| = \eta \text{ and } x'(t) > \eta,\ \forall\, t \in [t_0, t_1).$$

For the sake of brevity, we only consider the first case. Using the fact that

$$|x(t)| \leq \max(|\alpha(t)|/\phi_2(t), |\beta(t)|/\phi_2(t))\phi_2(t),$$

we get

$$\begin{aligned}
\int_{x'(t_0)}^{x'(t_1)} \frac{ds}{h(s)} &= \int_{t_0}^{t_1} \frac{x''(s)}{h(x'(s))}ds = \int_{t_0}^{t_1} \frac{k^2(s)x(s) - q(s)f_R(s, x(s), x'(s))}{h(x'(s))}ds \\
&\leq \int_{t_0}^{t_1} \frac{k^2(s)|x(s)| + q(s)\psi(s)h(x'(s))}{h(x'(s))}ds \\
&\leq \int_{t_0}^{t_1} \frac{h(x'(s))\left(k^2(s)|x(s)| + q(s)\psi(s)\right)}{h(x'(s))}ds \\
&\leq \int_{t_0}^{t_1} k^2(s)|x(s)| + \int_{t_0}^{t_1} q(s)\psi(s)ds \\
&\leq \max\left(\sup_{t\in[0,\infty)}\{|\beta(t)|\phi_2^{-1}(t)\}, \sup_{t\in[0,\infty)}\{|\alpha(t)|\phi_2^{-1}(t)\}\right). \\
&\quad \int_{t_0}^{t_1} k^2(s)\phi_2(s)ds + \int_{t_0}^{t_1} q(s)\psi(s))ds.
\end{aligned}$$

Hence

$$\int_{x'(t_0)}^{x'(t_1)} \frac{ds}{h(s)} \leq k_0 \max(\beta_0, \alpha_0) + \int_0^\infty q(s)\psi(s)ds \leq \int_\eta^R \frac{ds}{h(s)}.$$

Then $x'(t_1) \le R$. Since t_0 and t_1 are arbitrary, we obtain that if $x'(t) \ge \eta$, then $x'(t) \le R$, $t \in [0,\infty)$ yielding that $f_R(t,x(t),x'(t)) = f(t,x(t),x'(t))$. This means that x is a solution of problem (1.1), which completes the proof of the theorem. □

Remark 4.2. *The condition $h(s) \ge 1$ in $(\mathcal{H}_3)$ is not essential; in fact it is sufficient to suppose $h(s) \ge h_0$ for some $h_0 > 0$. Indeed, in this case $\frac{h(s)}{h_0} \ge 1$ and then we have to write in the above estimates:*

$$\begin{aligned}
\textstyle\int_{x'(t_0)}^{x'(t_1)} \frac{ds}{h(s)} &\le \int_{t_0}^{t_1} \frac{k^2(s)|x(s)| + q(s)\psi(s)h(x'(s))}{h(x'(s))} ds \\
&\le \int_{t_0}^{t_1} \frac{h(x'(s))(\frac{1}{h_0}k^2(s)|x(s)| + q(s)\psi(s))}{h(x'(s))} ds \\
&\le \textstyle\int_{t_0}^{t_1} \frac{1}{h_0}k^2(s)|x(s)| + \int_{t_0}^{t_1} q(s)\psi(s))ds \\
&\le \textstyle\frac{1}{h_0}k_0 \max(\beta_0,\alpha_0) + \int_0^\infty q(s)\psi(s))ds.
\end{aligned}$$

So we have just to modify (4.3) by

$$\int_\eta^R \frac{ds}{h(s)} > \frac{1}{h_0}k_0 \max(\beta_0,\alpha_0) + \int_0^\infty q(s)\psi(s))ds.$$

Our second existence result is

Theorem 4.3. *Assume that all conditions of Theorem 4.2 are satisfied but $(\mathcal{H}_2)$ replaced by*

