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Positive solutions for a system of nth-order nonlinear boundary value problems*

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Abstract

In this paper, we investigate the existence, multiplicity and uniqueness of positive solutions for the following system of nth-order nonlinear boundary value problems

$$\begin{cases} u^{(n)}(t) + f(t, u(t), v(t)) = 0, 0 < t < 1, \\ v^{(n)}(t) + g(t, u(t), v(t)) = 0, 0 < t < 1, \\ u(0) = u'(0) = \ldots = u^{(n-2)}(0) = u(1) = 0, \\ v(0) = v'(0) = \ldots = v^{(n-2)}(0) = v(1) = 0. \end{cases}$$

Based on a priori estimates achieved by using Jensen's integral inequality, we use fixed point index theory to establish our main results. Our assumptions on the nonlinearities are mostly formulated in terms of spectral radii of associated linear integral operators. In addition, concave and convex functions are utilized to characterize coupling behaviors of f and g, so that we can treat the three cases: the first with both superlinear, the second with both sublinear, and the last with one superlinear and the other sublinear.
Key words: Boundary value problem; Positive solution; Fixed point index; Jensen inequality; Concave and convex function.
MSC(2000): 34B10; 34B18; 34A34; 45G15; 45M20

1 Introduction

In this paper we study the existence, multiplicity and uniqueness of positive solutions for the following system of nth-order nonlinear boundary value problems

$$\begin{cases} u^{(n)}(t) + f(t, u(t), v(t)) = 0, 0 < t < 1, \\ v^{(n)}(t) + g(t, u(t), v(t)) = 0, 0 < t < 1, \\ u(0) = u'(0) = \ldots = u^{(n-2)}(0) = u(1) = 0, \\ v(0) = v'(0) = \ldots = v^{(n-2)}(0) = v(1) = 0, \end{cases} \tag{1.1}$$

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where $n \geq 2$, $f, g \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ ($\mathbb{R}^+ := [0, \infty)$).

The solvability of systems for nonlinear boundary value problems of second order ordinary differential equations has received a great deal of attention in the literature. For more details of recent development in the direction, we refer the reader to [1, 5, 10, 14–18, 21–26, 33, 34, 36, 39, 42] and references cited therein. A considerable number of these problems can be formulated as systems of integral equations by virtue of some suitable Green's functions. Therefore, it seems natural that many authors pay more attention to the systems for nonlinear integral equations, see for example [2,3,7,12,19,35,41]. Yang [35] considered the following system of Hammerstein integral equations

$$\begin{cases} u(x) = \int_G k(x,y) f(y, u(y), v(y)) \mathrm{d}y, \\ v(x) = \int_G k(x,y) g(y, u(y), v(y)) \mathrm{d}y. \end{cases} \tag{1.2}$$

where $G \subset \mathbb{R}^n$ is a bounded closed domain, $k \in C(G \times G, \mathbb{R}^+)$, and $f, g \in C(G \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$. By using fixed point index theory, he obtained some existence and multiplicity results of positive solutions for the system (1.2) where assumptions imposed on the nonlinearities f and g are formulated in terms of spectral radii of some related linear integral operators.

To the best of our knowledge, only a few papers deal with systems with high-order nonlinear boundary value problems, see for example [4, 6, 11, 13, 20, 27–31, 37, 38, 40, 43]. Based on a priori estimates achieved by Jensen's integral inequality, we use fixed point index theory to establish our main results. Our assumptions on the nonlinearities are mostly formulated in terms of spectral radii of associated linear integral operators. It is of interest to note that our nonlinearities are allowed to grow in distinct manners. Our work is motivated by [35], but our main results extend and improve the corresponding ones in [35].

The remainder of this paper is organized as follows. Section 2 provides some preliminary results required in the proofs of our main results. Section 3 is devoted to the existence, multiplicity and uniqueness of the positive solutions for the problem (1.1), respectively.

2 Preliminaries

We can obtain the system (1.1) which is equivalent to the system of nonlinear Hammerstein integral equations, (see [32])

$$\begin{cases} u(t) = \int_0^1 G(t,s) f(s, u(s), v(s)) \mathrm{d}s, \\ v(t) = \int_0^1 G(t,s) g(s, u(s), v(s)) \mathrm{d}s, \end{cases} \tag{2.1}$$

where

$$G(t,s) := \frac{1}{(n-1)!}\begin{cases}(1-s)^{n-1}t^{n-1}, & 0\le t\le s\le 1,\\ (1-s)^{n-1}t^{n-1}-(t-s)^{n-1}, & 0\le s\le t\le 1.\end{cases} \tag{2.2}$$

Lemma 2.1([32]) $G(t,s)$ has the following properties

(i) $0 \le G(t,s) \le y(s)$, $\forall t, s \in [0,1]$, where $y(s) := \frac{s(1-s)^{n-1}}{(n-2)!}$;

(ii) $G(t,s) \ge \gamma(t)y(s)$, $\forall t, s \in [0,1]$, where $\gamma(t) := \frac{1}{n-1}\min\{t^{n-1}, (1-t)t^{n-2}\}$.

