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Periodic solutions of second order differential equations with vanishing Green's functions

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Abstract. We study the existence and multiplicity of positive periodic solutions for second order differential equations with vanishing Green's functions. The proof relies on a fixed point theorem in cones. Some recent results in the literature are generalized.

Keywords: periodic solutions, differential equations, vanishing Green's functions, fixed point theorem in cones.

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1 Introduction

During the last two decades, the existence of periodic solutions for second order differential equation

$$x'' + a(t)x = f(t, x) \tag{1.1}$$

has been extensively studied in the literature for the regular cases as well as the singular cases, where $a \in X = \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ and the nonlinearity $f \in \mathbb{C}((\mathbb{R}/T\mathbb{Z}) \times (0, \infty), \mathbb{R})$. See, for example, [4–6, 15, 17, 18]. Some classical tools have been used in the study of periodic solutions of equation (1.1), including the method of upper and lower solutions [15], some fixed point theorems in cones for completely continuous operators [4, 17], Schauder's fixed point theorem [5, 18] and a nonlinear Leray–Schauder alternative principle [2, 6].

In the above mentioned works, when one tried to apply some fixed point theorems in cones, or the nonlinear alternative principle of Leray–Schauder, to study the existence of periodic solutions of equation (1.1), one major assumption is that the corresponding Green's function $G(t, s)$ for the linear differential equation

$$x'' + a(t)x = 0 \tag{1.2}$$

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is positive, which is equivalent to the strict anti-maximum principle for equation (1.2). Such an assumption plays an important role in constructing the following cone

$$K_1 = \left\{ x \in X : \min_{0 \le t \le T} x(t) \ge \sigma \|x\| \right\},$$

where

$$\sigma = m/M, \qquad m = \min_{0 \le s,t \le T} G(t,s), \qquad M = \max_{0 \le s,t \le T} G(t,s).$$

When the Green's function vanishes, we know that $m = 0$ and K_1 becomes the cone of non-negative functions, which is not effective in obtaining the desired estimates.

For example, when $a(t) = k^2$ with $k > 0$ and $k \neq \frac{2n\pi}{T} (n \in \mathbb{Z}^+)$, the Green's function is given as

$$G(t,s) = \begin{cases} \dfrac{\sin k(t-s) + \sin k(T-t+s)}{2k(1-\cos kT)}, & 0 \le s \le t \le T, \\ \dfrac{\sin k(s-t) + \sin k(T-s+t)}{2k(1-\cos kT)}, & 0 \le t \le s \le T. \end{cases}$$

Therefore the positiveness of the Green's function is equivalent to $k < \pi/T$. For the critical case $k = \pi/T$, the Green's function vanishes on the line $t = s$, and therefore the results in [2, 4, 6, 17] cannot deal with such a critical case. In this paper, we focus on the case $k \le \pi/T$ because we assume that the following condition holds

(A) The associated Green's function $G(t,s)$ of (1.2) is non-negative for all $(t,s) \in [0,T] \times [0,T]$.

In Section 2, we will make a brief comment on condition (A). We observe that even when the Green's function vanishes, the following fact also holds

$$\nu = \min_{0 \le s \le T} \int_0^T G(t,s)dt > 0.$$

Based on this fact, Graef, Kong and Wang in [8] introduced the following cone

$$K = \left\{ x \in X : x(t) \ge 0 \text{ and } \int_0^T x(t)dt \ge \frac{\nu}{M}\|x\| \right\}. \tag{1.3}$$

Using the above cone, it was proved in [8] that equation

$$x'' + a(t)x = g(t)f(x) \tag{1.4}$$

has at least one nontrivial T-periodic solution if $f : [0, +\infty) \to [0, +\infty)$ is continuous, $g : [0,T] \to [0, \infty)$ is continuous with $\min_t g(t) > 0$ and one of the following two conditions holds:

(i) $f_0 = \lim_{x \to 0} \dfrac{f(x)}{x} = \infty$ and $f_\infty = \lim_{x \to \infty} \dfrac{f(x)}{x} = 0$ (sublinear);

(ii) $f_0 = 0$, $f_\infty = \infty$ (superlinear), f is convex and nondecreasing.