$(\mathcal{H}_2)'$ $k_1 := \int_0^\infty k^2(t)\max(|\alpha(t)|,|\beta(t)|)\,dt < \infty$. *Then problem (1.1) has at least one solution x having the representation*

$$x(t) = \int_0^\infty G(t,s)q(t)f(s,x(s),x'(s))ds$$

and such that

$$\alpha(t) \le x(t) \le \beta(t), \quad t \in [0,\infty).$$

Proof. From (3.2), we can find real numbers $\widetilde{R} > \max\{|\alpha_1|,|\beta_1|\}$ and $\eta > 0$ such that

$$\int_\eta^{\widetilde{R}} \frac{ds}{h(s)} \ge k_1 + \int_0^\infty \psi(s)q(s)ds. \tag{4.4}$$

Then the proof runs parallel to the proof of Theorem 4.2 with R replaced by $\widetilde{R}$. However, in Case 3 of the proof of Theorem 4.2, we have the following estimates instead:

$$\begin{aligned}
\int_{x'(t_0)}^{x'(t_1)} \frac{ds}{h(s)} &= \int_{t_0}^{t_1} \frac{x''(s)}{h(x'(s))} ds = \int_{t_0}^{t_1} \frac{k^2(s)x(s)-q(s)f_R(s,x(s),x'(s))}{h(x'(s))} ds \\
&\leq \int_{t_0}^{t_1} \frac{k^2(s)|x(s)|+q(s)\psi(s)h(x'(s))}{h(x'(s))} ds \\
&\leq \int_{t_0}^{t_1} \frac{h(x'(s))(k^2(s)|x(s)|+q(s)\psi(s))}{h(x'(s))} ds \\
&\leq \int_{t_0}^{t_1} k^2(s)|x(s)| + \int_{t_0}^{t_1} q(s)\psi(s) ds \\
&\leq k_1 + \int_0^{\infty} q(s)\psi(s) ds \\
&\leq \int_{\eta}^{\widetilde{R}} \frac{ds}{h(s)}.
\end{aligned}$$

Finally, we complete the proof using (4.4). □

Remark 4.3. *Contrarily to* $(\mathcal{H}_2)$*, assumption* $(\mathcal{H}_2)'$ *allows the upper and lower solutions to be unbounded.*

5 A Uniqueness Result

The following result complements Theorems 4.2 and 4.3.

Theorem 5.1. *Assume that* $f = f(t, x, y)$ *is continuously differentiable in* x *and* y *for each* $t \geq 0$ *and satisfies either the conditions of Theorem* 4.2 *or Theorem* 4.3 *together with*

$(\mathcal{H}_5)$ $\quad f(t, x, y)$ *is nonincreasing in* x *for each* t *and* y *fixed.*

Then problem (1.1) *has a unique solution* x *such that*

$$\alpha(t) \leq x(t) \leq \beta(t), \quad \forall\, t \geq 0.$$

Proof. Suppose there exist two distinct solutions x_1, x_2 of problem (1.1) and let $z := x_1 - x_2$. By the mean value theorem, there exist θ, φ such that

$$f(t, x_2, x_2') = f(t, x_1, x_1') - z\frac{\partial f}{\partial x}(t, \theta, \varphi) - z'\frac{\partial f}{\partial y}(t, \theta, \varphi).$$

Assume that $z(t_1) > 0$ for some t_1 and that z has a positive maximum at some $t_0 < \infty$. Then, with $(\mathcal{H}_5)$, we have

$$\begin{aligned}
0 \geq z''(t_0) &= k^2(t_0)z(t_0) + q(t_0)\left[f(t_0, x_2(t_0), x_2'(t_0)) - f(t_0, x_1(t_0), x_1'(t_0))\right] \\
&= k^2(t_0)z(t_0) - q(t_0)\tfrac{\partial f}{\partial x}(t_0, \theta, \varphi))z(t_0) - q(t_0)\tfrac{\partial f}{\partial y}(t_0, \theta, \varphi)z'(t_0) \\
&= z(t_0)\left[k^2(t_0) - q(t_0)\tfrac{\partial f}{\partial x}(t_0, \theta, \varphi)\right] > 0,
\end{aligned}$$

leading to a contradiction. Hence $\sup z(t) = \lim_{t\to\infty} z(t)$. But $\lim_{t\to\infty} z(t) = \lim_{t\to\infty}[x_1(t) - x_2(t)] = 0$, which is again a contradiction, ending the proof of the theorem. $\square$

