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Existence of solutions for nonlinear fractional differential equations with impulses and anti-periodic boundary conditions

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Abstract

In this paper, we prove the existence of solutions for an anti-periodic boundary value problem of nonlinear impulsive fractional differential equations by applying some known fixed point theorems. Some examples are presented to illustrate the main results.

Keywords and Phrases: Anti-periodic boundary value problem; Impulse; Nonlinear fractional differential equations; Fixed point theorem.

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1 Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, etc. involves derivatives of fractional order([1]-[3]). Recently, many authors have studied fractional-order differential equations from two aspects, one is the theoretical aspects of existence and uniqueness of solutions, the other is the analytic and numerical methods for finding solutions. The interest in the study of fractional-order differential equations lies in the fact that fractional-order models are found to be more accurate than the classical integer-order models, that is, there are more degrees of freedom in the fractional-order models. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. In consequence, the subject of fractional differential equations is gaining more and more attention. For some recent development on the topic, see ([4]-[13]) and the references therein.

Impulsive differential equations arising from the real world describe the dynamics of processes in which sudden, discontinuous jumps occur. Such processes are naturally seen in biology, physics, engineering, etc. Due to their significance, it is important to study the solvability of impulsive differential equations. For the general theory and applications of impulsive differential equations, we refer the reader to the references ([14]-[18]). It is worthwhile mentioning that impulsive differential equations of fractional order have not been much studied and many aspects of these equations are yet to be explored. The recent results on impulsive fractional differential equations can be found in ([19]-[26]).

Anti-periodic problems constitute an important class of boundary value problems and have recently received considerable attention. Anti-periodic boundary conditions appear in physics in a variety of situations (see for example, in ([27]-[35]) and the references therein). For some recent work on anti-periodic boundary value problems of fractional differential equations, see ([36]-[40]) and the references therein.

Motivated by the above-mentioned work on anti-periodic and impulsive boundary value problems of fractional order, in this paper, we study the following problem

$$\begin{cases} {}^CD^\alpha u(t) = f(t, u(t)), \quad 1 < \alpha \le 2, \quad t \in J', \\ \triangle u(t_k) = I_k(u(t_k)), \quad \triangle u'(t_k) = I_k^*(u(t_k)), \quad k = 1, 2, \cdots, p, \\ \quad u(0) = -u(T), \qquad u'(0) = -u'(T), \end{cases} \tag{1.1}$$

where ${}^CD^\alpha$ is the Caputo fractional derivative, $f \in C(J\times\mathbb{R}, \mathbb{R})$, $I_k, I_k^* \in C(\mathbb{R}, \mathbb{R})$, $J = [0, T](T > 0)$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_p < t_{p+1} = T$, $J' = J\backslash\{t_1, t_2, \cdots, t_p\}$, $\triangle u(t_k) = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ denote the right and the left limits of $u(t)$ at $t = t_k (k = 1, 2, \cdots, p)$, respectively. $\triangle u'(t_k)$ have a similar meaning for $u'(t)$.

We organize the rest of this paper as follows: in Section 2, we present some necessary definitions and preliminary results that will be used to prove our main results. The proofs of our main results are given in Section 3. Section 4 contains some illustrative examples.

2 Preliminaries

Let $J_0 = [0, t_1], J_1 = (t_1, t_2], \cdots, J_{p-1} = (t_{p-1}, t_p], J_p = (t_p, T]$, and we introduce the spaces: $PC(J, \mathbb{R}) = \{u : J \to \mathbb{R} \mid u \in C(J_k),\ k = 0, 1, \cdots, p,$ and $u(t_k^+)$ exist, $k = 1, 2, \cdots, p, \}$ with the norm $\|u\| = \sup_{t\in J} |u(t)|$, and $PC^1(J, \mathbb{R}) = \{u : J \to \mathbb{R} \mid u \in C^1(J_k),\ k = 0, 1, \cdots, p,$ and $u(t_k^+)$, $u'(t_k^+)$ exist, $k = 1, 2, \cdots, p, \}$ with the norm $\|u\|_{PC^1} = \max\{\|u\|, \|u'\|\}$. Obviously, $PC(J, \mathbb{R})$ and $PC^1(J, \mathbb{R})$ are Banach spaces.