As an example, it was shown that equation

$$x'' + k^2 x = x^\alpha$$

has at least one nontrivial T-periodic solution if $0 < k \leq \pi/T$ and $\alpha \in (0,1) \cup (1,\infty)$. Such a result was generalized in [19] using fixed point index theory and used conditions related to the principal eigenvalue of the corresponding linear problem. Theorem 2.2 of [19] also has a result for existence of two positive solutions in a sublinear case. The result of [8] was extended in [12] to systems. For the superlinear case, the result in [8] was improved in [9], in which the convexity assumption was removed. A modification of the cone K was used in [14, 19] and some sharp existence conditions were given for (1.4) by assuming that the non-negative function g satisfies a weaker condition $\int_0^T g(t)dt > 0$. Such a cone was also used to deal with some singular case in [1]. We remark that existence results for (1.4) were proved in [13] using the Schauder fixed point theorem even when the Green's function is sign-changing.

The aim of this paper is to use the cone defined in (1.3), together with fixed point theorems in cones, to establish the existence of at least one or at least two positive T-periodic solutions for equation (1.1). Our main motivation is to obtain new existence results for the following differential equation

$$x'' + a(t)x = p(t)x^{\alpha} + \mu q(t)x^{\beta} + e(t), \tag{1.5}$$

where $a, p, q, e \in X$, $0 < \alpha < 1$, $\beta > \alpha$ and $\mu > 0$ is a parameter. Our new results generalize some recent results contained in [4, 8, 9, 17], because not only we can deal with the critical case, but also we can obtain the multiplicity result for the case $e \succ 0$, here the notation $e \succ 0$ means that $e(t) \geq 0$ for all $t \in [0,T]$ and $\bar{e} = \frac{1}{T}\int_0^T e(t)dt > 0$.

2 Preliminaries

First we make a brief comment on condition (A). When $a(t) = k^2$, condition (A) is equivalent to $0 < k^2 \leq (\frac{\pi}{T})^2$. For a non-constant function $a(t)$, there is an L^p-criterion proved in [17], which is given in the following lemma for the sake of completeness. Let $\mathbf{K}(q)$ denote the best Sobolev constant in the following inequality:

$$C\|u\|_q^2 \leq \|u'\|_2^2, \quad \text{for all } u \in H_0^1(0,T).$$

The explicit formula for $\mathbf{K}(q)$ is

$$\mathbf{K}(q) = \begin{cases} \frac{2\pi}{qT^{1+2/q}}\left(\frac{2}{2+q}\right)^{1-2/q}\left(\frac{\Gamma(\frac{1}{q})}{\Gamma(\frac{1}{2}+\frac{1}{q})}\right)^2 & \text{if } 1 \leq q < \infty, \\ \frac{4}{T} & \text{if } q = \infty, \end{cases}$$

where Γ is the Gamma function.

Lemma 2.1 ([17]). *Assume that $a(t) \succ 0$ and $a \in L^p[0,T]$ for some $1 \leq p \leq \infty$. Then condition* (A) *holds if $\|a\|_p \leq \mathbf{K}(2\tilde{p})$ with $\frac{1}{p} + \frac{1}{\tilde{p}} = 1$.*

We can obtain the first positive T-periodic solution of (1.1) as a consequence of [11, Lemma 2.8] or [20, Lemma 5.3]. The second positive T-periodic solution will be found based on the following well-known fixed point theorem in cones. Recall that a completely continuous operator means a continuous operator which transforms every bounded set into a relatively compact set. If D is a subset X, we write $D_K = D \cap K$ and $\partial_K D = \partial D \cap K$.

