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Fully Nonlinear Boundary Value Problems with Impulses

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Abstract

An impulsive boundary value problem with nonlinear boundary conditions for a second order ordinary differential equation is studied. In particular, sufficient conditions are provided so that a compression - expansion cone theoretic fixed point theorem can be applied to imply the existence of positive solutions. The nonlinear forcing term is assumed to satisfy usual sublinear or superlinear growth as $t \to \infty$ or $t \to 0^+$. The nonlinear impulse terms and the nonlinear boundary terms are assumed to satisfy the analogous asymptotic behavior.

Key words and phrases: boundary value problem with impulse, nonlinear boundary condition, compression - expansion fixed point theorem.
AMS (MOS) Subject Classifications: 34B10

1 Introduction

A well-known compression - expansion cone theoretic fixed point theorem due to Krasnosel'skii [18] and Guo [16] has been employed extensively to many types of boundary value problems. We refer the reader to [2] and its bibliography for some of the many citations. Early applications appear in the case of partial differential equations, [3, 4, 6], for example; in a landmark paper [14], Erbe and Wang introduced the applications to ordinary differential equations. For their primary applications, they showed that under the assumptions that the nonlinear term exhibits superlinear or sublinear growth, the fixed point theorem applies readily to boundary value problems whose solutions exhibit a natural type of concavity.

The purpose of this study is to consider a simple boundary value problem for a nonlinear ordinary differential equation, simple in the context that solutions exhibit a natural type of concavity and simple in the context that we restrict ourselves to a second order differential operator with boundary conditions of right focal type. In addition to assuming a nonlinearity in the differential equation, we shall assume nonlinear boundary conditions; moreover, we shall assume impact due to nonlinear impulse effects.

The contribution of this study is two-fold. First, methods of upper and lower solutions (which then employ the Schauder fixed point theorem) and monotone methods have routinely been employed to problems with impulse or nonlinear boundary conditions, see [8, 12, 13] for example. Applications of cone theoretic fixed point theorems to problems with impulse are more recent and we cite [1, 5, 15, 19, 20], for example; so in this article we include dependence on nonlinear boundary conditions as well. Recently, Shen and Wang [21] considered a second order impulsive problem with nonlinear boundary conditions and they employed the method of upper and lower solutions and the Schauder fixed point theorem. Second, assuming the nonlinear term in the differential equation satisfies standard superlinear or sublinear asymptotic conditions that imply existence of solutions, we stress that the nonlinear boundary terms or impulse terms can assume completely analogous conditions and obtain sufficient conditions for the existence of solutions. The article by Lin and Jiang [19] is closely related; however, they allow impulses only in the derivative and their solutions are piecewise smooth. Moreover, we exploit that on the boundary in which expansion is employed, only one of the nonlinear terms needs to exhibit the appropriate asymptotic growth.

Finally, we point out that applications of the cone theoretic fixed point theory are abundant with applications to multiple (countably infinite) fixed points ([9] or [10]), nonlinear eigenvalue problems [17], problems with nonlinear dependence on higher order derivatives [7], or conditions for nonexistence of positive solutions [22]. The theory applies readily to singular problems or nonautonomous problems. We do not pursue these variations or generalizations in this short article; we restrict ourselves to the simply posed problem.

2 Preliminaries

We begin with the statement of the Krasnosel'skii/Guo fixed point theorem [18], [16].

Theorem 2.1. *Let B be a Banach space, $P \subset B$ a cone in B. Assume Ω_1, Ω_2 are open balls in B with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$. Assume*

$$K : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$$

is a completely continuous operator such that

(i) $||Kx|| \leq ||x||, x \in P \cap \partial\Omega_1,$ *and* $||Kx|| \geq ||x||, x \in P \cap \partial\Omega_2$; *or*

(ii) $||Kx|| \geq ||x||, x \in \cap\partial\Omega_1,$ *and* $||Kx|| \leq ||x||, x \in P \cap \partial\Omega_2$.

Then K *has a fixed point in* $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Let $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$. We shall assume throughout that

$$f : [0,\infty) \to [0,\infty), \quad a : [0,\infty) \to [0,\infty), \quad b : [0,\infty) \to [0,\infty),$$

$$u_k : [0,\infty) \to [0,\infty), \quad -v_k : [0,\infty) \to [0,\infty), \quad k = 1, \dots, m.$$

We shall also assume throughout that f, a, b, u_k, v_k are all continuous functions on $[0,\infty)$.

