Electronic Journal of Qualitative Theory of Differential Equations
2011, No. **5**, 1-6; **http://www.math.u-szeged.hu/ejqtde/**

On the fixed point theorem of Krasnoselskii and Sobolev

Cristina G. Fuentes and Hernán R. Henríquez *

Abstract

We formulate a version of the fixed point theorem of Krasnoselskii and Sobolev in locally convex spaces. We apply this result to establish the existence of solutions of an integral equation defined in an abstract space.

Mathematics Subject Classification Numbers 2010: Primary 47H10; Secondary 45N05

Key Words and Phrases. Fixed points of maps, integral equations in abstract spaces.

1 Fixed Points of Monotone Operators

In this work we are concerned with the existence of fixed points for a class of monotone operators defined on locally convex spaces. For generalities about locally convex spaces we refer the reader to [1].

Let X be a Hausdorff locally convex space. Let $K \subseteq X$ be a closed convex cone such that $K \cap (-K) = \{0\}$. The cone K defines the partial ordering in X given by $x \leq y$ if $y - x \in K$. An operator $\Gamma : X \to X$ is called monotone if $x, y \in X$, $x \leq y$ implies that $\Gamma x \leq \Gamma y$. The operator Γ is said to be monotonically limit compact on a bounded set $M \subseteq X$ if for each sequence $(x_n)_n$ in M such that

$$x_0 \geq \Gamma x_1 \geq \Gamma^2 x_2 \cdots \Gamma^n x_n \geq \cdots$$

*Corresponding author. The research of this author was supported in part by CONICYT under Grant FONDECYT 1090009.

we have that the sequence $(\Gamma^n x_n)_n$ is convergent. We refer the reader to [2] for a discussion about the properties of monotonically limit compact operators. In particular, the following result [2, Theorem 38.2] has been established by Krasnoselskii and Sobolev.

Theorem 1.1. *Let X be a Banach space. Let $\Gamma : X \to X$ be a monotone operator which is monotonically limit compact on a closed bounded set $M \subseteq X$ such that $\Gamma M \subseteq M$. If there is an $x_0 \in M$ with $\Gamma x_0 \leq x_0$, then Γ has a fixed point in M.*

Henceforth we will assume that (X, τ) is a Hausdorff locally convex space which satisfies the following property:

(P) For every closed and bounded set $A \subseteq X$ the induced topology is metrizable and complete.

It is clear that every Fréchet space satisfies the property (P). In addition, if X is a reflexive separable Banach space, then $(X, \sigma(X, X^*))$ also satisfies the property (P).

The main result of this note is the following extension of Theorem 1.1.

Theorem 1.2. *Let X be a real Hausdorff locally convex space which satisfies the property (P). Let $\Gamma : X \to X$ be a monotone operator which is monotonically limit compact on a closed bounded set $M \subseteq X$ such that $\Gamma M \subseteq M$. If there is an $x_0 \in M$ with $\Gamma x_0 \leq x_0$, then Γ has a fixed point in M.*

Proof. We only need to modify slightly the construction carried out in the proof of [2, Theorem 38.2]. Let $N \subseteq X$ be a closed bounded absolutely convex set such that $M - M \subseteq N$. Let ρ be a metric on N which induces the relative topology τ in N. We can assume that ρ is bounded.

We set $M_0 = \{x \in M : \Gamma x \leq x\}$. It is immediate that $\Gamma M_0 \subseteq M_0$.

For $x \in M_0$ and $j \in \mathbb{N}$ we define

$$\alpha_j(x) = \sup\{\rho(\Gamma^j w - \Gamma^j v, 0) : v, w \in M_0,\ \Gamma^j v \leq \Gamma^j w \leq x\}.$$

It is easy to see that $\alpha_j(x)$ makes sense and that the sequence $(\alpha_j(x))_j$ is nonincreasing. Hence we can define $\alpha(x) = \lim_{j\to\infty} \alpha_j(x)$. It follows from the definition that $\alpha(\Gamma x) \leq \alpha(x)$.

We assert that $\inf\{\alpha(u) : u \in M_0, u \leq x\} = 0$ for any $x \in M_0$. In fact, if we assume that this assertion is false, proceeding inductively we can construct a sequence

$$x \geq \Gamma^2 w_1 \geq \Gamma^2 v_1 \geq \Gamma^4 w_2 \geq \Gamma^4 v_2 \geq \cdots$$

such that $\rho(\Gamma^{2j}w_j - \Gamma^{2j}v_j, 0) > \beta$ for some $\beta > 0$. This implies the sequence does not converge, which contradicts our hypothesis that Γ is a monotonically limit compact operator.