Lemma 2.2. [7] *Let X be a Banach space and K $(\subset X)$ be a cone. Assume that Ω^1, Ω^2 are open subsets of X with $0 \in \Omega_1$, $\Omega_K^1 \neq \emptyset$, $\overline{\Omega^1}_K \subset \Omega_K^2$. Let*

$$\mathcal{A} : \overline{\Omega^2}_K \to K$$

be a continuous and completely continuous operator such that one of the following conditions is satisfied

(i) $\|\mathcal{A}u\| \geq \|u\|$, $u \in \partial_K \Omega^1$ *and* $\|\mathcal{A}u\| \leq \|u\|$, $u \in \partial_K \Omega^2$.

(ii) $\|\mathcal{A}u\| \leq \|u\|$, $u \in \partial_K \Omega^1$ *and* $\|\mathcal{A}u\| \geq \|u\|$, $u \in \partial_K \Omega^2$.

Then $\mathcal{A}$ has at least one fixed point in $\overline{\Omega^2}_K \setminus \Omega^1_K$.

3 Main results

In this section, we state and prove the main results of this paper. We will use the notations

$$\omega(t) = \int_0^T G(t,s)ds, \qquad \omega^* = \max_t \omega(t).$$

Recall that we suppose that $\nu = \min_t \omega(t) > 0$.

Lemma 3.1. *Suppose that $a(t)$ satisfies* (A). *Assume further that*

(H_1) *There exists a continuous function $\phi \succ 0$ such that $f(t,x) \geq \phi(t)$ for all $(t,x) \in [0,T] \times [0,\infty)$.*

(H_2) *There exists a positive constant r such that*

$$0 \leq f(t,x) < r/\omega^* \quad \textit{for all } (t,x) \in [0,T] \times [0,r].$$

Then equation (1.1) *has at least one T-periodic solution x with $0 < \|x\| < r$.*

Proof. Let K be the cone in X defined by (1.3). Define the operator $\mathcal{A}: X \to X$ as

$$\mathcal{A}x(t) = \int_0^T G(t,s)f(s,x(s))ds. \tag{3.1}$$

Since f is continuous and non-negative in $(t,x) \in [0,T] \times [0,\infty)$, using the similar proof in [8], we can obtain that $\mathcal{A}$ maps the set $\{x \in X : x(t) \geq 0\}$ into K. Moreover, T-periodic solutions of (1.1) are fixed points of the operator $\mathcal{A}$.

Since (H_2) holds, as a consequence of [11, Lemma 2.8] or [20, Lemma 5.3], we can obtain that equation (1.1) has a non-negative T-periodic solution x with $\|x\| < r$.

By the condition (H_1), we obtain that

$$\begin{aligned}\int_0^T x(t)dt &= \int_0^T \int_0^T G(t,s)f(s,x(s))dsdt \\ &\geq \int_0^T \int_0^T G(t,s)\phi(s)ds \\ &= \int_0^T \phi(s) \int_0^T G(t,s)dtds \\ &\geq \nu \int_0^T \phi(s)ds > 0,\end{aligned}$$

which implies that x is a positive T-periodic solution of (1.1). □

Example 3.2. Let $a(t)$ satisfy (A) and consider the differential equation

$$x'' + a(t)x = x^\alpha + \mu x^\beta + e(t), \tag{3.2}$$

where $e \in X$ and $e \succ 0$, $0 < \alpha < 1$, $\beta > \alpha$, $\mu > 0$ is a positive parameter.

(i) If $\beta < 1$, then equation (3.2) has at least one nontrivial T-periodic solution for each $\mu > 0$.

(ii) If $\beta \geq 1$, then equation (3.2) has at least one nontrivial T-periodic solution for each $0 < \mu < \tilde{\mu}$, where $\tilde{\mu}$ is a positive constant given by (3.3).