We shall study the boundary value problem with impulse,

$$x''(t) + f(x(t)) = 0, \quad t_k < t < t_{k+1}, \quad k = 0, \dots, m, \tag{2.1}$$

$$\begin{cases} \Delta x(t_k) & = u_k(x(t_k^-)), \\ \Delta x'(t_k) & = v_k(x(t_k^-)), \end{cases}, \quad k = 1, \dots, m, \tag{2.2}$$

$$x(0) = a(x), \quad x'(1) = b(x), \tag{2.3}$$

where $\Delta x(t) = \lim_{s\to t^+} x(s) - \lim_{s\to t^-} x(s)$ and $x(t_k^-) = \lim_{s\to t_k^-} x(s)$.

Let $PC[0,1]$ denote the set of piecewise continuous functions on $[0,1]$. For $x \in PC[0,1]$, for each $t_k \leq t \leq t_{k+1}$, $k = 0, \dots, m$, define

$$\overline{x}_k(t) = \begin{cases} \lim_{s\to t^+} x(s), & t_k \leq t < t_{k+1}, \\ \lim_{s\to t_{k+1}^-} x(s), & t = t_{k+1}. \end{cases}$$

Define the Banach space B by

$$B = \{x \in PC[0,1] : \overline{x}_k \in C[t_k, t_{k+1}], k = 0, \dots m\},$$

with $||x|| = \max_{k=0,\dots m} \max_{t\in[t_k,t_{k+1}]} |\overline{x}_k(t)|$. Define an operator K on B by

$$Kx(t) = b(x)t + a(x) + I(t,x) + \int_0^1 G(t,s)f(x(s))ds, \quad 0 \leq t \leq 1, \tag{2.4}$$

where

$$I(t,x) = -\sum_{j=i+1}^{m} v_j(x(t_j))t - \sum_{j=1}^{i} v_j(x(t_j))t_j + \sum_{j=1}^{i} u_j(x(t_j)),$$

if $t_i \le t < t_{i+1}, i = 0, \dots m-1$,

$$I(t,x) = \sum_{j=1}^{m} (-v_j(x(t_j))t_j + u_j(x(t_j))),$$

if $t_m \le t \le 1$, and

$$G(t,s) = \begin{cases} s & : \ 0 \le s < t \le 1, \\ t & : \ 0 \le t < s \le 1. \end{cases} \tag{2.5}$$

Lemma 2.2. *x is a solution of the boundary value problem, (2.1), (2.2), (2.3), if, and only if, $x \in B$ and $x(t) = Kx(t), 0 \le t \le 1$.*

Assuming the continuity of each of the nonlinear terms, f, a, b, u_k, v_k, $k = 1, \dots, m$, standard applications of the Arzela-Ascoli theorem on each subinterval, $[t_k, t_{k+1}]$, imply that $K : B \to B$ is a completely continuous operator.

If $x \in C^2[0,1]$, $x''(t) \le 0, 0 < t < 1$, $x(0) = x'(1) = 0$, then

$$x(t) \ge t_m \max_{0 \le t \le 1} |x(t)| = t_m x(1), \quad t_m \le t \le 1.$$

Direct calculations also show that

$$G(t,s) \ge t_m G(s,s) = t_m \max_{0 \le t \le 1} G(t,s), \quad t_m \le t \le 1.$$

Motivated by these inequalities, define the cone $P \subset B$ by

$$P = \{x \in B : \ x \text{ is nonnegative and nondecreasing, and } x(t) \ge t_m ||x||, t_m \le t \le 1\}.$$

Lemma 2.3. *$K : P \to P$ is completely continuous.*

Proof. We have already addressed the complete continuity of K. We address $K(P) \subset P$. The conditions on the nonlinear terms imply that if x is nonnegative then Kx is nonnegative. For $t_i < t < t_{i+1}$, $i = 1, \dots, m$,

$$\frac{d}{dt} Kx(t) = b(x) - \sum_{j=i+1}^{m} v_j(x(t_j)) + \int_0^1 G_t(t,s) f(x(s)) ds \ge 0.$$

The $-v_i$ terms are all nonnegative and so, Kx is nondecreasing. Finally, note that

$$Kx(t) \le b(x) + a(x) + \sum_{j=1}^{m} -v_j(x(t_j))t_j + \sum_{j=1}^{m} u_j(x(t_j)) + \int_0^1 G(s,s) f(x(s)) ds, \quad 0 \le t \le 1.$$

In particular,

$$||Kx|| \le b(x) + a(x) + \sum_{j=1}^{m} -v_j(x(t_j))t_j + \sum_{j=1}^{m} u_j(x(t_j)) + \int_0^1 G(s,s)f(x(s))ds.$$