Proceeding similarly we can construct a sequence $(x_n)_n$ in M_0 such that

$$x_0 \geq \Gamma x_1 \geq \Gamma^2 x_2 \geq \cdots$$

and $\alpha(\Gamma^n x_n) < \frac{1}{n}$ for $n \in \mathbb{N}$. Using again that Γ is a monotonically limit compact operator, we can affirm that there exists $z \in M$ such that $\Gamma^n x_n \to z$ as $n \to \infty$. Since $\Gamma^n x_n \geq \Gamma^{n+1} x_n \geq \Gamma z$ we have that $z \geq \Gamma z$. Moreover, for each $n \in \mathbb{N}$ there exists $m_n \in \mathbb{N}$ such that $\rho(\Gamma^{n+m_n} x_{n+m_n} - \Gamma^{n+m_n} z, 0) \leq \frac{1}{n}$. In view of that

$$z - \Gamma z \leq \Gamma^{n+m_n} x_{n+m_n} - \Gamma^{n+m_n} z$$

it follows that $z \leq \Gamma z$. Consequently, $z = \Gamma z$. □

2 Applications

The fixed point Theorem 1.2 can be applied to study the existence of solutions of many ordinary or integral equations, with or without delay, defined by a non-continuous function, on an unbounded interval of $\mathbb{R}$.

In this section we will show the usefulness of the Theorem 1.2 to establish sufficient conditions for the existence of solutions of an abstract integral equation on $\mathbb{R}$.

To specify the problem under consideration, we will assume that $(X, \|\cdot\|)$ is a Banach space and that $K \subseteq X$ is a closed convex cone such that $K \cap (-K) = \{0\}$. We will assume that K is a fully regular cone, which means that every norm bounded nondecreasing sequence in X is convergent. We denote by $\leq$ the partial ordering induced by K in X such as was established in the Section 1. Let $C(\mathbb{R}, X)$ be the space consisting of the continuous functions $x : \mathbb{R} \to X$ endowed with the compact-open topology. It is well known that $C(\mathbb{R}, X)$ is a Fréchet space. Moreover,

$$\widetilde{K} = \{x \in C(\mathbb{R}, X) : x(t) \in K\}$$

is a closed convex cone such that $\widetilde{K} \cap (-\widetilde{K}) = \{0\}$. Hence we can consider $C(\mathbb{R}, X)$ as an ordered locally convex space.

We consider the following integral equation

$$x(t) = h(t) + \int_{-\infty}^{t} k(t,s) f(s, x(s)) ds, \quad t \in \mathbb{R}, \tag{2.1}$$

where $x, h : \mathbb{R} \to X$ and $k : \Delta = \{(t,s) : t \in \mathbb{R}, -\infty < s \le t\} \to [0,\infty)$ are continuous functions. Moreover, the values $h(s) \le 0$ for all $s \in \mathbb{R}$, and the function $f : \mathbb{R} \times X \to X$ verifies the following conditions:

(f1) The function f is Borel-measurable.

(f2) For every separable set $A \subseteq X$ the set $f(\mathbb{R} \times A)$ is separable.

(f3) There exists a locally integrable function $g : \mathbb{R} \to [0,\infty)$ such that $\|f(s,x)\| \le g(s)$ for all $s \in \mathbb{R}$, $x \in X$, and $\int_{-\infty}^{t} k(t,s)g(s)ds < \infty$.

(f4) If $x, y \in X$, $x \le y$, then $f(s,x) \le f(s,y)$ for all $s \in \mathbb{R}$.

For example, a function f which is the pointwise limit of a sequence of continuous functions satisfies conditions (f1) and (f2).

Remark 2.1. *If f satisfies conditions (f1)-(f3) and $x : \mathbb{R} \to X$ is a continuous function, it follows from (f1), (f2) and [3, Proposition 2.2.6] that the function $s \to f(s, x(s))$ is strongly measurable on $(-\infty, t]$ for every $t \in \mathbb{R}$. Combining this property with (f3) and the Lebesgue dominated convergence theorem [3, Théorème 2.4.7] we infer that the function $s \to f(s, x(s))$ is integrable on $(-\infty, t]$ for every $t \in \mathbb{R}$.*

Theorem 2.3. *Under the above assumptions, there exists a continuous solution of the equation (2.1).*

Proof. We define the map $\Gamma : C(\mathbb{R}, X) \to C(\mathbb{R}, X)$ by

$$\Gamma x(t) = h(t) + \int_{-\infty}^{t} k(t,s) f(s, x(s))ds, \quad t \in \mathbb{R}. \tag{2.2}$$

