Remark on regularity criteria of a weak solution to the 3D MHD equations

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Abstract. We study the regularity conditions for a weak solution to 3D MHD equations in a whole space $\mathbb{R}^3$, based on the papers [C. He, Y. Wang, *J. Differential Equations* 238(2007), No. 1, 1–17] and [W. Wang, *J. Math. Anal. Appl.* 328(2007), No. 2, 1082–1086].

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1 Introduction

We study the following 3D Hall-MHD equations:

$$(1.1) \quad \begin{cases} u_t - \triangle u + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla \pi = 0, \\ b_t - \triangle b + (u \cdot \nabla)b - (b \cdot \nabla)u = 0, \\ \operatorname{div} u = 0, \quad \operatorname{div} b = 0, \end{cases} \quad \text{in } Q_T := \mathbb{R}^3 \times [0, T),$$

where $u \colon Q_T \to \mathbb{R}^3$ is the flow velocity vector, $b \colon Q_T \to \mathbb{R}^3$ is the magnetic vector and $\pi = p + \frac{|b|^2}{2} \colon Q_T \to \mathbb{R}$ is the total pressure. We consider the initial value problem of (1.1), which requires initial conditions

$$(1.2) \qquad u(x,0) = u_0(x) \quad \text{and} \quad b(x,0) = b_0(x), \qquad x \in \mathbb{R}^3$$

The initial conditions satisfy the compatibility condition, i.e.

$$\nabla \cdot u_0(x) = 0, \quad \text{and} \quad \nabla \cdot b_0(x) = 0.$$

The notion of weak solutions will be introduced in Definition 2.2 of Section 2.

The MHD equations describe the motions of the interactions of electrically conducting fluid flows and the electromagnetic forces, e.g., plasma and liquid metals (see e.g., [3]).

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Definition 1.1. A weak solution pair (u, b) of the 3D MHD equations (1.1)–(1.2) is regular in Q_T provided that $\|u\|_{L^\infty(Q_T)} + \|b\|_{L^\infty(Q_T)} < \infty$.

It is well-known that global weak solutions for MHD exist in finite energy space (see [4]) and classical solutions can exist locally in time in $\mathbb{R}^3$. In other words, the weak solutions exist globally in time (see [4]), however, as shown in [14], if weak solutions (u, b) are fulfilled in $L^\infty(0, T; H^1(\mathbb{R}^3))$, they become regular.

In particular, for the regularity issue, lots of contributions have been made so far (see e.g. [1,5–8,18,19]).

We list only some results relevant to our concerns. In view of the regularity conditions in Lorentz space, He and Wang proved in [9] that a weak solution pair (u, b) become regular in the presence of a certain type of the integral conditions, typically referred to as Serrin's condition, namely,

$$u \in L^{q,\infty}(0, T; L^{p,\infty}(\mathbb{R}^3)) \quad \text{with} \quad 3/p + 2/q \le 1, \quad 3 < p \le \infty,$$

or

$$\nabla u \in L^{q,\infty}(0, T; L^{p,\infty}(\mathbb{R}^3)) \quad \text{with} \quad 3/p + 2/q \le 2, \quad \frac{3}{2} < p \le \infty.$$

On the other hand, Wang proved in [17] that a weak solution pair (u, b) become regular if u satisfies

$$u \in L^2(0, T; BMO(\mathbb{R}^3)).$$

Our study is motivated by these viewpoints, we obtain the regularity conditions for a weak solution to 3D MHD equations (1.1)–(1.2) in a whole space. Our proof is based on a priori estimate for the gradient of the velocity field. In the argument of proof, the pressure term is vanished due to the divergence structure of u and b.

Our main results reads as follows.

Theorem 1.2. *Suppose that (u, b) be a weak solution of 3D MHD equations (1.1)–(1.2) with initial condition $u_0, b_0 \in H^1(\Omega)$. If (u, b) satisfies*

$$\nabla u \in L^q((0, T); L^{p,\infty}(\mathbb{R}^3)), \qquad \frac{3}{p} + \frac{2}{q} = 2, \qquad \frac{3}{2} < p \le \infty,$$

then (u, b) is regular in Q_T.

