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Blow-up phenomena for a pseudo-parabolic system with variable exponents

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Abstract. In this paper, we consider a pseudo-parabolic system with nonlinearities of variable exponent type

$$\begin{cases} u_t - \Delta u_t - \operatorname{div}(|\nabla u|^{m(x)-2}\nabla u) = |uv|^{p(x)-2}uv^2 & \text{in } \Omega \times (0,T), \\ v_t - \Delta v_t - \operatorname{div}(|\nabla v|^{n(x)-2}\nabla v) = |uv|^{p(x)-2}u^2v & \text{in } \Omega \times (0,T) \end{cases}$$

associated with initial and Dirichlet boundary conditions, where the variable exponents $p(\cdot)$, $m(\cdot)$, $n(\cdot)$ are continuous functions on $\overline{\Omega}$. We obtain an upper bound and a lower bound for blow-up time if variable exponents $p(\cdot)$, $m(\cdot)$, $n(\cdot)$ and the initial data satisfy some conditions.

Keywords: pseudo-parabolic system, blow-up, upper bound, lower bound, variable exponent.

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1 Introduction

In this paper, we consider the initial-boundary value problem

$$\begin{cases} u_t - \Delta u_t - \operatorname{div}(|\nabla u|^{m(x)-2}\nabla u) = |uv|^{p(x)-2}uv^2 & \text{in } \Omega \times (0,T), \\ v_t - \Delta v_t - \operatorname{div}(|\nabla v|^{n(x)-2}\nabla v) = |uv|^{p(x)-2}u^2v & \text{in } \Omega \times (0,T), \\ u = 0, \ v = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \ v(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$, the nonlinear term $\operatorname{div}(|\nabla u|^{m(x)-2}\nabla u)$ is called $m(x)$-Laplace operator, and the variable exponents $p(\cdot)$, $m(\cdot)$, $n(\cdot)$ are continuous functions on $\overline{\Omega}$, later specified.

It is well known that nonlinear pseudo-parabolic equations appear in the study of various problems of the hydrodynamics, filtration theory, electrorheological fluids and others

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(see [1, 4, 6]). Recently, Di et al. [2] has been studied the following initial-boundary value problem

$$u_t - \nu\Delta u_t - \operatorname{div}(|\nabla u|^{m(x)-2}\nabla u) = |u|^{p(x)-2}u \quad in\ \Omega \times (0,T) \tag{1.2}$$

with Dirichlet boundary condition. By means of a differential inequality technique, they obtained an upper bound and a lower bound for blow-up time if variable exponents $p(\cdot)$, $m(\cdot)$ and the initial data satisfy some conditions. Obviously, if $\nu = 1$, $m(x) = 2$, $p(x) = p$, (1.2) reduces to the following pseudo-parabolic equation

$$u_t - \Delta u - \Delta u_t = |u|^{p-2}u \quad \text{in } \Omega \times (0,T). \tag{1.3}$$

As for (1.3), there are many results concerning asymptotic behavior [7, 14], the existence and uniqueness [1, 13] of solutions, blow-up [8, 14] property and so on. Especially, Xu [14] prove that the solutions blow up in finite time in $H_0^1(\Omega)$-norm. Luo [8] obtain an upper bound and a lower bound of the blow-up rate. More generally, Peng et al. [10] considered the following initial-boundary value problem

$$u_t - \nu\Delta u_t - \operatorname{div}(\rho(|\nabla u|^2)\nabla u) = f(u) \quad \text{in } \Omega \times (0,T).$$

A lower bound for blow-up time is determined if blow-up does occur. Furthermore, they establish an upper bound for blow-up time to a special class.

As we know, on the bounds, has been less studied the case of blow-up time to the system (1.1). Our objective in this paper is to study the blow-up phenomenon of solutions of the system (1.1) in the framework of the Lebesgue and Sobolev spaces with variable exponents. In details, this paper is organized as follows: in Section 2, we introduce the function spaces of Orlicz–Sobolev type and present a brief description of their main properties. In Section 3, a criterion for blow-up to the system (1.1) that leads to the upper bound for blow-up time is obtained. In Section 4, we give the lower bound of blow-up time to the system (1.1).