Definition 2.1 *A function $u \in PC^1(J, \mathbb{R})$ with its Caputo derivative of order α existing on J is a solution of* (1.1) *if it satisfies* (1.1).

We need the following known results to prove the existence of solutions for (1.1).

Theorem 2.1 [17] *Let E be a Banach space. Assume that Ω is an open bounded subset of E with $\theta \in \Omega$ and let $T : \overline{\Omega} \to E$ be a completely continuous operator such that*

$$\|Tu\| \leq \|u\|, \qquad \forall u \in \partial\Omega.$$

Then T has a fixed point in $\overline{\Omega}$.

Theorem 2.2 [17] *Let E be a Banach space. Assume that $T : E \to E$ is a completely continuous operator and the set $V = \{u \in E \mid u = \mu Tu, 0 < \mu < 1\}$ is bounded. Then T has a fixed point in E.*

Lemma 2.1 *For a given $y \in C[0,T]$, a function u is a solution of the impulsive anti-periodic boundary value problem*

$$\begin{cases} {}^C D^\alpha u(t) = y(t), \quad 1 < \alpha \leq 2, \quad t \in J', \\ \triangle u(t_k) = I_k(u(t_k)), \quad \triangle u'(t_k) = I_k^*(u(t_k)), \quad k = 1, 2, \cdots, p, \\ u(0) = -u(T), \qquad u'(0) = -u'(T), \end{cases} \tag{2.1}$$

if and only if u is a solution of the impulsive fractional integral equation

$$u(t) = \begin{cases} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - \frac{1}{2}\int_{t_p}^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \frac{T-2t}{4}\int_{t_p}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + \mathscr{A}, \ t \in J_0; \\ \int_{t_k}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - \frac{1}{2}\int_{t_p}^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \frac{T-2t}{4}\int_{t_p}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds \\ + \sum_{i=1}^{k} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + I_i(u(t_i)) \Big] + \sum_{i=1}^{k-1} (t_k - t_i) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + I_i^*(u(t_i)) \Big] \\ + \sum_{i=1}^{k} (t - t_k) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + I_i^*(u(t_i)) \Big] + \mathscr{A}, \quad t \in J_k, \ k = 1, 2, \cdots, p. \end{cases} \tag{2.2}$$

where

$$\begin{aligned} \mathscr{A} = \ & -\frac{1}{2}\sum_{i=1}^{p} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + I_i(u(t_i)) \Big] \\ & -\frac{1}{2}\sum_{i=1}^{p-1} (t_p - t_i) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + I_i^*(u(t_i)) \Big] \\ & -\sum_{i=1}^{p} \frac{T - 2t_p + 2t}{4} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + I_i^*(u(t_i)) \Big] \end{aligned}$$

Proof. Let u be a solution of (2.1). Then, for $t \in J_0$, there exist constants $c_1, c_2 \in \mathbb{R}$ such that

$$u(t) = I^\alpha y(t) - c_1 - c_2 t = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds - c_1 - c_2 t. \tag{2.3}$$

$$u'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} y(s)ds - c_2.$$

For $t \in J_1$, there exist constants $d_1, d_2 \in \mathbb{R}$, such that

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} y(s)ds - d_1 - d_2(t - t_1), \\ u'(t) &= \frac{1}{\Gamma(\alpha-1)} \int_{t_1}^t (t-s)^{\alpha-2} y(s)ds - d_2, \end{aligned}$$

Then we have

$$u(t_1^-) = \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} y(s)ds - c_1 - c_2 t_1, \qquad u(t_1^+) = -d_1,$$
$$u'(t_1^-) = \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1 - s)^{\alpha-2} y(s)ds - c_2, \qquad u'(t_1^+) = -d_2,$$