Combining (i) and (ii), we can easily see

$$G(t,s) \ge \gamma(t)G(\tau,s), \forall t, s, \tau \in [0,1] \tag{2.3}$$

and $\gamma(t)$ is positive on $[0,1]$. Let

$$E := C[0,1], \|u\| := \max_{t\in[0,1]}|u(t)|, P := \{u \in E : u(t) \ge 0, \forall t \in [0,1]\}.$$

Then $(E, \|\cdot\|)$ is a real Banach space and P a cone on E. We denote $B_\rho := \{u \in E : \|u\| < \rho\}$ for $\rho > 0$ in the sequel. The norm on $E\times E$ is defined by $\|(u,v)\| := \max\{\|u\|,\|v\|\}, (u,v) \in E\times E$. Note $E\times E$ is a real Banach space under the above norm, and $P\times P$ is a positive cone on $E\times E$. Let

$$K := \max_{t,s\in[0,1]} G(t,s) > 0,\ K_1 := \max_{t\in[0,1]}\int_0^1 G(t,s)\mathrm{d}s > 0.$$

Define the operators $A_i (i = 1,2)$ and A by

$$\begin{aligned}A_1(u,v)(t) &:= \int_0^1 G(t,s)f(s,u(s),v(s))\mathrm{d}s,\\ A_2(u,v)(t) &:= \int_0^1 G(t,s)g(s,u(s),v(s))\mathrm{d}s,\\ A(u,v)(t) &:= (A_1(u,v), A_2(u,v))(t).\end{aligned}$$

Now $A_i : P\times P \to P (i = 1,2)$ and $A : P\times P \to P\times P$ are completely continuous operators. Note that $(u,v) \in P\times P$ is called a positive solution of (1.1) provided $(u,v) \in P\times P$ solves (1.1) and $(u,v) \ne 0$. Clearly, $(u,v) \in P\times P$ is a positive solution of (1.1) if and only if $(u,v) \in (P\times P)\setminus\{0\}$ is a fixed point of A.

We also denote the linear integral operator L by

$$(Lu)(t) := \int_0^1 G(t,s)u(s)\mathrm{d}s.$$

Then $L : E \to E$ is a completely continuous positive linear operator. We can easily prove the spectral radius of L, denoted by $r(L)$, is positive. Now the well-known Krein-Rutman theorem [9] asserts that there exist two functions $\varphi \in P \setminus \{0\}$ and $\psi \in L(0,1)\backslash\{0\}$ with $\psi(x) \geq 0$ for which

$$\int_0^1 G(t,s)\varphi(s)\mathrm{d}s = r(L)\varphi(t), \int_0^1 G(t,s)\psi(t)\mathrm{d}t = r(L)\psi(s), \int_0^1 \psi(t)\mathrm{d}t = 1. \tag{2.4}$$

Put

$$P_0 := \left\{ u \in P : \int_0^1 \psi(t)u(t)\mathrm{d}t \geq \omega\|u\| \right\}, \tag{2.5}$$

where $\psi(t)$ is determined by (2.4) and $\omega := \int_0^1 \gamma(t)\psi(t)\mathrm{d}t > 0$. Clearly, P_0 is also a cone on E. The following is a result that is of vital importance in our proofs and can be proved as Lemma 4 in [35].

Lemma 2.2 $L(P) \subset P_0$.

Lemma 2.3 ([8]) Suppose $\Omega \subset E$ is a bounded open set and $A : \overline{\Omega} \cap P \to P$ is a completely continuous operator. If there exists $u_0 \in P \setminus \{0\}$ such that $u - Au \neq \nu u_0, \forall \nu \geq 0, u \in \partial\Omega \cap P$, then $i(A, \Omega \cap P, P) = 0$.

Lemma 2.4 ([8]) Let $\Omega \subset E$ be a bounded open set with $0 \in \Omega$. Suppose $A : \overline{\Omega} \cap P \to P$ is a completely continuous operator. If $u \neq \nu Au, \forall u \in \partial\Omega \cap P, 0 \leq \nu \leq 1$, then $i(A, \Omega \cap P, P) = 1$.

Lemma 2.5 If $p : \mathbb{R}^+ \to \mathbb{R}^+$ is concave, then p is nondecreasing. In addition, if there exist $0 \leq x_1 < x_2$ such that $p(x_1) = p(x_2)$, then

$$p(x) \equiv p(x_1) = p(x_2), \forall x \geq x_1. \tag{2.6}$$

Moreover, the following inequality holds:

$$p(a + b) \leq p(a) + p(b), \quad \forall a, b \in \mathbb{R}^+. \tag{2.7}$$

Proof. For any $x_2 > x_1 \geq 0$, the concavity of p implies

$$p(x) \leq p(x_2) + \frac{p(x_2) - p(x_1)}{x_2 - x_1}(x - x_2), \forall x > x_2 \tag{2.8}$$

and thus $p(x_1) \leq p(x_2)$ by nonnegativity of p. In addition, if $p(x_1) = p(x_2)$, then (2.6) holds, as is seen from (2.8). The proof of (2.7) can be found in [35, Lemma 5]. The proof is completed.

Lemma 2.6 Let

$$w_0(t) := \int_0^1 G(t,s)\mathrm{d}s = \frac{t^{n-1} - t^n}{n!}.$$

Then for each $w \in P\backslash\{0\}$, there are positive numbers $b_w \geq a_w$ such that

$$a_w w_0(t) \leq \int_0^1 G(t,s)w(s)\mathrm{d}s \leq b_w w_0(t), t \in [0,1].$$

Let $\lambda_1 := \frac{1}{r(L)}$. We now list our hypotheses.

(H1) There exist $p, q \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

(1) p is concave on $\mathbb{R}^+$.

(2) $f(t, u, v) \geq p(v) - c$, $g(t, u, v) \geq q(u) - c$, $\forall (t, u, v) \in [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+$.