Remark 1.3. Motivated by the work of He and Wang [9], we obtain weak type regularity condition with respect to the space variables only for the gradient of the velocity field. Substituting a priori estimate for the heat kernel method in [9], we can obtain the desired result.

Remark 1.4. We do not yet obtain the result in [2,12].

Theorem 1.5. *Suppose that (u, b) be a weak solution of 3D MHD equations (1.1)–(1.2) with initial condition $u_0, b_0 \in H^1(\Omega)$. If (u, b) satisfies one of the following two conditions:*

a.
$$\int_0^T \|\nabla u\|_{BMO(\mathbb{R}^3)} \, dt < \infty,$$

b.
$$\int_0^T \|\nabla u\|_{BMO^{-1}(\mathbb{R}^3)}^2 + \|\nabla b\|_{BMO^{-1}(\mathbb{R}^3)}^2 \, dt < \infty.$$

then (u, b) is regular in Q_T.

Remark 1.6. This result is improved the work of He and Wang [9] with respect to the gradient of the velocity field. Moreover, using the estimate in [16, Lemma A.5], we obtain $BMO^{-1}(\mathbb{R}^3)$-regularity condition.

This paper is organized as follows: in Section 2 we recall the notion of weak solutions and review some known results. In Section 3, we present the proofs of the Theorem 1.5.

2   Preliminaries

In this section we collect notations and definitions used throughout this paper. We also recall some lemmas, which are useful to our analysis. For $1 \leq q \leq \infty$, $W^{k,q}(\mathbb{R}^3)$ indicates the usual Sobolev space with standard norm $\|\cdot\|_{k,q}$, i.e.

$$W^{k,q}(\mathbb{R}^3) = \{u \in L^q(\mathbb{R}^3) : D^\alpha u \in L^q(\mathbb{R}^3), \ 0 \leq |\alpha| \leq k\}.$$

In case that $q = 2$, we write $W^{k,q}(\mathbb{R}^3)$ as $H^k(\mathbb{R}^3)$. All generic constants will be denoted by C, which may vary from line to line.

2.1   BMO and Lorentz spaces

The John–Nirenberg space or the space of the Bounded Mean Oscillation (in short BMO space) [10] consists of all functions f which are integrable on every ball $B_R(x) \subset \mathbb{R}^3$ and satisfy:

$$\|f\|_{BMO}^2 = \sup_{x \in \mathbb{R}^3} \sup_{x \in \mathbb{R}} \frac{1}{B(x,R)} \int_{B(x,R)} |f(y) - f_{B_R}(y)| \, dy < \infty.$$

Here, f_{B_R} denote for the average of f over all ball $B_R(x)$ in $\mathbb{R}^3$. It will be convenient to define BMO in terms of its dual space, $\mathcal{H}^1$. On the other hand, following [11] let w be the solution to the heat equation $w_t - \Delta w = 0$ with initial data v. Then

$$\|v\|_{BMO}^2 = \sup_{x,R} \frac{1}{B(x,R)} \int_{B(x,R)} \int_0^{R^2} |w|^2 \, dt \, dy,$$

and define the space BMO^{-1}-norms by

$$\|v\|_{BMO^{-1}}^2 = \sup_{x,R} \frac{1}{B(x,R)} \int_{B(x,R)} \int_0^{R^2} |\nabla w|^2 \, dt \, dy.$$

We note that suppose that u be a tempered distribution. Then $u \in BMO^{-1}$ if and only if there exist $f^i \in BMO$ with $u = \sum \partial_i f^i$ in [11, Theorem 1].