In view of $\triangle u(t_1) = u(t_1^+) - u(t_1^-) = I_1(u(t_1))$, and $\triangle u'(t_1) = u'(t_1^+) - u'(t_1^-) = I_1^*(u(t_1))$, we have

$$-d_1 = \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} y(s)ds - c_1 - c_2 t_1 + I_1(u(t_1)),$$
$$-d_2 = \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1 - s)^{\alpha-2} y(s)ds - c_2 + I_1^*(u(t_1)).$$

Consequently,

$$\begin{aligned} u(t) = & \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t - s)^{\alpha-1} y(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} y(s)ds \\ & + \frac{t - t_1}{\Gamma(\alpha-1)} \int_0^{t_1} (t_1 - s)^{\alpha-2} y(s)ds + I_1(u(t_1)) + (t - t_1) I_1^*(u(t_1)) - c_1 - c_2 t, \qquad t \in J_1. \end{aligned}$$

By a similar process, we can get

$$\begin{aligned} u(t) = & \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \sum_{i=1}^{k} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + I_i(u(t_i)) \Big] \\ & + \sum_{i=1}^{k-1} (t_k - t_i) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + I_i^*(u(t_i)) \Big] \\ & + \sum_{i=1}^{k} (t - t_k) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + I_i^*(u(t_i)) \Big] - c_1 - c_2 t, \quad t \in J_k, \ k = 1, 2, \cdots, p. \end{aligned} \tag{2.4}$$

By conditions $u(0) = -u(T)$ and $u'(0) = -u'(T)$, we have

$$\begin{aligned} c_1 = & \frac{1}{2} \int_{t_p}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - \frac{T}{4} \int_{t_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds \\ & + \frac{1}{2} \sum_{i=1}^{p} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + I_i(u(t_i)) \Big] \\ & + \frac{1}{2} \sum_{i=1}^{p-1} (t_p - t_i) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + I_i^*(u(t_i)) \Big] \\ & + \sum_{i=1}^{p} \frac{T - 2t_p}{4} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + I_i^*(u(t_i)) \Big] \end{aligned}$$

and

$$c_2 = \frac{1}{2} \int_{t_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + \frac{1}{2} \sum_{i=1}^{p} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds + I_i^*(u(t_i)) \Big]$$

Substituting the value of $c_i (i = 1, 2)$ in (2.3) and (2.4), we can get (2.2). Conversely, assume that u is a solution of the impulsive fractional integral equation (2.2), then by a direct computation, it follows that the solution given by (2.2) satisfies (2.1). □

Remark 2.1 *The first three terms of the solution* (2.2) *correspond to the solution for the problem without impulses.*

3 Main results

Define an operator $T : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ as

$$\begin{aligned} Tu(t) = &\int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds - \frac{1}{2}\int_{t_p}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds \\ &+ \frac{T-2t}{4}\int_{t_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds + \sum_{i=1}^{k}\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds + I_i(u(t_i))\Big] \\ &+ \sum_{i=1}^{k-1}(t_k-t_i)\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds + I_i^*(u(t_i))\Big] \\ &+ \sum_{i=1}^{k}(t-t_k)\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds + I_i^*(u(t_i))\Big] \\ &- \frac{1}{2}\sum_{i=1}^{p}\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds + I_i(u(t_i))\Big] \\ &- \frac{1}{2}\sum_{i=1}^{p-1}(t_p-t_i)\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds + I_i^*(u(t_i))\Big] \\ &- \sum_{i=1}^{p}\frac{T-2t_p+2t}{4}\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds + I_i^*(u(t_i))\Big]. \end{aligned} \tag{3.1}$$

Then the problem (1.1) has a solution if and only if the operator T has a fixed point.