Let $t \in [t_m, 1]$. Then

$$\begin{aligned} Kx(t) &= b(x)t + a(x) + \sum_{j=1}^{m}(-v_j(x(t_j))t_j + u_j(x(t_j))) + \int_0^1 G(t,s)f(x(s))ds \\ &\ge t_m(b(x) + a(x) + \sum_{j=1}^{m}(-v_j(x(t_j))t_j + u_j(x(t_j))) + \int_0^1 G(s,s)f(x(s))ds) \\ &\ge t_m||Kx||. \end{aligned}$$

□

3 Applications of the Fixed Point Theorem

As stated in the introduction, we note that sublinear or superlinear growth of f couples very nicely with the compression - expansion fixed point theorem. So in that context, let

$$f_0 = \lim_{x \to 0^+} \frac{f(x)}{x}, \quad f_\infty = \lim_{x \to \infty} \frac{f(x)}{x},$$

$$a_0 = \lim_{x \to 0^+} \frac{a(x)}{x}, \quad a_\infty = \lim_{x \to \infty} \frac{a(x)}{x}, \quad b_0 = \lim_{x \to 0^+} \frac{b(x)}{x}, \quad b_\infty = \lim_{b \to \infty} \frac{b(x)}{x}$$

$$u_{i0} = \lim_{x \to 0^+} \frac{u_i(x)}{x}, \quad u_{i\infty} = \lim_{x \to \infty} \frac{u_i(x)}{x}, \quad i = 1, \dots, m,$$

$$v_{i0} = \lim_{x \to 0^+} \frac{v_i(x)}{x}, \quad v_{i\infty} = \lim_{x \to \infty} \frac{v_i(x)}{x}, \quad i = 1, \dots, m.$$

Note that

$$\left| \int_0^1 G(t,s)ds \right| \le \int_0^1 G(s,s)ds = \frac{1}{2}, \quad 0 \le t \le 1, \tag{3.1}$$

$$\left| \int_{t_m}^1 G(t_m,s)ds \right| = t_m(1 - t_m). \tag{3.2}$$

Theorem 3.1. *Assume $f_0 = a_0 = b_0 = u_{i0} = v_{i0} = 0, i = 1 \dots, m$, and assume $f_\infty = \infty$. Then the BVP with impulse, (2.1), (2.2), (2.3), has a nontrivial solution, $x \in P$.*

Proof. First, choose $H_1 > 0$ such that if $0 < x \leq H_1$, then $f(x) \leq \eta_1 x$ where $\eta_1 = \frac{2}{3}$. Second, choose $H_2 > 0$ such that if $0 < x \leq H_2, -v_j(x) + u_j(x) \leq \frac{x}{3m}$. Finally, choose $H_3 > 0$ such that if $0 < x \leq H_3$, then $a(x) + b(x) \leq \frac{x}{3}$. Set $H = \min\{H_1, H_2, H_3\}$.

Define $\Omega_1 = \{x \in B : ||x|| < H\}$. Let $x \in P \cap \partial\Omega_1$. Then apply (3.1) to see that

$$\begin{aligned}|Kx(t)| &\leq (b(x) + a(x)) + \sum_{j=1}^{m}(-v_j(x(t_j)) + u_j(x(t_j))) + \int_0^1 G(s,s)f(x(s))ds \\ &\leq (\frac{1}{3} + \frac{1}{3} + \frac{1}{3})H = H;\end{aligned}$$

in particular, $||Kx|| \leq ||x||$ for $x \in P \cap \partial\Omega_1$.

To construct Ω_2, employ $f_\infty = \infty$ and choose $H_4 > 0$ such that if $x \geq H_4$, then $f(x) \geq \mu x$, where

$$\mu \geq (t_m \int_{t_m}^1 G(t_m, s)ds)^{-1} = (t_m^2(1 - t_m))^{-1}. \tag{3.3}$$

Set $\hat{H} = \max\{2H, t_m^{-1}H_4\}$ and define $\Omega_2 = \{x \in B : ||x|| < \hat{H}\}$. Let $x \in P \cap \partial\Omega_2$. Then $x(s) \geq H_4$, if $t_m \leq s \leq 1$ and

$$\begin{aligned}Kx(t_m) &\geq \int_{t_m}^1 G(t_m, s)f(x(s))ds \geq \mu \int_{t_m}^1 G(t_m, s)x(s)ds \\ &\geq \mu(t_m \int_{t_m}^1 G(t_m, s)ds)||x|| \geq ||x||;\end{aligned}$$

in particular, $||Kx|| \geq ||x||$ for $x \in P \cap \partial\Omega_2$.