(3) $p(Kq(u)) \geq \mu_1 \lambda_1^2 K u - c, \mu_1 > 1, \forall u \in \mathbb{R}^+$.

(H2) There exist $\xi, \eta \in C(\mathbb{R}^+, \mathbb{R}^+)$ and a sufficiently small constant $r > 0$ such that

(1) ξ is convex and strictly increasing on $\mathbb{R}^+$.

(2) $f(t, u, v) \leq \xi(v)$, $g(t, u, v) \leq \eta(u)$, $\forall (t, u, v) \in [0, 1] \times [0, r] \times [0, r]$.

(3) $\xi(K\eta(u)) \leq \mu_2 K \lambda_1^2 u, \mu_2 < 1, \forall u \in [0, r]$.

(H3) There exist $p, q \in C(\mathbb{R}^+, \mathbb{R}^+)$ and a sufficiently small constant $r > 0$ such that

(1) p is concave on $\mathbb{R}^+$.

(2) $f(t, u, v) \geq p(v)$, $g(t, u, v) \geq q(u)$, $\forall (t, u, v) \in [0, 1] \times [0, r] \times [0, r]$.

(3) $p(Kq(u)) \geq \mu_3 K \lambda_1^2 u, \mu_3 > 1, \forall u \in [0, r]$.

(H4) There exist $\xi, \eta \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

(1) ξ is convex and strictly increasing on $\mathbb{R}^+$.

(2) $f(t, u, v) \leq \xi(v)$, $g(t, u, v) \leq \eta(u)$, $\forall (t, u, v) \in [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+$.

(3) $\xi(K\eta(u)) \leq \mu_4 K \lambda_1^2 u + c, \mu_4 < 1, \forall u \in \mathbb{R}^+$.

(H5) There is $N > 0$ such that the inequalities $f(t, u, v) < \frac{N}{K_1}, g(t, u, v) < \frac{N}{K_1}$ hold whenever $u, v \in [0, N]$ and $t \in [0, 1]$.

(H6) There are $\rho > 0$ and $\sigma \in (0, \frac{1}{2})$ such that the inequality $f(t, u, v) > \frac{2^{n-1}(n+1)!}{n-1}\rho$, $g(t, u, v) > \frac{2^{n-1}(n+1)!}{n-1}\rho$ hold whenever $u, v \in [\theta\rho, \rho]$ and $t \in [\sigma, 1 - \sigma]$, where $\theta = \min\{\gamma(\sigma), \gamma(1 - \sigma)\}$.

(H7) $f(t, u, v)$ and $g(t, u, v)$ are increasing in u, v, that is, the inequalities $f(t, u_1, v_1) \leq f(t, u_2, v_2)$ and $g(t, u_1, v_1) \leq g(t, u_2, v_2)$ hold for $(u_1, v_1) \in \mathbb{R}^+$ and $(u_2, v_2) \in \mathbb{R}^+$ satisfying $u_1 \leq u_2$ and $v_1 \leq v_2$.

(H8) $f(t, \lambda u, \lambda v) > \lambda f(t, u, v)$ and $g(t, \lambda u, \lambda v) > \lambda g(t, u, v)$ for each $\lambda \in (0, 1)$, $u, v \in \mathbb{R}^+$, and $t \in [0, 1]$.

3 Main Results

We adopt the convention in the sequel that $c_1, c_2, \ldots$ stand for different positive constants.

Theorem 3.1 Suppose that (H1), (H2) are satisfied, then (1.1) has at least one positive solution.

Proof. By (2) of (H1) and the definition of A_i $(i = 1, 2)$, we have

$$A_1(u,v)(t) \geq \int_0^1 G(t,s)p(v(s))\mathrm{d}s - c_1,\ A_2(u,v)(t) \geq \int_0^1 G(t,s)q(u(s))\mathrm{d}s - c_1, \quad (3.1)$$

for all $(t,u,v) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+$. We claim the set

$$\mathscr{M}_1 := \{(u,v) \in P \times P : (u,v) = A(u,v) + \nu(\varphi,\varphi), \nu \geq 0\} \quad (3.2)$$

is bounded, where φ is defined by (2.4). Indeed, if $(u,v) \in \mathscr{M}_1$, then $u \geq A_1(u,v)$ and $v \geq A_2(u,v)$. In view of (3.1), we get

$$u(t) \geq \int_0^1 G(t,s)p(v(s))\mathrm{d}s - c_1,\ v(t) \geq \int_0^1 G(t,s)q(u(s))\mathrm{d}s - c_1. \quad (3.3)$$

By the concavity of p and the second inequality of (3.3), together with Jensen's inequality, we obtain

$$\begin{aligned} p(v(t)) &\geq p(v(t)+c_1) - p(c_1) \geq p\left(\int_0^1 G(t,s)q(u(s))\mathrm{d}s\right) - p(c_1) \\ &\geq \int_0^1 p(G(t,s)q(u(s)))\mathrm{d}s - p(c_1) \geq K^{-1}\int_0^1 G(t,s)p(Kq(u(s)))\mathrm{d}s - p(c_1). \end{aligned} \quad (3.4)$$

Substitute this into the first inequality of (3.3) and use (3) of (H1) to obtain

$$\begin{aligned} u(t) &\geq \int_0^1 G(t,s)\left[K^{-1}\int_0^1 G(s,\tau)\Big[\mu_1\lambda_1^2 Ku(\tau) - c\Big]\mathrm{d}\tau - p(c_1)\right]\mathrm{d}s - c_1 \\ &\geq \mu_1\lambda_1^2\int_0^1\int_0^1 G(t,s)G(s,\tau)u(\tau)\mathrm{d}\tau\mathrm{d}s - c_2. \end{aligned}$$