Lemma 3.1 *The operator $T : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is completely continuous.*

Proof. It is obvious that T is continuous in view of continuity of f, I_k and I_k^*.

Let $\Omega \subset PC(J,\mathbb{R})$ be bounded. Then, there exist positive constants $L_i > 0 (i = 1, 2, 3)$ such that $|f(t,u)| \le L_1, |I_k(u)| \le L_2$ and $|I_k^*(u)| \le L_3, \forall u \in \Omega$. Thus, $\forall u \in \Omega$, we have

$$\begin{aligned} |Tu(t)| \le &\int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s))|ds + \frac{1}{2}\int_{t_p}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s))|ds \\ &+ \frac{|T-2t|}{4}\int_{t_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s))|ds + \sum_{i=1}^{k}\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s))|ds + |I_i(u(t_i))|\Big] \\ &+ \sum_{i=1}^{k-1}(t_k-t_i)\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s))|ds + |I_i^*(u(t_i))|\Big] \\ &+ \sum_{i=1}^{k}(t-t_k)\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s))|ds + |I_i^*(u(t_i))|\Big] \\ &+ \frac{1}{2}\sum_{i=1}^{p}\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s))|ds + |I_i(u(t_i))|\Big] \\ &+ \frac{1}{2}\sum_{i=1}^{p-1}(t_p-t_i)\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s))|ds + |I_i^*(u(t_i))|\Big] \\ &+ \sum_{i=1}^{p}\frac{|T-2t_p+2t|}{4}\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s))|ds + |I_i^*(u(t_i))|\Big] \\ \le\ & L_1\int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \frac{L_1}{2}\int_{t_p}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \frac{TL_1}{4}\int_{t_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds \\ &+ \frac{3}{2}\sum_{i=1}^{p}\Big[L_1\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} ds + L_2\Big] + \frac{3T}{2}\sum_{i=1}^{p-1}\Big[L_1\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + L_3\Big] \\ &+ \frac{7T}{4}\sum_{i=1}^{p}\Big[L_1\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + L_3\Big] \end{aligned}$$

$$\leq \quad \frac{3(1+p)T^\alpha L_1}{2\Gamma(\alpha+1)} + \frac{(13p-5)T^\alpha L_1}{4\Gamma(\alpha)} + \frac{3pL_2}{2} + \frac{(13p-6)TL_3}{4}, \tag{3.2}$$

which implies

$$\|Tu\| \leq \frac{3(1+p)T^\alpha L_1}{2\Gamma(\alpha+1)} + \frac{(13p-5)T^\alpha L_1}{4\Gamma(\alpha)} + \frac{3pL_2}{2} + \frac{(13p-6)TL_3}{4} := L.$$

On the other hand, for any $t \in J_k, 0 \leq k \leq p$, we have

$$\begin{aligned}
& |(Tu)'(t)| \\
\leq\ & \int_{t_k}^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|ds + \sum_{i=1}^k \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|ds + |I_i^*(u(t_i))| \Big] \\
& + \frac{1}{2} \int_{t_p}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|ds + \frac{1}{2}\sum_{i=1}^p \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|ds + |I_i^*(u(t_i))| \Big] \\
\leq\ & L_1 \int_{t_k}^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds + \frac{L_1}{2}\int_{t_p}^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds + \frac{3}{2}\sum_{i=1}^p \Big[L_1 \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds + L_3 \Big] \\
\leq\ & \frac{3(1+p)T^{\alpha-1}L_1}{2\Gamma(\alpha)} + \frac{3pL_3}{2} := \overline{L}.
\end{aligned}$$

Hence, for $t_1, t_2 \in J_k,\ t_1 < t_2,\ 0 \leq k \leq p$, we have

$$|(Tu)(t_2) - (Tu)(t_1)| \leq \int_{t_1}^{t_2} |(Tu)'(s)|ds \leq \overline{L}(t_2 - t_1).$$