2 Function spaces

As in [2], we first recall some known results about the Lebesgue and Sobolev spaces with variable exponents (see [3, 5, 11, 12]) which will be needed in this paper.

Let $r(\cdot) : \Omega \to [1,\infty)$ be a measurable function, where Ω is a domain of $\mathbb{R}^n$. We denote by $r_- = \operatorname{ess\,inf}_{x\in\Omega} r(x)$ and $r_+ = \operatorname{ess\,sup}_{x\in\Omega} r(x)$. The variable exponent Lebesgue space $L^{r(\cdot)}(\Omega)$ consists of all measurable functions u defined on Ω for which

$$\rho_{r(\cdot)} = \int_\Omega |u(x)|^{r(x)}dx < \infty.$$

The set $L^{r(\cdot)}(\Omega)$ equipped with the Luxembourg norm $\|u\|_{r(\cdot)} = \inf\{\lambda > 0 : \rho_{r(\cdot)}(u/\lambda) \le 1\}$ is a Banach space (see [3]). The variable exponent Sobolev space $W^{1,r(\cdot)}(\Omega)$ is defined by

$$\begin{cases} W^{1,r(\cdot)}(\Omega) = \{u \in L^{r(\cdot)}(\Omega) : |\nabla u(x)|^{r(x)} \in L^1(\Omega)\}, \\ \|u\|_{W^{1,r(\cdot)}(\Omega)} = \|u\|_{1,r(\cdot)} = \|\nabla u\|_{r(\cdot)} + \|u\|_{r(\cdot)}. \end{cases}$$

$W_0^{1,r(\cdot)}(\Omega)$ is defined as the closure in $W^{1,r(\cdot)}(\Omega)$ of $C_0^\infty(\Omega)$. $W^{1,r'(\cdot)}(\Omega)$ is the dual space of $W^{1,r(\cdot)}(\Omega)$ where $r'(\cdot)$ is the function such that $\frac{1}{r(\cdot)} + \frac{1}{r'(\cdot)} = 1$.

Let the variable exponent $p(\cdot)$ satisfy the Zhikov–Fan conditions:

$$|p(x) - p(y)| \leq \frac{A}{\log(\frac{1}{|x-y|})}, \quad \text{for all } x, y \in \Omega \text{ with } |x-y| < \delta, \tag{2.1}$$

where $A > 0$ and $0 < \delta < 1$.

Now, we present some useful lemmas which will be used later.

Lemma 2.1 (see [3,5]). *We have the following results.*

(1) If Ω has a finite measure and $q_1(\cdot)$, $q_2(\cdot)$ are variable exponents satisfying $q_1(x) \leq q_2(x)$ almost everywhere in Ω, then there is a continuous embedding from $L^{q_2(\cdot)}(\Omega) \hookrightarrow L^{q_1(\cdot)}(\Omega)$.

(2) Let the variable exponent $p(\cdot)$ satisfy (2.1), then $\|u\|_{p(\cdot)} \leq C\|\nabla u\|_{p(\cdot)}$ for all $u \in W_0^{1,p(\cdot)}(\Omega)$, where Ω is bounded.

(3) Let the variable exponents $q_1(\cdot) \in C(\overline{\Omega})$, $q_2 : \Omega \to [1, \infty)$ be a measurable function and satisfy

$$\operatorname*{ess\,inf}_{x \in \overline{\Omega}} (q_1^*(x) - q_2(x)) > 0, \quad \text{where } q_1^* = \begin{cases} \dfrac{nq_1(x)}{n - q_1(x)}, & \text{if } q_1(x) < n, \\ +\infty, & \text{if } q_1(x) \geq n. \end{cases}$$

Then, the Sobolev embedding $W_0^{1,q_1(\cdot)}(\Omega) \hookrightarrow L^{q_2(\cdot)}(\Omega)$ is continuous and compact.