Multiply both sides of the above by $\psi(t)$ and integrate over [0,1] and use (2.4) to obtain

$$\int_0^1 u(t)\psi(t)\mathrm{d}t \geq \mu_1\int_0^1 u(t)\psi(t)\mathrm{d}t - c_2.$$

Consequently, $\int_0^1 u(t)\psi(t)\mathrm{d}t \leq \frac{c_2}{\mu_1 - 1}$. By Lemma 2.2 and (2.5), we obtain

$$\|u\| \leq \frac{c_2}{\omega(\mu_1 - 1)}, \forall (u,v) \in \mathscr{M}_1. \quad (3.5)$$

Multiply both sides of the first inequality of (3.3) by $\psi(t)$ and integrate over [0,1] and use (2.4) to obtain

$$\|u\| \geq \int_0^1 u(t)\psi(t)\mathrm{d}t \geq \lambda_1^{-1}\int_0^1 p(v(t))\psi(t)\mathrm{d}t - c_1.$$

Therefore, $\int_0^1 p(v(t))\psi(t)\mathrm{d}t \leq \lambda_1(\|u\| + c_1)$. Without loss of generality, we may assume $v \not\equiv 0$, then $\|v\| > 0$. From (2.5), we obtain

$$\|v\| \leq \frac{1}{\omega}\int_0^1 v(t)\psi(t)\mathrm{d}t \leq \frac{\|v\|}{\omega p(\|v\|)}\int_0^1 \psi(t)\frac{v(t)}{\|v\|}p(\|v\|)\mathrm{d}t \leq \frac{\|v\|}{\omega p(\|v\|)}\int_0^1 \psi(t)p(v(t))\mathrm{d}t.$$

Consequently,

$$p(\|v\|) \le \frac{1}{\omega}\int_0^1 \psi(t)p(v(t))\mathrm{d}t \le \lambda_1\omega^{-1}(\|u\|+c_1).$$

By (3) of (H1), we have $\lim_{z\to\infty} p(z)=\infty$, and thus there exists $c_3>0$ such that $\|v\|\le c_3, \forall (u,v)\in \mathscr{M}_1$. Combining this and (3.5), we find $\mathscr{M}_1$ is bounded in $P\times P$, as claimed. Taking $R>\sup\mathscr{M}_1$, then we have

$$(u,v)\ne A(u,v)+\nu(\varphi,\varphi), \forall (u,v)\in \partial B_R\cap(P\times P), \nu\ge 0.$$

Lemma 2.3 implies

$$i(A, B_R\cap(P\times P), P\times P)=0. \tag{3.6}$$

On the other hand, by (2) of (H2), we find

$$A_1(u,v)(t)\le \int_0^1 G(t,s)\xi(v(s))\mathrm{d}s,\ A_2(u,v)(t)\le \int_0^1 G(t,s)\eta(u(s))\mathrm{d}s, \tag{3.7}$$

for any $(t,u,v)\in[0,1]\times[0,r]\times[0,r]$. Now we show

$$(u,v)\ne \nu A(u,v), \forall (u,v)\in \partial B_r\cap(P\times P), \nu\in[0,1]. \tag{3.8}$$

If the claim is false, there exist $(u_1,v_1)\in\partial B_r\cap(P\times P)$ and $\nu_1\in[0,1]$ such that $(u_1,v_1)=\nu_1A(u_1,v_1)$. Therefore, $u_1\le A_1(u_1,v_1)$ and $v_1\le A_2(u_1,v_1)$. In view of (3.7), we have

$$u_1(t)\le \int_0^1 G(t,s)\xi(v_1(s))\mathrm{d}s,\ \ v_1(t)\le\int_0^1 G(t,s)\eta(u_1(s))\mathrm{d}s.$$

Consequently, the convexity of ξ and Jensen's inequality imply

$$\xi(v_1(t))\le \xi\left(\int_0^1 G(t,s)\eta(u_1(s))\mathrm{d}s\right)\le K^{-1}\int_0^1 G(t,s)\xi(K\eta(u_1(s)))\mathrm{d}s. \tag{3.9}$$

Therefore,

$$u_1(t)\le K^{-1}\int_0^1\int_0^1 G(t,s)G(s,\tau)\xi(K\eta(u_1(\tau)))\mathrm{d}\tau\mathrm{d}s.$$

Multiply both sides of the above by $\psi(t)$ and integrate over [0,1] and use (2.4) and (3) of (H2) to obtain

$$\int_0^1 u_1(t)\psi(t)\mathrm{d}t\le\mu_2\int_0^1 u_1(t)\psi(t)\mathrm{d}t.$$

Since $\mu_2<1$, from which we find $\int_0^1 u_1(t)\psi(t)\mathrm{d}t=0$, thus $u_1=0$. We have from (3.9) and (3) of (H2)

$$\xi(v_1(t))\le K^{-1}\int_0^1 G(t,s)\xi(K\eta(u_1(s)))\mathrm{d}s\le \mu_2\lambda_1^2\int_0^1 G(t,s)u_1(s)\mathrm{d}s=0.$$

Since ξ is strictly increasing, then $v_1 = 0$, which is a contradiction to $(u_1, v_1) \in \partial B_r \cap (P \times P)$. Hence, (3.8) is true. So, we have from Lemma 2.4 that

$$i(A, B_r \cap (P \times P), P \times P) = 1. \tag{3.10}$$

Combining (3.6) and (3.10) gives

$$i(A, (B_R \backslash \overline{B}_r) \cap (P \times P), P \times P) = 0 - 1 = -1.$$

Therefore the operator A has at least one fixed point on $(B_R \backslash \overline{B}_r) \cap (P \times P)$. Equivalently, (1.1) has at least one positive solution. This completes the proof.