6 Applications

6.1 Example 1

Consider the boundary value problem

$$\begin{cases} x''(t) - k^2(t)x(t) + q(t)[x(t)(x')^\theta(t) + c(t)] = 0, \quad t > 0, \\ x(0) = 0, \quad x(+\infty) = 0, \end{cases} \tag{6.1}$$

where $\theta = \frac{1}{2p+1}$ and p is a positive integer. The positive functions $c = c(t)$ and $q = q(t)$ satisfy $q(t), \phi_2(t)q(t) \in C(0, +\infty) \cap L^1(0, \infty)$ and $0 \leq c(t) \leq -\phi_2(t)(\phi_2')^\theta(t)$. The function k verifies $(\mathcal{H}_0)$.

Then $\alpha(t) \equiv 0$ and $\beta(t) = \phi_2(t)$ are respectively lower solution and upper solution with $\alpha \leq \beta$. Moreover

$$\alpha_0 = \sup_{t\in[0,\infty)} \{|\alpha(t)|\phi_2^{-1}(t)\} = 0 \quad \text{and} \quad \beta_0 = \sup_{t\in[0,\infty)} \{|\beta(t)|\phi_2^{-1}(t)\} = 1.$$

Then $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ are satisfied. As for $(\mathcal{H}_3)$, one may take $\psi(t) = \phi_2(t)$ and $h(y) = (|y| + 1 + \gamma) \geq 1$ with $\gamma := -(\phi_2')^\theta(0)$ so that, for $0 \leq x \leq \phi_2(t)$, we have

$$\begin{aligned} |f(t, x, y)| &= |xy^\theta + c(t)| \leq \phi_2(t)|y|^\theta - \phi_2(t)(\phi_2')^\theta(t) \\ &\leq \phi_2(t)(|y|^\theta - (\phi_2')^\theta(0)) \leq \phi_2(t)(|y| + 1 + \gamma) \end{aligned}$$

as well as

$$\int_0^\infty \psi(s)q(s)ds < \infty, \text{ and } \int_0^\infty \frac{ds}{h(s)} = \int_0^\infty \frac{ds}{s + 1 + \gamma} = +\infty.$$

Regarding $(\mathcal{H}_4)$, we have

$$\begin{cases} \alpha_1 := & \sup_{t\in\mathbb{R}^+} \alpha'(t) = 0 \\ \beta_1 := & \inf_{t\in\mathbb{R}^+} \phi_2'(t) = \phi_2'(0) = -\int_0^\infty \phi_2''(s)ds = -\int_0^\infty k^2(s)\phi_2(s)ds = -k_0. \end{cases}$$

In addition, for any $y \in \mathbb{R}$ and $t \in (0, \infty)$, we have

$$\begin{cases} y < \beta'(t) \Longrightarrow f(t, \beta(t), y) & = \phi_2(t)y^\theta + c(t) \\ & \leq \phi_2(t)\phi_2'^\theta(t) + c(t) = f(t, \beta(t), \beta'(t)) \\ & \\ y > \alpha'(t) \Longrightarrow y > 0 \Longrightarrow & f(t, \alpha(t), y) = c(t) = f(t, \alpha(t), \alpha'(t)). \end{cases}$$

Therefore, Theorem 4.2 yields that problem (6.1) has at least one solution x such that

$$0 \leqslant x(t) \leqslant \phi_2(t), \ \forall\, t \geq 0.$$

6.2 Example 2

Further to $(\mathcal{H}_1)-(\mathcal{H}_4)$, it is clear that the nonlinearity $f(t,x,y) = -e^x e^{-y} + c(t)$ satisfies $(\mathcal{H}_5)$. Arguing as in Example 1, we can see that Theorems 4.2 and 5.1 imply that the boundary value problem

$$\begin{cases} x''(t) - k^2(t)x(t) + q(t)[-e^{x(t)}e^{-x'(t)} + c(t)] = 0, \quad t > 0, \\ x(0) = x(+\infty) = 0, \end{cases}$$