Let $m(\varphi, t)$ be the Lebesgue measure of the set $\{x \in \mathbb{R}^3 : |\varphi(x)| > t\}$, i.e.

$$m(\varphi, t) := m\{x \in \mathbb{R}^3 : |\varphi(x)| > t\}.$$

We consider the Lorentz space $L^{p,q}(\mathbb{R}^3)$ with $1 \leq p, q \leq \infty$ with the norm [15]

$$\|\varphi\|_{L^{p,q}(\mathbb{R}^3)} = \begin{cases} \left(\int_0^\infty t^q (m(\varphi,t))^{q/p} \dfrac{dt}{t} \right)^{1/q} < \infty, & \text{for } 1 \leq q < \infty, \\[2ex] \sup_{t \geq 0} \left\{ t(m(\varphi,t))^{\frac{1}{p}} \right\} < \infty, & \text{for } q = \infty. \end{cases}$$

In particular, in the case of bounded domains Ω, the Lorentz space $L^{p,\infty}(\mathbb{R}^3)$ is also called weak $L^p(\Omega)$ space with the norm, which is equivalent to the norm

$$\|f\|_{L^{q,\infty}(\Omega)} = \sup_{0<|\Omega|<\infty} |\Omega|^{1/q-1} \int_\Omega |f(x)|\, dx.$$

Following [15], the Lorentz space $L^{p,q}(\mathbb{R}^3)$ may be defined by real interpolation methods

$$L^{p,q}(\mathbb{R}^3) = (L^{p_1}(\mathbb{R}^3), L^{p_2}(\mathbb{R}^3))_{\alpha,q}, \tag{2.1}$$

with

$$\frac{1}{p} = \frac{1-\alpha}{p_1} + \frac{\alpha}{p_2}, \qquad 1 \le p_1 < p < p_2 \le \infty.$$

From the interpolation method above, we note that

$$L^{\frac{2p}{p-1},2}(\mathbb{R}^3) = \left(L^2(\mathbb{R}^3), L^6(\mathbb{R}^3)\right)_{\frac{3}{2p},2}.$$

We also need the Hölder inequality in Lorentz spaces (see [13] for a proof).

Lemma 2.1. *Assume $1 \le p_1, p_2 \le \infty$, $1 \le q_1, q_2 \le \infty$ and $u \in L^{p_1,q_1}(\mathbb{R}^3)$, $v \in L^{p_2,q_2}(\mathbb{R}^3)$. Then $uv \in L^{p_3,q_3}(\mathbb{R}^3)$ with $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q_3} \le \frac{1}{q_1} + \frac{1}{q_2}$, and the inequality*

$$\|uv\|_{L^{p_3,q_3}(\mathbb{R}^3)} \le C\|u\|_{L^{p_1,q_1}(\mathbb{R}^3)}\|v\|_{L^{p_2,q_2}(\mathbb{R}^3)}$$

is valid.

We recall first the definition of weak solutions.

Definition 2.2 (Weak solutions). Let $u_0, b_0 \in L^2_\sigma(\mathbb{R}^3)$. We say that (u,b) is a weak solution of (1.1) if u and b satisfy the following:

(i) $u \in L^\infty([0,T); L^2(\mathbb{R}^3)) \cap L^2([0,T); H^1(\mathbb{R}^3))$, $\quad b \in L^\infty([0,T); L^2(\mathbb{R}^3)) \cap L^2([0,T); H^1(\mathbb{R}^3))$.

(ii) (u,b) satisfies (1.1) in the sense of distribution; that is

$$\int_0^T \int_{\mathbb{R}^3} \left(\frac{\partial \phi}{\partial t} + \Delta\phi + (u\cdot\nabla)\phi\right) u\, dx\, dt + \int_{\mathbb{R}^3} u_0\phi(x,0)\, dx = \int_0^T \int_{\mathbb{R}^3} (b\cdot\nabla)\phi\, b\, dx\, dt,$$

$$\int_0^T \int_{\mathbb{R}^3} \left(\frac{\partial\phi}{\partial t} + \Delta\phi + (u\cdot\nabla)\phi\right) b\, dx\, dt + \int_{\mathbb{R}^3} b_0\phi(x,0)\, dx$$

$$= \int_0^T \int_{\mathbb{R}^3} (b\cdot\nabla)\phi\, u\, dx\, dt + \gamma\int_0^T \int_{\mathbb{R}^3} (\nabla\times b)\times b\cdot(\nabla\times\phi)\, dx\, dt,$$

for all $\phi \in C_0^\infty(\mathbb{R}^3 \times [0,T))$ with div $\phi = 0$, and

$$\int_{\mathbb{R}^3} u\cdot\nabla\psi\, dx = 0, \qquad \int_{\mathbb{R}^3} b\cdot\nabla\psi\, dx = 0,$$

for every $\psi \in C_0^\infty(\mathbb{R}^3)$. $\qquad\qquad\square$

3 Proofs of the theorems

Proof of Theorem 1.2.