Theorem 3.2 Suppose that (H3), (H4) are satisfied, then (1.1) has at least one positive solution.

Proof. By (2) of (H3), we find

$$A_1(u, v) \geq \int_0^1 G(t, s) p(v(s)) \mathrm{d}s, \;\; A_2(u, v) \geq \int_0^1 G(t, s) q(u(s)) \mathrm{d}s, \tag{3.11}$$

for any $(t, u, v) \in [0, 1] \times [0, r] \times [0, r]$. Let

$$\mathscr{M}_2 := \{(u, v) \in \overline{B}_r \cap (P \times P) : (u, v) = A(u, v) + \nu(\varphi, \varphi), \nu \geq 0\} \tag{3.12}$$

where φ is defined by (2.4). We shall prove $\mathscr{M}_2 \subset \{0\}$. Indeed, if $(u, v) \in \mathscr{M}_2$, then $u \geq A_1(u, v)$ and $v \geq A_2(u, v)$. In view of (3.11), we get

$$u(t) \geq \int_0^1 G(t, s) p(v(s)) \mathrm{d}s, \;\; v(t) \geq \int_0^1 G(t, s) q(u(s)) \mathrm{d}s. \tag{3.13}$$

By the concavity of p and the second inequality of (3.13), together with Jensen's inequality, we obtain

$$p(v(t)) \geq p\left(\int_0^1 G(t, s) q(u(s)) \mathrm{d}s\right) \geq K^{-1} \int_0^1 G(t, s) p(Kq(u(s))) \mathrm{d}s \tag{3.14}$$

From the first inequality of (3.13), we have

$$u(t) \geq K^{-1} \int_0^1 \int_0^1 G(t, s) G(s, \tau) p(Kq(u(\tau))) \mathrm{d}\tau \mathrm{d}s.$$

Multiply both sides of the above by $\psi(t)$ and integrate over [0,1] and use (2.4) and (3) of (H3) to obtain

$$\int_0^1 u(t)\psi(t) \mathrm{d}t \geq \mu_3 \int_0^1 u(t)\psi(t) \mathrm{d}t. \tag{3.15}$$

Since $\mu_3 > 1$, thus we obtain $\int_0^1 u(t)\psi(t)\mathrm{d}t = 0$, then $u \equiv 0$. Also, We have from (3.13) that $\int_0^1 G(t, s) p(v(s)) \mathrm{d}s = 0$, then $p(v(t)) = 0$. We find from Lemma 2.5 that $v \equiv 0$. As a result, $\mathscr{M}_2 \subset \{0\}$ holds. Lemma 2.3 implies

$$i(A, B_r \cap (P \times P), P \times P) = 0. \tag{3.16}$$

On the other hand, by (2) of (H4), we find

$$A_1(u,v) \le \int_0^1 G(t,s)\xi(v(s))\mathrm{d}s, \ A_2(u,v) \le \int_0^1 G(t,s)\eta(u(s))\mathrm{d}s, \tag{3.17}$$

for all $(t,u,v) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+$. We shall show there exists an adequately big positive number $R > 0$ such that the following claim holds.

$$(u,v) \neq \nu A(u,v), \forall (u,v) \in \partial B_R \cap (P \times P), \nu \in [0,1]. \tag{3.18}$$

If the claim is false, there exist $(u_1,v_1) \in \partial B_R \cap (P \times P)$ and $\nu_1 \in [0,1]$ such that $(u_1,v_1) = \nu_1 A(u_1,v_1)$. Therefore, $u_1 \le A_1(u_1,v_1)$ and $v_1 \le A_2(u_1,v_1)$. In view of (3.17), we have

$$u_1(t) \le \int_0^1 G(t,s)\xi(v_1(s))\mathrm{d}s, \ v_1(t) \le \int_0^1 G(t,s)\eta(u_1(s))\mathrm{d}s.$$

Subsequently, Jensen's inequality implies

$$\xi(v_1(t)) \le \xi\left(\int_0^1 G(t,s)\eta(u_1(s))\mathrm{d}s\right) \le K^{-1}\int_0^1 G(t,s)\xi(K\eta(u_1(s)))\mathrm{d}s. \tag{3.19}$$

Therefore,

$$u_1(t) \le K^{-1}\int_0^1\int_0^1 G(t,s)G(s,\tau)\xi(K\eta(u_1(\tau)))\mathrm{d}\tau\mathrm{d}s.$$

Multiply both sides of the above by $\psi(t)$ and integrate over [0,1] and use (2.4) and (3) of (H4) to obtain

$$\int_0^1 u_1(t)\psi(t)\mathrm{d}t \le \mu_4\int_0^1 u_1(t)\psi(t)\mathrm{d}t + c_4.$$

Therefore, $\int_0^1 u_1(t)\psi(t)\mathrm{d}t \le \frac{c_4}{1-\mu_4}$. From (2.5), we get

$$\|u_1\| \le \frac{c_4}{\omega(1-\mu_4)}. \tag{3.20}$$

By (3.19) and (3) of (H4), we obtain

$$\xi(v_1(t)) \le \mu_4\lambda_1^2\int_0^1 G(t,s)u_1(s)\mathrm{d}t + c_5 \le \mu_4\lambda_1^2\|u_1\|K_1 + c_5.$$