Apply Theorem 2.1 and the proof is complete. □

Note that in Theorem 3.1 each of the superlinear conditions at $x = 0$ was employed whereas only one of the superlinear conditions at ∞ was employed. Variations of Theorem 3.1 are easily obtained if superlinear boundary conditions or superlinear impulse conditions at ∞ are employed.

For example, consider a more specific boundary value problem, (2.1), (2.2), with boundary conditions

$$x(0) = x^2(t_m), \quad x'(1) = b(x). \tag{3.4}$$

Theorem 3.2. *Assume $f_0 = b_0 = u_{i0} = v_{i0} = 0, i = 1\ldots, m$. Then the BVP with impulse, (2.1), (2.2), (3.4), has a nontrivial solution, $x \in P$.*

Proof. The details to construct Ω_1 (and H) are precisely as in the proof of Theorem 3.1. To construct Ω_2, employ the condition, $a_\infty = \infty$, where $a(x(t_m)) = x^2(t_m)$. Choose $H_4 > 0$ such that if $x \geq H_4$, then $x^2 \geq \mu x$, where

$$\mu = (t_m)^{-1}.$$

Set $\hat{H} = \max\{2H, (t_m)^{-1}H_4\}$. Then, if $x \in P$ and $||x|| = \hat{H}$, then

$$|Kx(t)| = x^2(t_m) \geq (t_m)^{-1}x(t_m) \geq ||x||.$$

$\square$

Analogous results can be stated if b or if any one of the impulse functions satisfy superlinear growth at $t \to \infty$.

We now address the sublinear case. The sublinear case is the more delicate case since one must consider bounded nonlinearities or unbounded nonlinearities on unbounded domains with separate arguments. Since we consider multiple nonlinearities and essentially construct Ω_2 as an intersection of unbounded sets, we shall assume a monotonicity condition in unbounded nonlinearities that is not needed by Erbe and Wang [14].

Definition 3.3. *We shall say that h satisfies Hypothesis H if $h : [0,\infty) \to [0,\infty)$ is continuous, $h_\infty = \lim_{x\to\infty} \frac{h(x)}{x} = 0$, and either h is bounded or h is unbounded as $t \to \infty$ and h is eventually monotone increasing.*

Theorem 3.4. *Assume that each of $f, a, b, u_i, v_i, i = 1\ldots, m$, satisfies Hypothesis H and assume $f_0 = \infty$. Then the BVP with impulse, (2.1), (2.2), (2.3), has a nontrivial solution, $x \in P$.*

Proof. Precisely as in (3.3), choose $H > 0$ such that such that if $x \leq H$, then $f(x) \geq \mu x$, where

$$\mu \geq (t_m \int_{t_m}^1 G(t_m, s)ds)^{-1} = (t_m^2(1 - t_m))^{-1}.$$

Let $\Omega_1 = \{x \in B : ||x|| < H\}$. If $x \in P$ and $||x|| = H$,

$$Kx(t_m) \geq \int_{t_m}^1 G(t_m, s)f(x(s))ds \geq ||x||$$

and $||Kx|| \geq ||x||$ for $x \in P \cap \partial\Omega_1$.

To construct Ω_2, we consider f, the impulses, and the boundary conditions independently.

Suppose f is bounded; assume $f(x) \leq M$ for all $x \in (0, \infty)$. Choose $\hat{H}_2 \geq \max\{2H, 3M \int_0^1 G(s,s)ds\} = \max\{2H, \frac{3M}{2}\}$. Then, if $x \in P$, and $||x|| = \hat{H}_2$,

$$\int_0^1 G(t,s)f(x(s))ds \leq M \int_0^1 G(s,s)ds \leq \frac{\hat{H}_2}{3} = \frac{||x||}{3}.$$

Now assume f is unbounded. Let $\tilde{H}_2 > 0$ be such that $f(x) \leq \frac{2x}{3}$ for $x \geq \tilde{H}_2$. Let $\overline{H}_2 \geq \max\{2H, \tilde{H}_2\}$, such that $f(x) \leq f(\overline{H}_2)$ if $0 < x < \overline{H}_2$. (Note, as in [14], monotonicity, introduced in the definition of Hypothesis H, is not needed to imply the existence of $\overline{H}_2$ such that $f(x) \leq f(\overline{H}_2)$ if $0 < x < \overline{H}_2$; monotonicity will be applied later.) Then, if $x \in P$, $||x|| = \overline{H}_2$,

$$\begin{aligned}\int_0^1 G(t,s)f(x(s))ds &\leq \int_0^1 G(s,s)f(x(s))ds \leq \int_0^1 G(s,s)f(\overline{H}_2)ds \\ &\leq \frac{2\overline{H}_2}{3}\int_0^1 G(s,s)ds = \frac{\overline{H}_2}{3} = \frac{||x||}{3}.\end{aligned}$$