Testing $-\Delta u$ and $-\Delta b$ to the fluid equation and by the magnetic equation of (1.1), respectively, using the integrating by parts, integrating on domain, we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}&(\|\nabla u(\tau)\|^2_{L^2(\mathbb{R}^3)} + \|\nabla b(\tau)\|^2_{L^2(\mathbb{R}^3)}) + \int_{\mathbb{R}^3}(|\Delta u|^2 + |\Delta b|^2)\,dx \\
&\leq -\int_{\mathbb{R}^3}\nabla[(u\cdot\nabla)u]:\nabla u\,dx + \int_{\mathbb{R}^3}\nabla[(b\cdot\nabla)b]:\nabla u\,dx \\
&\quad -\int_{\mathbb{R}^3}\nabla[(u\cdot\nabla)b]\cdot\nabla b\,dx + \int_{\mathbb{R}^3}\nabla[(b\cdot\nabla)u]:\nabla b\,dx \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}
\tag{3.1}
$$

We estimate separately the terms in the right hand side of (3.1). The first term I_1 is computed as follows:

$$
I_1 = \int_{\mathbb{R}^2}(\nabla u\cdot\nabla)u:\nabla u \leq \|\nabla u\|^3_{L^3},
$$

where the divergence free condition of u is used.

On the other hand, we observe that

$$
I_2 + I_4 = \int_{\mathbb{R}^2}(\nabla b\cdot\nabla)b\cdot\nabla u\,dx + \int_{\mathbb{R}^2}(\nabla b\cdot\nabla)u\cdot\nabla b\,dx.
$$

Indeed,

$$
\begin{aligned}
\int(b\cdot\nabla)\nabla b\cdot\nabla u\,dx &+ \int(b\cdot\nabla)\nabla u\cdot\nabla b\,dx \\
&= \sum_{j=1}^{3}\int b_j\left(\frac{\partial\nabla b}{\partial x_j}\nabla u\,dx + \frac{\partial\nabla u}{\partial x_j}\nabla b\right)dx = -\sum_{j=1}^{3}\int b_j\left(\frac{\partial(\nabla b\nabla u)}{\partial x_j}\right)dx = 0,
\end{aligned}
$$

where we use the product rule and $\operatorname{div} b = 0$. Summing up the terms I_1–I_4, we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}&(\|\nabla u(\tau)\|^2_{L^2} + \|\nabla b(\tau)\|^2_{L^2}) + \int_{\mathbb{R}^3}(|\Delta u|^2 + |\Delta b|^2)\,dx \\
&\leq \int_{\mathbb{R}^2}|\nabla u|^3\,dx - \int_{\mathbb{R}^2}(\nabla b\cdot\nabla)b\cdot\nabla u\,dx \\
&\quad -\int_{\mathbb{R}^2}(\nabla b\cdot\nabla)u\cdot\nabla b\,dx - \int_{\mathbb{R}^3}\nabla[(u\cdot\nabla)b]\cdot\nabla b\,dx.
\end{aligned}
\tag{3.2}
$$

First of all, using the interpolation (2.1), Lemma 2.1, Hölder and Young's inequalities, we estimate the second term as follows:

$$
\begin{aligned}
\int_{\mathbb{R}^3}|\nabla b|^2|\nabla u|\,dx &\leq \|\nabla u\|_{L^{q,\infty}}\||\nabla b|^2\|_{L^{\frac{q}{q-1},1}} \\
&= \|\nabla u\|_{L^{q,\infty}}\|\nabla b\|^2_{L^{\frac{2q}{q-1},1}} \\
&\leq C\|\nabla u\|_{L^{q,\infty}}\|\nabla b\|^{2\theta}_{L^2}\|\nabla^2 b\|^{2(1-\theta)}_{L^2} \\
&\leq C\|\nabla u\|^{\frac{2q}{2q-3}}_{L^{q,\infty}}\|\nabla b\|^2_{L^2} + \frac{1}{16}\|\nabla^2 b\|^2_{L^2},
\end{aligned}
$$

where $\theta = 1 - \frac{3}{2q}$. Similarly, we have

$$
\int_{\mathbb{R}^3}|\nabla u|^3\,dx \leq C\|\nabla u\|^{\frac{2q}{2q-3}}_{L^{q,\infty}}\|\nabla u\|^2_{L^2} + \frac{1}{16}\|\nabla^2 u\|^2_{L^2}.
$$