Since ξ is strictly increasing, then there exists $c_6 > 0$ such that $\|v_1\| \le c_6$. Taking $R > \max\left\{c_6, \frac{c_4}{\omega(1-\mu_4)}\right\}$, which is a contradiction to $(u_1,v_1) \in \partial B_R \cap (P \times P)$. As a result, (3.18) is true. So, we have from Lemma 2.4 that

$$i(A, B_R \cap (P \times P), P \times P) = 1. \tag{3.21}$$

Combining (3.16) and (3.21) gives

$$i(A, (B_R \backslash \overline{B}_r) \cap (P \times P), P \times P) = 1 - 0 = 1.$$

Therefore the operator A has at least one fixed point on $(B_R\backslash\overline{B}_r)\cap(P\times P)$. Equivalently, (1.1) has at least one positive solution. This completes the proof.

Theorem 3.3 Suppose that (H1), (H3) and (H5) are satisfied, then (1.1) has at least two positive solutions.

Proof. By (H5), we have

$$A_1(u,v)(t) < \int_0^1 \frac{N}{K_1}G(t,s)\mathrm{d}s \le N, A_2(u,v)(t) < \int_0^1 \frac{N}{K_1}G(t,s)\mathrm{d}s \le N,$$

for any $(t,u,v)\in[0,1]\times\partial B_N\times\partial B_N$, from which we obtain

$$\|A(u,v)\| < \|(u,v)\|, \quad \forall(u,v)\in\partial B_N\cap(P\times P).$$

This leads to

$$(u,v)\neq\nu A(u,v),\forall(u,v)\in\partial B_N\cap(P\times P),\nu\in[0,1]. \tag{3.22}$$

Now Lemma 2.4 implies

$$i(A,B_N\cap(P\times P),P\times P)=1. \tag{3.23}$$

On the other hand, by (H1) and (H3) (see the proofs of Theorems 3.1 and 3.2), we may take $R>N$ and $r\in(0,N)$ so that (3.6) and (3.16) hold. Combining (3.6), (3.16) and (3.23), we conclude

$$i(A,(B_R\backslash\overline{B}_N)\cap(P\times P),P\times P)=0-1=-1,$$

$$i(A,(B_N\backslash\overline{B}_r)\cap(P\times P),P\times P)=1-0=1.$$

Consequently, A has at least two fixed points in $(B_R\backslash\overline{B}_N)\cap(P\times P)$ and $(B_N\backslash\overline{B}_r)\cap(P\times P)$, respectively. Equivalently, (1.1) has at least two positive solutions $(u_1,v_1)\in(P\times P)\backslash\{0\}$ and $(u_2,v_2)\in(P\times P)\backslash\{0\}$. This completes the proof.

Theorem 3.4 Suppose that (H2), (H4) and (H6) are satisfied, then (1.1) has at least two positive solutions.

Proof. By (H6), we have

$$\begin{aligned}\|A_1(u,v)\| &= \max_{0\le t\le 1}A_1(u,v)(t)\ge \max_{t\in[\sigma,1-\sigma]}A_1(u,v)(t)\\ &= \max_{t\in[\sigma,1-\sigma]}\int_0^1 G(t,s)f(s,u(s),v(s))\mathrm{d}s\\ &\ge \max_{t\in[\sigma,1-\sigma]}\int_0^1 \gamma(t)y(s)f(s,u(s),v(s))\mathrm{d}s\\ &> \left(\tfrac{1}{2}\right)^{n-1}\int_0^1 y(s)\tfrac{2^{n-1}(n+1)!}{n-1}\rho\mathrm{d}s=\|u\|,\forall u\in\partial B_\rho\cap(P\times P).\end{aligned}$$

Similarly, $\|A_2(u,v)\|>\|v\|$, $\forall v\in\partial B_\rho\cap(P\times P)$. Consequently,

$$\|A(u,v)\|>\|(u,v)\|,\forall(u,v)\in\partial B_\rho\cap(P\times P).$$

This yields

$$(u,v)\neq A(u,v)+\nu(\varphi,\varphi),\forall(u,v)\in\partial B_\rho\cap(P\times P),\nu\geq 0.$$

Lemma 2.3 gives

$$i(A,B_\rho\cap(P\times P),P\times P)=0. \tag{3.24}$$

On the other hand, by (H2) and (H4) (see the proofs of Theorems 3.1 and 3.2), we may take $R>\rho$ and $r\in(0,\rho)$ so that (3.10) and (3.21) hold. Combining (3.10), (3.21) and (3.24), we conclude

$$i(A,(B_R\backslash\overline{B}_\rho)\cap(P\times P),P\times P)=1-0=1,$$

$$i(A,(B_\rho\backslash\overline{B}_r)\cap(P\times P),P\times P)=0-1=-1.$$

Consequently, A has at least two fixed points in $(B_R\backslash\overline{B}_\rho)\cap(P\times P)$ and $(B_\rho\backslash\overline{B}_r)\cap(P\times P)$, respectively. Equivalently, (1.1) has at least two positive solutions $(u_1,v_1)\in(P\times P)\backslash\{0\}$ and $(u_2,v_2)\in(P\times P)\backslash\{0\}$. This completes the proof.

Theorem 3.5 If (H3), (H4), (H7) and (H8) hold, then (1.1) has exactly one positive solution.