Proof. We will apply Theorem 3.1. To this end, we take

$$\phi(t) = e(t), \quad f(t,x) = x^\alpha + \mu x^\beta + e(t).$$

Clearly, (H_1) is satisfied since $e \succ 0$. Moreover, since $0 < \alpha < \beta$, the existence condition (H_2) becomes

$$\mu < \frac{r - e^*\omega^* - \omega^* r^\alpha}{\omega^* r^\beta}$$

for some $r > 0$. So equation (3.2) has at least one T-periodic solution for

$$0 < \mu < \tilde{\mu} := \sup_{r>0} \frac{r - e^*\omega^* - \omega^* r^\alpha}{\omega^* r^\beta}. \tag{3.3}$$

Note that $\tilde{\mu} = \infty$ if $\beta < 1$ and $\tilde{\mu} < \infty$ if $\beta \geq 1$. We have (i) and (ii). $\square$

Theorem 3.3. *Suppose that $a(t)$ satisfies* (A) *and $f(t,x)$ satisfies* (H_2). *Assume further that*

(H_3) *There exist continuous, non-negative functions $g(x)$ and $h_1(x)$ such that*

$$f(t,x) \geq g(x) + h(x) \quad \textit{for all } (t,x) \in [0,T] \times (0,\infty),$$

where $g(x) > 0$ is non-decreasing and convex, $h(x)/g(x)$ is non-increasing in $x \in (0,\infty)$.

(H_4) *There exists a constant $R > r$ such that*

$$g\left(\frac{\nu}{MT}R\right)\left\{1 + \frac{h(R)}{g(R)}\right\} \geq \frac{R}{\nu}.$$

Then equation (1.1) *has at least one T-periodic solution $\tilde{x}$ with $r \leq \|\tilde{x}\| \leq R$.*

Proof. Let K be the cone in X defined by (1.3). Define the open sets

$$\Omega^1 = \{x \in X : \|x\| < r\}, \qquad \Omega^2 = \{x \in X : \|x\| < R\},$$

and define the operator $\mathcal{A} : \overline{\Omega}^2_K \to K$ as (3.1). For each $x \in \overline{\Omega}^2_K \setminus \Omega^1_K$, we have $r \leq \|x\| \leq R$. Thus the operator $\mathcal{A} : \overline{\Omega}^2_K \setminus \Omega^1_K \to K$ is well defined and is continuous and completely continuous since f is continuous.

First we claim that $\|\mathcal{A}x\| \leq \|x\|$ for $x \in \partial_K\Omega^1$. In fact, if $x \in \partial_K\Omega^1$, then $\|x\| = r$ and we have $f(t,x) < r/\omega^*$. Therefore,

$$\mathcal{A}x(t) = \int_0^T G(t,s)f(s,x(s))ds \leq \int_0^T G(t,s)r/\omega^* ds \leq r,$$

and thus $\|\mathcal{A}x\| \leq \|x\|$ for $x \in \partial_K\Omega^1$.

Next we prove that $\|\mathcal{A}x\| \geq \|x\|$ for $x \in \partial_K\Omega^2$. In fact, if $x \in \partial_K\Omega^2$, then $\|x\| = R$ and

$$\int_0^T x(t)dt \geq \frac{\nu}{M}R.$$

Thus

$$\begin{aligned}\int_0^T \mathcal{A}x(t)dt &= \int_0^T \int_0^T G(t,s)f(s,x(s))dsdt \\ &= \int_0^T f(s,x(s)) \int_0^T G(t,s)dtds \\ &\geq \nu \int_0^T f(s,x(s))ds \\ &\geq \nu \int_0^T g(x(s)) \left\{1 + \frac{h(x(s))}{g(x(s))}\right\} ds \\ &\geq \nu \left\{1 + \frac{h(R)}{g(R)}\right\} \int_0^T g(x(s))ds.\end{aligned}$$

Since g is convex, using Jensen's inequality [16, Theorem 3.3], we have

$$\int_0^T g(x(s))ds \geq Tg\left(\frac{1}{T}\int_0^T x(s)ds\right) \geq Tg\left(\frac{\nu}{MT}R\right).$$