3 Upper bound for blow-up time

Since $p(\cdot)$, $m(\cdot)$, $n(\cdot)$ are continuous functions on $\overline{\Omega}$, we denote by

$$\ell_+ = \max_{\bar{\Omega}} \ell(x), \qquad \ell_- = \min_{\bar{\Omega}} \ell(x)$$

where ℓ stands for $p(\cdot)$, $m(\cdot)$ and $n(\cdot)$ respectively. Assume that

$$p_- > \max\{m_+, n_+\}, \qquad \min\{m_-, n_-\} \geq 2, \tag{3.1}$$

and

$$m_+ \geq n_-, \qquad n_+ \geq m_-. \tag{3.2}$$

Firstly, we start with the following local existence theorem for the solutions of system (1.1) which can be obtained by Faedo–Galerkin method.

Theorem 3.1. *Let the variable exponent $p(\cdot)$ satisfy the Zhikov–Fan conditions (2.1) and (3.1) hold. Then for any $u_0 \in W_0^{1,m(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega)$, $v_0 \in W_0^{1,n(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega)$, there exists a number $T_0 \in (0, T]$ such that the system (1.1) has a unique solution*

$$u \in L^\infty([0, T_0]; W_0^{1,m(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega)), \qquad u_t \in L^2([0, T_0]; W_0^{1,2}(\Omega)),$$
$$v \in L^\infty([0, T_0]; W_0^{1,n(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega)), \qquad v_t \in L^2([0, T_0]; W_0^{1,2}(\Omega)),$$

satisfying

$$\begin{aligned}
&(u_t,\varphi)+(\nabla u_t,\nabla\varphi)+(|\nabla u|^{m(x)-2}\nabla u,\nabla\varphi)=(|uv|^{p(x)-2}uv^2,\varphi),\\
&\quad\forall\varphi\in W_0^{1,m(\cdot)}(\Omega)\cap L^{p(\cdot)}(\Omega),\\
&(v_t,\psi)+(\nabla v_t,\nabla\psi)+(|\nabla v|^{n(x)-2}\nabla v,\nabla\psi)=(|uv|^{p(x)-2}u^2v,\psi),\\
&\quad\forall\psi\in W_0^{1,n(\cdot)}(\Omega)\cap L^{p(\cdot)}(\Omega),
\end{aligned} \tag{3.3}$$

where $(u_t,\varphi)=\int_\Omega u_t\varphi dx$.

Next, we seek the upper bound for the blow-up time of the system (1.1).

Theorem 3.2. *Assume that* (2.1), (3.1) *and* (3.2) *hold. Let* $u_0\in W_0^{1,m(\cdot)}(\Omega)\cap L^{p(\cdot)}(\Omega)$, $v_0\in W_0^{1,n(\cdot)}(\Omega)\cap L^{p(\cdot)}(\Omega)$ *such that* $\|u_0\|_{H_0^1},\|v_0\|_{H_0^1}>0$ *and*

$$\int_\Omega\left[\frac{|u_0v_0|^{p(x)}}{p(x)}-\left(\frac{|\nabla u_0|^{m(x)}}{m(x)}+\frac{|\nabla v_0|^{n(x)}}{n(x)}\right)\right]dx\geq 0. \tag{3.4}$$

Then, the solution (u,v) *of the system* (1.1) *blows up in finite time* T^* *in* $H_0^1(\Omega)$*-norm. Moreover, an upper bound for blow-up time is given by*

$$T^*\leq\frac{b(F(0))^{1-\frac{1}{b}}}{(b-1)\beta}, \tag{3.5}$$

where β *and* b *are suitable positive constants given later and* $F(0)=\|u_0\|^2_{H_0^1}+\|v_0\|^2_{H_0^1}$.

Proof. Replacing φ by u_t, ψ by v_t in (3.3) respectively, and adding, we have

$$\begin{aligned}
&\int_\Omega(|u_t|^2+|\nabla u_t|^2+|v_t|^2+|\nabla v_t|^2)dx+\frac{d}{dt}\int_\Omega\left(\frac{1}{m(x)}|\nabla u|^{m(x)}+\frac{1}{n(x)}|\nabla v|^{n(x)}\right)dx\\
&\quad=\frac{d}{dt}\int_\Omega\frac{1}{p(x)}|uv|^{p(x)}dx.
\end{aligned} \tag{3.6}$$