Proof. We first show the problem (1.1) has at most one positive solution. Indeed, if (u_1,v_1) and (u_2,v_2) are two positive solutions of (1.1), then for $i=1,2$, we get

$$u_i(t)=\int_0^1 G(t,s)f(s,u_i(s),v_i(s))\mathrm{d}s, v_i(t)=\int_0^1 G(t,s)g(s,u_i(s),v_i(s))\mathrm{d}s.$$

Lemma 2.6 implies that eight positive numbers $b_i\geq a_i$ $(i=1,2,3,4)$ such that $a_1w_0\leq u_1\leq b_1w_0$, $a_2w_0\leq u_2\leq b_2w_0$, $a_3w_0\leq v_1\leq b_3w_0$ and $a_4w_0\leq v_2\leq b_4w_0$. Therefore $u_2\geq\frac{a_2}{b_1}u_1$ and $v_2\geq\frac{a_4}{b_3}v_1$. Let

$$\mu_0:=\sup\{\mu>0:u_2\geq\mu u_1,v_2\geq\mu v_1\}.$$

We obtain by $\mu_0>0$ that $u_2\geq\mu_0u_1$ and $v_2\geq\mu_0v_1$. We claim that $\mu_0\geq 1$. Suppose the contrary. Then $\mu_0<1$ and

$$u_2(t)\geq\int_0^1 G(t,s)f(s,\mu_0u_1(s),\mu_0v_1(s))\mathrm{d}s, v_2(t)\geq\int_0^1 G(t,s)g(s,\mu_0u_1(s),\mu_0v_1(s))\mathrm{d}s.$$

Let

$$h_1(t):=f(t,\mu_0u_1(t),\mu_0v_1(t))-\mu_0f(t,u_1(t),v_1(t)),$$

and

$$h_2(t):=g(t,\mu_0u_1(t),\mu_0v_1(t))-\mu_0g(t,u_1(t),v_1(t)).$$

(H8) implies $h_i \in P\backslash\{0\}$ $(i = 1, 2)$. By Lemma 2.6, there are two positive numbers ε_i such that

$$\int_0^1 G(t,s)h_i(s)\mathrm{d}s \geq \varepsilon_i w_0(t).$$

Therefore,

$$u_2(t) \geq \int_0^1 G(t,s)h_1(s)\mathrm{d}s + \mu_0 u_1(t) \geq \frac{\varepsilon_1}{b_1}u_1(t) + \mu_0 u_1(t),$$

and

$$v_2(t) \geq \int_0^1 G(t,s)h_2(s)\mathrm{d}s + \mu_0 v_1(t) \geq \frac{\varepsilon_2}{b_3}v_1(t) + \mu_0 v_1(t),$$

contradicting the definition of μ_0. As a result of this, we have $\mu_0 \geq 1$, and thus $u_2 \geq u_1$. Similarly $u_1 \geq u_2$. Therefore $u_1 = u_2$. Similarly $v_1 = v_2$. Thus (1.1) has at most one positive solution. Combining this and Theorem 3.2, we find (1.1) has exactly one positive solution. This completes the proof.

References

[1] R.P. Agarwal, D. O'Regan, A coupled system of boundary value problems, Appl. Anal. 69 (1998) 381-385.

[2] R.P. Agarwal, D. O'Regan, P.J.Y. Wong, Constant-sign solutions of a system of Fredholm integral equations, Acta Appl. Math. 80 (2004) 57-94.

[3] R.P. Agarwal, D. O'Regan, P.J.Y. Wong, Eigenvalues of a system of Fredholm integral equations, Math. Comput. Modelling 39 (2004) 1113-1150.

[4] R.P. Agarwal, D. O'Regan, P.J.Y. Wong, Constant-sign solutions of systems of higher order boundary value problems with integrable singularities, Mathematical and Computer Modelling 44 (2006) 983-1008.

[5] X. Cheng, C. Zhong, Existence of positive solutions for a second order ordinary differential system, J. Math. Anal. Appl. 312 (2005) 14-23.

[6] E. Cetin, S. G. Topal, Existence of multiple positive solutions for the system of higher order boundary value problems on time scales, Mathematical and Computer Modelling (In press).

[7] D. Franco, G. Infante, D. O'Regan, Nontrivial solutions in abstract cones for Hammerstein integral systems, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 14 (2007) 837-850.

[8] D. Guo, Nonlinear Functional Analysis, Jinan: Science and Technology Press of Shandong, 1985 (in Chinese).

[9] M.G. Krein, M.A. Rutman, Linear operators leaving invariant a cone in a Banach space, Transl. Amer. Math. Soc. 10 (1962) 199-325.

[10] L. Hu, L. Wang, Multiple positive solutions of boundary value problems for systems of nonlinear second-order differential equations, J. Math. Anal. Appl. 335 (2007) 1052-1060.

[11] J. Henderson, S. K. Ntouyas, Positive solutions for systems of nth order three-point nonlocal boundary value problems, EJQTDE 18 (2007) 1-12.

[12] G. Infante, P. Pietramala, Existence and multiplicity of non-negative solutions for systems of perturbed Hammerstein integral equations, Nonlinear Analysis 71 (2009) 1301-1310.

[13] P. Kang, J. Xu, Z. Wei, Positive solutions for $2p$-order and $2q$-order systems of singular boundary value problems with integral boundary conditions, Nonlinear Analysis 72 (2010) 2767-2786.

[14] Y. Liu, B. Yan, Multiple solutions of singular boundary value problems for differential systems, J. Math. Anal. Appl. 287 (2003) 40-556.