where the positive functions $c = c(t)$ and $q = q(t)$ satisfy $q(t)$, $\phi_2(t)q(t) \in C(0,+\infty) \cap L^1(0,\infty)$ and $1 \leq c(t) \leq e^{\phi_2(t)}e^{-\phi_2'(t)}$, $t \geq 0$, has exactly one solution x satisfying

$$0 \leqslant x(t) \leqslant \phi_2(t), \; \forall\, t \geq 0.$$

References

[1] R.P. Agarwal, O.G. Mustafa, and Yu.V. Rogovchenko, *Existence and asymptotic behavior of solutions of boundary value problem on an infinite interval,* Math. and Comput. Model. 41 (2005) 135–157.

[2] R.P. Agarwal, D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations,* Kluwer Academic Publisher, Dordrecht, 2001.

[3] R.P. Agarwal, D. O'Regan, and P.J.Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations,* Kluwer Academic Publisher, Dordrecht, 1999.

[4] C. Corduneanu, *Integral Equations and Stability of Feedback Systems,* Academic Press, New York, 1973.

[5] K. Deimling, *Nonlinear Functional Analysis,* Springer-Verlag, Berlin, Heidelberg, 1985.

[6] S. Djebali, O. Kavian and T. Moussaoui, *Qualitative properties and existence of solutions for a generalized Fisher-like equation,* Iranian J. of Math. Sci. Infor. **4(2)** (2009), 65–81.

[7] S. Djebali and K. Mebarki, *Multiple positive solutions for singular BVPs on the positive half-line,* Comp. Math. Appl., **55(12)** (2008) 2940–2952

[8] S. Djebali and K. Mebarki, *On the singular generalized Fisher-like equation with derivative depending nonlinearity,* Appl. Math. and Comput. **205** (2008) 336–351

[9] S. Djebali and T. Moussaoui, *A class of second order BVPs on infinite intervals,* Elec. Jour. Qual. Theo. Diff. Eq. **4** (2006) 1–19.

[10] S. Djebali and S. Zahar, *Bounded solutions for a derivative dependent boundary value problem on a the half-line,* Dynamic Syst. and Appli. 19 (2010) 545–556.

[11] S. Djebali, O. Saifi, and S. Zahar, *Singular BVPs with variable coefficient on the positive half-line,* submitted

[12] H. Lian, P. Wang and W. Ge, *Unbounded upper and lower solutions method for Sturm-Liouville boundary value problem on infinite intervals,* Nonlin. Anal., **70** (2009), 2627–2633.

[13] R. Ma and B. Zhu, *Existence of positive solutions for a semipositone boundary value problem on the half line,* Comput. Math. with Appl., **58(8)** (2009) 1672–1686.

[14] D. O'Regan, *Existence Theory for Ordinary Differential Equations,* Kluwer, Dordrecht, 1997.

[15] D. O'Regan, B. Yan, and R.P. Agarwal, *Solutions in weighted spaces of singular boundary value problems on the half-line,* J. Comput. Appl. Math. **205** (2007) 751–763.

[16] Y. Tian, W. Ge, *Positive solutions for multi-point boundary value problem on the half-line,* J. Math. Anal. Appl. **325** (2007) 1339–1349.

[17] B. Yan, D. O'Regan, and R.P. Agarwal, *Unbounded solutions for singular boundary value problems on the semi-infinite interval: Upper and lower solutions and multiplicity,* J. Comput. Appl. Math. **197** (2006) 365–386.

[18] E. Zeidler, *Nonlinear Functional Analysis and its Applications. Vol. I: Fixed Point Theorems,* Springer-Verlag, New York, 1986.

Smaïl DJEBALI (e-mail: djebali@ens-kouba.dz)
Department of Mathematics, E.N.S. P.B. 92 Kouba, 16050. Algiers, ALGERIA

Samira ZAHAR (e-mail: zahar_samira@yahoo.fr)
Department of Mathematics, University Abd-Errahmane Mira, 06000. Bejaia, Algeria

(Received September 5, 2010)