Let us define the energy as follows

$$E(t)=\int_\Omega\left(\frac{1}{m(x)}|\nabla u|^{m(x)}+\frac{1}{n(x)}|\nabla v|^{n(x)}-\frac{1}{p(x)}|uv|^{p(x)}\right)dx. \tag{3.7}$$

Hence, by (3.6) and (3.7), we have

$$E'(t)=-\int_\Omega(|u_t|^2+|\nabla u_t|^2+|v_t|^2+|\nabla v_t|^2)dx\leq 0. \tag{3.8}$$

We define an auxiliary function

$$F(t)=\int_\Omega u^2dx+\int_\Omega|\nabla u|^2dx+\int_\Omega v^2dx+\int_\Omega|\nabla v|^2dx. \tag{3.9}$$

Multiplying u and v on two sides of two equations of the system (1.1) respectively, and integrating by part, we have

$$\int_\Omega uu_tdx+\int_\Omega\nabla u\cdot\nabla u_tdx+\int_\Omega|\nabla u|^{m(x)}dx=\int_\Omega|uv|^{p(x)}dx \tag{3.10}$$

and

$$\int_\Omega vv_t dx + \int_\Omega \nabla v \cdot \nabla v_t dx + \int_\Omega |\nabla v|^{n(x)} dx = \int_\Omega |uv|^{p(x)} dx. \tag{3.11}$$

Adding (3.10) and (3.11), we get

$$\begin{aligned}\int_\Omega uu_t dx + \int_\Omega \nabla u \cdot \nabla u_t dx + \int_\Omega vv_t dx + \int_\Omega \nabla v \cdot \nabla v_t dx \\ = -\int_\Omega (|\nabla u|^{m(x)} + |\nabla v|^{n(x)}) dx + 2\int_\Omega |uv|^{p(x)} dx. \end{aligned} \tag{3.12}$$

By differentiating $F(t)$ with respect to t, we have

$$\begin{aligned} F'(t) &= 2\int_\Omega uu_t dx + 2\int_\Omega \nabla u \cdot \nabla u_t dx + 2\int_\Omega vv_t dx + 2\int_\Omega \nabla v \cdot \nabla v_t dx \\ &= 4\int_\Omega |uv|^{p(x)} dx - 2\int_\Omega (|\nabla u|^{m(x)} + |\nabla v|^{n(x)}) dx \\ &= 4\int_\Omega p(x)\left[\frac{|uv|^{p(x)}}{p(x)} - \left(\frac{|\nabla u|^{m(x)}}{m(x)} + \frac{|\nabla v|^{n(x)}}{n(x)}\right)\right] dx + 4\int_\Omega p(x)\left(\frac{1}{m(x)} - \frac{1}{p(x)}\right)|\nabla u|^{m(x)} dx \\ &\quad + 4\int_\Omega p(x)\left(\frac{1}{n(x)} - \frac{1}{p(x)}\right)|\nabla v|^{n(x)} dx + 2\int_\Omega (|\nabla u|^{m(x)} + |\nabla v|^{n(x)}) dx. \end{aligned} \tag{3.13}$$

Thanks to $E'(t) \le 0$, we have

$$\begin{aligned} &\int_\Omega p(x)\left[\frac{|uv|^{p(x)}}{p(x)} - \left(\frac{|\nabla u|^{m(x)}}{m(x)} + \frac{|\nabla v|^{n(x)}}{n(x)}\right)\right] dx \\ &\ge \int_\Omega p(x)\left[\frac{|u_0 v_0|^{p(x)}}{p(x)} - \left(\frac{|\nabla u_0|^{m(x)}}{m(x)} + \frac{|\nabla v_0|^{n(x)}}{n(x)}\right)\right] dx \\ &\ge \int_\Omega p_-\left[\frac{|u_0 v_0|^{p(x)}}{p(x)} - \left(\frac{|\nabla u_0|^{m(x)}}{m(x)} + \frac{|\nabla v_0|^{n(x)}}{n(x)}\right)\right] dx \\ &\ge 0. \end{aligned} \tag{3.14}$$