[15] H. Lü, H. Yu, Y. Liu, Positive solutions for singular boundary value problems of a coupled system of differential equations, J. Math. Anal. Appl. 302 (2005) 14-29.

[16] L. Liu, B. Liu , Y. Wu, Positive solutions of singular boundary value problems for nonlinear differential systems, Applied Mathematics and Computation 186 (2007) 1163-1172.

[17] B. Liu, L. Liu, Y. Wu, Positive solutions for singular systems of three-point boundary value problems, Computers and Mathematics with Applications 53 (2007) 1429-1438.

[18] R. Ma, Multiple nonnegative solutions of second-order systems of boundary value problems, Nonlinear Anal. 42 (2000) 1003-1010.

[19] J. Sun, X. Liu, Computation for topological degree and its applications, J. Math. Anal. Appl. 202 (1996) 785-796.

[20] H. Su, Z. Wei, X. Zhang, J. Liu, Positive solutions of n-order and m-order multi-point singular boundary value system, Applied Mathematics and Computation 188 (2007) 1234-1243.

[21] A. Saadatmandi, J. Farsangi, Chebyshev finite difference method for a nonlinear system of second-order boundary value problems Applied Mathematics and Computation 192 (2007) 586-591.

[22] A. Tamilselvan, N. Ramanujam, V. Shanthi, A numerical method for singularly perturbed weakly coupled system of two second order ordinary differential equations with discontinuous source term, J. Comput. Appl. Math. 202 (2007) 203-216.

[23] H.B. Thompson, C. Tisdell, Systems of difference equations associated with boundary value problems for second order systems of ordinary differential equations, J. Math. Anal. Appl. 248 (2000) 333-347.

[24] H.B. Thompson, C. Tisdell, Boundary value problems for systems of difference equations associated with systems of second-order ordinary differential equations, Appl. Math. Lett. 15 (6) (2002) 761-766.

[25] T. Valanarasu, N. Ramanujam, An asymptotic initial value method for boundary value problems for a system of singularly perturbed second-order ordinary differential equations, Appl. Math. Comput. 147 (6) (2004) 227-240.

[26] Z. Wei, Positive solution of singular Dirichlet boundary value problems for second order ordinary differential equation system, J. Math. Anal. Appl. 328 (2007) 1255-1267.

[27] Z. Wei, M. Zhang, Positive solutions of singular sub-linear boundary value problems for fourth-order and second-order differential equation systems, Applied Mathematics and Computation 197 (2008) 135-148.

[28] P.J.Y. Wong, Multiple fixed-sign solutions for a system of higher order three-point boundary-value problems with deviating arguments, Computers and Mathematics with Applications 55 (2008) 516-534.

[29] Jiafa Xu, Zhilin Yang, Three positive solutions for a system of singular generalized Lidstone problems, Electron. J. Diff. Equ., Vol. 2009(2009), No. 163, pp. 1-9.

[30] Jiafa Xu, Zhilin Yang, Positive solutions of boundary value problem for system of nonlinear nth order ordinary differential equations, J. Sys. Sci. & Math. Scis. 30(5) (2010,5), 633-641 (in Chinese).

[31] Jiafa Xu, Zhilin Yang, Positive solutions for a system of generalized Lidstone problems, J Appl Math Comput DOI 10.1007/s12190-010-0418-3.

[32] D. Xie, C. Bai, Y. Liu, C. Wang, Positive solutions for nonlinear semipositone nth-Order boundary value problems, Electronic Journal of Qualitative Theory of Differential Equations 2008, No. 7, 1-12.

[33] X. Xu, Existence and multiplicity of positive solutions for multi-parameter three-point differential equations system, J. Math. Anal. Appl. 324 (2006) 472-490.

[34] Z. Yang, J. Sun, Positive solutions of boundary value problems for systems of nonlinear second order ordinary differential equations, Acta Math. Sinica 47 (1) (2004) 111-118 (in Chinese).

[35] Z. Yang, D. O'Regan, Positive solvability of systems of nonlinear Hammerstein integral equations, J. Math. Anal. Appl. 311 (2005), 600-614.

[36] Z. Yang, Positive solutions to a system of second-order nonlocal boundary value problems, Nonlinear Anal. 62 (2005), 1251-1265.

[37] X. Yang, Existence of positive solutions for $2m$-order nonlinear differential systems, Nonlinear Anal. 61 (2005) 77-95.

[38] Ru. Y, An. Y, Positive solutions for $2p$-order and $2q$-order nonlinear ordinary differential systems, J. Math. Anal. Appl. 324 (2006) 1093-1104.

[39] Q. Yao, Positive solutions to a semilinear system of second order two-point boundary value problems, Ann. Differential Equations 22 (1) (2006) 87-94.

[40] R. Yang, Y. Zhang, Positive solutions for $2p$ and $2q$ order simultaneous nonlinear ordinary differential system, J. Sys. Sci. and Math. Scis. 29(12) (2009, 12), 1571-1578 (in Chinese).

[41] Z. Zhang, Existence of non-trivial solution for superlinear system of integral equations and its applications, Acta Math. Sinica 15 (1999) 153-162 (in Chinese).

[42] Y. Zhou, Y. Xu, Positive solutions of three-point boundary value problems for systems of nonlinear second order ordinary differential equations, J. Math. Anal. Appl. 320 (2006) 578-590.

[43] X. Zhang, Y. Xu, Multiple positive solutions of singularly perturbed differential systems with different orders, Nonlinear Analysis 72 (2010) 2645-2657.

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