Therefore,

$$\begin{aligned}\|\mathcal{A}x\| &\geq \frac{1}{T}\int_0^T \mathcal{A}x(t)dt \\ &\geq \frac{\nu}{T}\left\{1 + \frac{h(R)}{g(R)}\right\} \int_0^T g(x(s))ds \\ &\geq \frac{\nu}{T}\left\{1 + \frac{h(R)}{g(R)}\right\} Tg\left(\frac{\nu}{MT}R\right) \\ &\geq R.\end{aligned}$$

Now Lemma 2.2 guarantees that $\mathcal{A}$ has at least one fixed point $\tilde{x} \in \overline{\Omega}_K^2 \setminus \Omega_K^1$ with $r \leq \|\tilde{x}\| \leq R$. Clearly, $\tilde{x}$ is a T-periodic solution of (1.1). □

Example 3.4. Let us consider the differential equation (3.2) again, where $a(t)$ satisfies (A), $0 < \alpha < 1 < \beta$, $\mu > 0$ is a positive parameter and $e \in X$ is nonnegative. Then equation (3.2) has at least one positive T-periodic solutions for each $0 < \mu < \tilde{\mu}$, where $\tilde{\mu}$ is the constant given as (3.3) in Example 3.2.

Proof. We will apply Theorem 3.3. To this end, we take

$$g(x) = \mu x^\beta, \qquad h(x) = x^\alpha.$$

As in Example 3.2, we know that (H_1) and (H_2) are satisfied for all $\mu < \tilde{\mu}$. Moreover, since $\beta > 1$, it is easy to see that (H_3) is satisfied and condition (H_4) becomes

$$\mu \geq \frac{(TM)^\beta R - \nu^{1+\beta} R^\alpha}{\nu^{1+\beta} R^\beta} \tag{3.4}$$

for some $R > 0$. Since $\beta > 1$, the right-hand side of (3.4) goes to 0 as $R \to +\infty$. Thus, for any given $0 < \mu < \tilde{\mu}$, it is always possible to find $R \gg r$ such that (3.3) is satisfied. Now all conditions of Theorem 3.3 are satisfied. Thus, equation (3.2) has a positive T-periodic solution $\tilde{x}$. □

Remark 3.5. In Lemma 3.1, condition (H_1) guarantees that the periodic solution obtained is nontrivial, while (H_1) is not required in Theorem 3.3. For the equation (3.2), we require that the function $e \succ 0$ in Example 3.2, while e is only required to be nonnegative in Example 3.4.

The following multiplicity result is a direct consequence of Lemma 3.1 and Theorem 3.3.

Theorem 3.6. *Suppose that $a(t)$ satisfies* (A) *and $f(t,x)$ satisfies* (H_1)–(H_2)–(H_3)–(H_4). *Then equation* (1.1) *has at least two positive T-periodic solutions x and $\tilde{x}$ with $0 < \|x\| < r \le \|\tilde{x}\| \le R$.*

Example 3.7. Let us assume that $a(t)$ satisfy (A), $0 < \alpha < 1 < \beta$ and $e \succ 0$. Then equation (3.2) has at least two positive T-periodic solutions for each $0 < \mu < \tilde{\mu}$, where $\tilde{\mu}$ is the constant given as (3.3).

Remark 3.8. It is easy to obtain results analogous to equation (3.2) for the general equation (1.5) with p, q being positive T-periodic continuous functions, but the notation becomes cumbersome. Here we consider only (3.2) for simplicity.

Remark 3.9. We generalize the results in [9] because we can obtain two T-periodic solutions in Example 3.7. In [14], based on bifurcation techniques, the existence of one or two positive solutions was proved. However, our method is different from [14].

Remark 3.10. Similar hypotheses to those in Theorem 3.3 have been used in [2,3,10] to study the existence and multiplicity of periodic solutions of singular differential equations.

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