This implies that T is equicontinuous on all $J_k, k = 0, 1, 2, \cdots, p$. The Arzela-Ascoli Theorem implies that $T : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is completely continuous. □

Theorem 3.1 *Let* $\lim_{u\to 0} \frac{f(t,u)}{u} = 0, \lim_{u\to 0} \frac{I_k(u)}{u} = 0$ *and* $\lim_{u\to 0} \frac{I_k^*(u)}{u} = 0$, *then problem* (1.1) *has at least one solution.*

Proof. Since $\lim_{u\to 0} \frac{f(t,u)}{u} = 0,\ \lim_{u\to 0} \frac{I_k(u)}{u} = 0$ and $\lim_{u\to 0} \frac{I_k^*(u)}{u} = 0$, then there exists a constant $r > 0$ such that $|f(t,u)| \leq \delta_1|u|,\ |I_k(u)| \leq \delta_2|u|$ and $|I_k^*(u)| \leq \delta_3|u|$ for $0 < |u| < r$, where $\delta_i > 0 (i = 1, 2, 3)$ satisfy

$$\frac{3(1+p)T^\alpha \delta_1}{2\Gamma(\alpha+1)} + \frac{(13p-5)T^\alpha \delta_1}{4\Gamma(\alpha)} + \frac{3p\delta_2}{2} + \frac{(13p-6)T\delta_3}{4} \leq 1. \tag{3.3}$$

Let $\Omega = \{u \in PC(J, \mathbb{R}) \mid \|u\| < r\}$. Take $u \in PC(J, \mathbb{R})$, such that $\|u\| = r$, which means $u \in \partial\Omega$. Then, by the process used to obtain (3.2), we have

$$|Tu(t)| \leq \Big\{ \frac{3(1+p)T^\alpha \delta_1}{2\Gamma(\alpha+1)} + \frac{(13p-5)T^\alpha \delta_1}{4\Gamma(\alpha)} + \frac{3p\delta_2}{2} + \frac{(13p-6)T\delta_3}{4} \Big\} \|u\|. \tag{3.4}$$

Thus, (3.4) shows $\|Tu\| \leq \|u\|$, $u \in \partial\Omega$.

Therefore, by Theorem 2.1, we know that T has at least one fixed point, which in turn implies that (1.1) has at least one solution $u \in \overline{\Omega}$. □

Theorem 3.2 *Assume that*

(H_1) *There exist positive constants* $L_i (i = 1, 2, 3)$ *such that*

$$|f(t,u)| \le L_1, \quad |I_k(u)| \le L_2, \quad |I_k^*(u)| \le L_3, \quad \text{for } t \in J,\ u \in \mathbb{R} \text{ and } k = 1, 2, \cdots, p.$$

Then problem (1.1) *has at least one solution.*

Proof. Now, we show the set $V = \{u \in PC(J, \mathbb{R}) \mid u = \mu Tu, 0 < \mu < 1\}$ is bounded.

Let $u \in V$, then $u = \mu Tu, 0 < \mu < 1$. For any $t \in J$ we have

$$\begin{aligned}
u(t) = & \int_{t_k}^{t} \frac{\mu(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds - \frac{\mu}{2}\int_{t_p}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds \\
& -\frac{(T-2t)\mu}{4}\int_{t_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds + \mu\sum_{i=1}^{k}\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds + I_i(u(t_i))\Big] \\
& +\mu\sum_{i=1}^{k-1}(t_k-t_i)\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds + I_i^*(u(t_i))\Big] \\
& +\mu\sum_{i=1}^{k}(t-t_k)\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds + I_i^*(u(t_i))\Big] \\
& -\frac{\mu}{2}\sum_{i=1}^{p}\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds + I_i(u(t_i))\Big] \\
& -\frac{\mu}{2}\sum_{i=1}^{p-1}(t_p-t_i)\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds + I_i^*(u(t_i))\Big] \\
& -\mu\sum_{i=1}^{p}\frac{T-2t_p+2t}{4}\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))ds + I_i^*(u(t_i))\Big].
\end{aligned} \tag{3.5}$$