Using the estimates above, (3.1) becomes

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \frac{3}{4}\int(|\nabla^2 u|^2 + |\nabla^2 b|^2)\,dx \le C\|\nabla u\|_{L^{q,\infty}}^{\frac{2q}{2q-3}}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2). \quad (3.3)$$

Under the given condition, we apply Gronwall's inequality to estimate (3.3)

$$\sup_{0<\tau\le T}(\|\nabla u(\tau)\|_{L^2}^2 + \|\nabla b(\tau)\|_{L^2}^2) + \int_0^T\int(|\nabla^2 u|^2 + |\nabla^2 b|^2)\,dx\,dt$$

$$\le C(\|\nabla u(0)\|_{L^2}^2 + \|\nabla b(0)\|_{L^2}^2).$$

$\square$

Proof of Theorem 1.5.   In this proof, we only need to estimate the convection terms in the previous proof as follows

a.

$$\int_{\mathbb{R}^3}|\nabla b|^2|\nabla u|\,dx \le \|\nabla u\|_{BMO}\||\nabla b|^2\|_{\mathcal{H}^1}$$

$$\le \|\nabla u\|_{BMO}\|\nabla b\|_{L^2}\|\nabla b\|_{L^2}$$

$$= \|\nabla u\|_{BMO}\|\nabla b\|_{L^2}^2.$$

Similarly, we obtain

$$\int_{\mathbb{R}^3}|\nabla u|^3\,dx \le C\|\nabla u\|_{BMO}\|\nabla u\|_{L^2}^2.$$

Using the estimates above, (3.1) becomes

$$\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \int(|\nabla^2 u|^2 + |\nabla^2 b|^2)\,dx$$

$$\le C\|\nabla u\|_{BMO}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2). \quad (3.4)$$

b. Following [16, Lemma A.5], we note that

$$\|u\|_{L^4}^2 = \|uu\|_{L^2} \le C\|\nabla u\|_{L^2}\|u\|_{BMO^{-1}}.$$

Using this estimate, we have

$$\int_{\mathbb{R}^3}|\nabla b|^2|\nabla u|\,dx \le \|\nabla u\|_{L^2}\|\nabla b\|_{L^4}^2$$

$$\le C\|\nabla u\|_{L^2}\|\nabla^2 b\|_{L^2}\|\nabla b\|_{BMO^{-1}}$$

$$\le C\|\nabla u\|_{L^2}^2\|\nabla b\|_{BMO^{-1}}^2 + \frac{1}{8}\|\nabla^2 b\|_{L^2}^2.$$

Similarly, we obtain

$$\int_{\mathbb{R}^3}|\nabla u|^3\,dx \le C\|\nabla u\|_{L^2}^2\|\nabla u\|_{BMO^{-1}}^2 + \frac{1}{8}\|\nabla^2 u\|_{L^2}^2.$$

Using the estimates above, (3.1) becomes

$$\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \int(|\nabla^2 u|^2 + |\nabla^2 b|^2)\,dx$$

$$\le C(\|\nabla u\|_{BMO^{-1}}^2 + \|\nabla b\|_{BMO^{-1}}^2)(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2). \quad (3.5)$$

So then, we apply Gronwall's inequality to estimates (3.4) and (3.5) to find

$$\sup_{0<\tau\leq T}(\|\nabla u(\tau)\|_{L^2}^2 + \|\nabla b(\tau)\|_{L^2}^2) + \int_0^T \int (|\nabla^2 u|^2 + |\nabla^2 b|^2)\, dx\, dt \leq C.$$

$\square$

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