It follows from the preceding remark that Γ is well defined. Moreover, if $x(\cdot) \le y(\cdot)$, it follows from (f4) that $f(s, x(s)) \le f(s, y(s))$, which means that $k(t,s)[f(s,y(s)) - f(s, x(s))] \in K$. Let $a < t$ be fixed. Since K is a closed convex cone, applying the mean value theorem for the Bochner integral we obtain that

$$\int_{a}^{t} k(t,s)[f(s,y(s)) - f(s,x(s))]ds \in (t-a)\overline{c(K)} \subseteq K,$$

where $c(\cdot)$ denotes the convex hull. This implies that

$$\begin{aligned}&\int_{-\infty}^{t} k(t,s)[f(s,y(s)) - f(s,x(s))]ds\\ &= \lim_{a \to -\infty} \int_{a}^{t} k(t,s)[f(s,y(s)) - f(s,x(s))]ds \in K.\end{aligned}$$

Consequently, $\Gamma x \leq \Gamma y$ and Γ is a monotone operator.

Let $\alpha(t) = \|h(t)\| + \int_{-\infty}^{t} k(t,s)g(s)ds$ and let M be the set consisting of functions $x \in C(\mathbb{R}, X)$ such that $\|x(t)\| \leq \alpha(t)$ for all $t \in \mathbb{R}$. It is clear that M is a closed bounded subset of $C(\mathbb{R}, X)$. Moreover, $h \in M$ and $\Gamma h \leq h$. It follows also from the definition of α that $\Gamma M \subseteq M$. In fact, if $x \in M$, then

$$\begin{aligned} \|\Gamma x(t)\| &\leq \|h(t)\| + \int_{-\infty}^{t} k(t,s)\|f(s,x(s))\|ds \\ &\leq \|h(t)\| + \int_{-\infty}^{t} k(t,s)g(s)ds = \alpha(t). \end{aligned}$$

Finally, we will show that Γ is monotonically limit compact on M. We take a sequence $(x_n)_n$ in M such that

$$x_0 \geq \Gamma x_1 \geq \Gamma^2 x_2 \cdots \Gamma^n x_n \geq \cdots$$

If $y_n(t) = \Gamma^n x_n(t)$, using that K is a fully regular cone we obtain that $y_n(t) \to y(t)$ as $n \to \infty$. Furthermore, it follows from the properties of f that the set $\{y_n : n \in \mathbb{N}\}$ is equicontinuous on bounded intervals. In fact, in general, for any $x \in C(\mathbb{R}, X)$ and $t_1 \leq t_2$ we have

$$\begin{aligned} \|\Gamma x(t_2) - \Gamma x(t_1)\| &\leq \|h(t_2) - h(t_1)\| + \int_{t_1}^{t_2} k(t_2,s)\|f(s,x(s))\|ds \\ &\quad + \int_{-\infty}^{t_1} |k(t_2,s) - k(t_1,s)|\|f(s,x(s))\|ds \\ &\leq \|h(t_2) - h(t_1)\| + \int_{-\infty}^{t_1} |k(t_2,s) - k(t_1,s)|g(s)ds \\ &\quad + \int_{t_1}^{t_2} k(t_2,s)g(s)ds \end{aligned}$$

which shows that the continuity of Γx on a bounded interval does not depend on x. Therefore, the function y is continuous. Moreover, for any compact interval $I \subseteq \mathbb{R}$ we can consider $\{y_n : n \in \mathbb{N}\} \subseteq C(I, X)$ and using the Ascoli-Arzela theorem we infer that $y_n(t) \to y(t)$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$. Consequently, the sequence $(y_n)_n$ converges to y in the space $C(\mathbb{R}, X)$.

It follows from the Theorem 1.2 that Γ has a fixed point in M. This completes the proof. □

A particular case of the Theorem 2.3 is obtained when $X = \mathbb{R}$ and K is the cone of positive elements of $\mathbb{R}$. In this case, the condition (f2) can be omitted.

References

[1] H. Jarchow, *Locally Convex Spaces*, B. G. Teubner, Stuttgart, 1981.

[2] M. A. Krasnoselskii and P. P. Zabreiko, *Geometrical Methods of Nonlinear Analysis*. Springer-Verlag, Berlin, 1984.

[3] C. M. Marle, *Mesures et Probabilités*. Hermann, Paris, 1974.

Cristina G. Fuentes and Hernán R. Henríquez,
Departamento de Matemática,
Universidad de Santiago-USACH,
Casilla 307, Correo 2, Santiago, Chile.

E-mail addresses: cristina.fuentes.gomez@gmail.com
hernan.henriquez@usach.cl

(Received October 26, 2010)