Combining (H_1) and (3.5), by the process used to obtain (3.2), we have

$$\begin{aligned}
& |u(t)| = \mu|Tu(t)| \\
\le & \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s))|ds + \frac{1}{2}\int_{t_p}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s))|ds \\
& +\frac{|T-2t|}{4}\int_{t_p}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s))|ds + \sum_{i=1}^{k}\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s))|ds + |I_i(u(t_i))|\Big] \\
& +\sum_{i=1}^{k-1}(t_k-t_i)\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s))|ds + |I_i^*(u(t_i))|\Big] \\
& +\sum_{i=1}^{k}(t-t_k)\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s))|ds + |I_i^*(u(t_i))|\Big] \\
& +\frac{1}{2}\sum_{i=1}^{p}\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s))|ds + |I_i(u(t_i)|)\Big] \\
& +\frac{1}{2}\sum_{i=1}^{p-1}(t_p-t_i)\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s))|ds + |I_i^*(u(t_i))|\Big] \\
& +\sum_{i=1}^{p}\frac{|T-2t_p+2t|}{4}\Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s))|ds + |I_i^*(u(t_i))|\Big] \\
\le & \frac{3(1+p)T^\alpha L_1}{2\Gamma(\alpha+1)} + \frac{(13p-5)T^\alpha L_1}{4\Gamma(\alpha)} + \frac{3pL_2}{2} + \frac{(13p-6)TL_3}{4}
\end{aligned}$$

which implies that for any $t \in J$, it hold that

$$\|u\| \le \frac{3(1+p)T^\alpha L_1}{2\Gamma(\alpha+1)} + \frac{(13p-5)T^\alpha L_1}{4\Gamma(\alpha)} + \frac{3pL_2}{2} + \frac{(13p-6)TL_3}{4}.$$

So, the set V is bounded. By Theorem 2.2, we know that T has at least one fixed point, which implies that (1.1) has at least one solution. $\square$

4 Examples

Example 4.1 *Consider the following impulsive anti-periodic fractional boundary value problem*

$$\begin{cases} {}^C D^\alpha u(t) = e^{u^2(t)} + t^3 \sin u^2(t) - 1, & 0 < t < 1, \ t \neq \frac{1}{3}, \\ \triangle u(\frac{1}{3}) = 5\ln(1 + u^4(t)), & \triangle u'(\frac{1}{3}) = \frac{\sin^4 u(t)}{3}, \\ u(0) = -u(1), & u'(0) = -u'(1), \end{cases} \tag{4.1}$$

where $1 < \alpha \leq 2$ *and* $p = 1$.

Clearly, all the assumptions of Theorem 3.1 hold. Thus, by the conclusion of Theorem 3.1 we can get that the above impulsive anti-periodic fractional boundary value problem (4.1) has at least one solution.

Example 4.2 *Consider the following impulsive anti-periodic fractional boundary value problem*

$$\begin{cases} {}^C D^\alpha u(t) = \frac{\ln(1 + 3e^t)e^{-u^2(t)}}{3 + \sin^3 u(t)}, & 0 < t < 1, \ t \neq \frac{1}{2}, \\ \triangle u(\frac{1}{2}) = \frac{7 + 2\cos u^2(t)}{3 + u^2(t)}, & \triangle u'(\frac{1}{2}) = 8\arctan^2[\ln(1 + 2u^2(t))], \\ u(0) = -u(1), & u'(0) = -u'(1), \end{cases} \tag{4.2}$$

where $1 < \alpha \leq 2$ *and* $p = 1$.

It can easily be found that $L_1 = \frac{\ln(1+3e)}{2}, L_2 = 3, L_3 = 2\pi^2$. Thus, the conclusion of Theorem 3.2 applies and the impulsive fractional boundary value problem (4.2) has at least one solution.

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