Set $H_2 \geq \hat{H}_2$ if f is bounded; set $H_2 \geq \overline{H}_2$ if f is unbounded. (It is precisely here, to set $H_2 \geq \overline{H}_2$, that the monotonicity condition is used.)

The analyses for the impulse effect and the boundary conditions are similar. We provide the details for the boundary conditions of which there are two. The details for the impulse effect are completely analogous; we do not provide those details as there are $2m$ related conditions.

If b is bounded, say $b(x) \leq M_b$ for all $x \in (0, \infty)$. Choose $\hat{H}_3 \geq \max\{2H, 6M_b\}$. Then if $x \in P$ and $||x|| = \hat{H}_3$, then $b(x) \leq \frac{\hat{H}_3}{6}$. If a is bounded, say $a(x) \leq M_a$ for all $x \in (0, \infty)$. Choose $\hat{H}_4 \geq \max\{2H, 6M_a\}$. Then if $x \in P$ and $||x|| = \hat{H}_4$, then $a(x) \leq \frac{\hat{H}_4}{6}$.

If b is unbounded, let $\overline{H}_5 \geq 2H$ be such that if $x \geq \overline{H}_5$, $b(x) \leq \frac{x}{6}$, and $b(x) \leq b(\overline{H}_5)$, for $0 < x \leq \overline{H}_5$. If a is unbounded, let $\overline{H}_6 \geq 2H$ be such that if $x \geq \overline{H}_6$, $a(x) \leq \frac{x}{6}$, and $a(x) \leq a(\overline{H}_6)$, for $0 < x \leq \overline{H}_6$.

Now assume for example in a specific boundary value problem, (2.1), (2.2), (2.3), that f is unbounded, b is bounded and a is unbounded. Set $H_2 \geq \max\{\overline{H}_2, \hat{H}_3, \overline{H}_6\}$; if $x \in P$ and $||x|| = H_2$, then

$$\int_0^1 G(t,s)f(x(s))ds \leq \frac{||x||}{3}$$

and

$$b(x)t + a(x) \leq b(x) + a(x) \leq \frac{||x||}{6} + \frac{||x||}{6} = \frac{||x||}{3}.$$

We briefly indicate how to finish the argument. Choose an impulse effect, v_j, say. If v_j is bounded, say $-v_j(x) \leq M_v$ for all $x \in (0, \infty)$. Choose $\hat{H}_v \geq \max\{2H, 6mM_v\}$. If v_j is unbounded, let $\overline{H}_v \geq 2H$ be such that if $x \geq \overline{H}_v$, then $-v_j(x) \leq \frac{x}{6m}$, and $-v_j(x) \leq v_j(\overline{H}_v)$, for $0 < x \leq \overline{H}_v$.

Thus, for a specific boundary value problem, (2.1), (2.2), (2.3), construct $2m+3$ radii depending on the boundedness or unboundedness nature each of the $2m+3$ nonlinear terms. Choose H_2 greater than or equal to the maximum of $2H$ and the maximum of these $2m+3$ radii.

□

As pointed out in Theorem 3.2, there are many related theorems to Theorem 3.4 in which we employ only one sublinear condition at 0.

As a closing comment, consider nonhomogeneous boundary conditions or nonhomogeneous impulse effects. So, for example, consider a more specific boundary value problem, (2.1), (2.2), with boundary conditions

$$x(0) = a > 0, \quad x'(1) = b(x). \tag{3.5}$$

Theorem 3.5. *Assume that each of $f, b, u_i, v_i, i = 1 \ldots, m$, satisfies Hypothesis H. Then the BVP with impulse, (2.1), (2.2), (3.5), has a nontrivial solution, $x \in P$.*

Theorem 3.5 can also be obtained by the Schauder fixed point theorem as well. The domain $\overline{\Omega}_2$, constructed in the proof of Theorem 3.4 provides the compact domain that is needed for the application of the Schauder fixed point theorem. So, a fixed point exists. All of the sign assumptions on the nonlinear or nonhomogeneous terms give that the fixed point is nonnegative. Finally, $a > 0$ implies the fixed point is nontrivial.

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