By (3.13) and (3.14), we see

$$\begin{aligned} F'(t) &\ge 4\int_\Omega p_-\left(\frac{1}{m_+} - \frac{1}{p_-}\right)|\nabla u|^{m(x)} dx + 4\int_\Omega p_-\left(\frac{1}{n_+} - \frac{1}{p_-}\right)|\nabla v|^{n(x)} dx + 2\int_\Omega (|\nabla u|^{m(x)} + |\nabla v|^{n(x)}) dx \\ &= C_1\int_\Omega |\nabla u|^{m(x)} dx + C_2\int_\Omega |\nabla v|^{n(x)} dx, \end{aligned}$$

where $C_1 = 2 + 4p_-(\frac{1}{m_+} - \frac{1}{p_-})$, $C_2 = 2 + 4p_-(\frac{1}{n_+} - \frac{1}{p_-})$. Define the sets $\Omega_+ = \{x \in \Omega \mid |\nabla u| \ge 1, |\nabla v| \ge 1\}$ and $\Omega_- = \{x \in \Omega \mid |\nabla u| < 1, |\nabla v| < 1\}$. By the fact that $\|\nabla u\|_2 \le C\|\nabla u\|_r$ for all $r \ge 2$, it follows

$$\begin{aligned} F'(t) &\ge C_1\left(\int_{\Omega_-} |\nabla u|^{m_+} dx + \int_{\Omega_+} |\nabla u|^{m_-} dx\right) + C_2\left(\int_{\Omega_-} |\nabla v|^{n_+} dx + \int_{\Omega_+} |\nabla v|^{n_-} dx\right) \\ &\ge C_3\left[\left(\int_{\Omega_-} |\nabla u|^2 dx\right)^{\frac{m_+}{2}} + \left(\int_{\Omega_+} |\nabla u|^2 dx\right)^{\frac{m_-}{2}}\right] + C_4\left[\left(\int_{\Omega_-} |\nabla v|^2 dx\right)^{\frac{n_+}{2}} + \left(\int_{\Omega_+} |\nabla v|^2 dx\right)^{\frac{n_-}{2}}\right]. \end{aligned}$$

This implies that

$$\begin{aligned}(F'(t))^a &\geq C_5 \int_{\Omega_-} (|\nabla u|^2 + |\nabla v|^2) dx \geq 0,\\ (F'(t))^b &\geq C_6 \int_{\Omega_+} (|\nabla u|^2 + |\nabla v|^2) dx \geq 0,\end{aligned} \tag{3.15}$$

where $a = \max(\frac{2}{m_+}, \frac{2}{n_+})$, $b = \max(\frac{2}{m_-}, \frac{2}{n_-})$. The Poincaré inequality gives $\|\nabla u\|_2^2 \geq \lambda_1 \|u\|_2^2$, where λ_1 is the first eigenvalue of the problem

$$\begin{cases} \triangle \omega + \lambda \omega = 0, & \text{in } \Omega, \\ \omega = 0, & \text{on } \partial\Omega. \end{cases}$$

Thus, the follow relations

$$\begin{aligned}\|\nabla u\|_2^2 &= \frac{1}{1+\lambda_1}\|\nabla u\|_2^2 + \frac{\lambda_1}{1+\lambda_1}\|\nabla u\|_2^2\\ &\geq \frac{\lambda_1}{1+\lambda_1}\|u\|_2^2 + \frac{\lambda_1}{1+\lambda_1}\|\nabla u\|_2^2 = \frac{\lambda_1}{1+\lambda_1}\|u\|_{H_0^1}^2,\\ \|\nabla v\|_2^2 &= \frac{1}{1+\lambda_1}\|\nabla v\|_2^2 + \frac{\lambda_1}{1+\lambda_1}\|\nabla v\|_2^2\\ &\geq \frac{\lambda_1}{1+\lambda_1}\|v\|_2^2 + \frac{\lambda_1}{1+\lambda_1}\|\nabla v\|_2^2 = \frac{\lambda_1}{1+\lambda_1}\|v\|_{H_0^1}^2\end{aligned} \tag{3.16}$$

hold, where $\|u\|_p = (\int_\Omega u^p dx)^{\frac{1}{p}}$ and $\|u\|_{H_0^1}^2 = \|u\|_2^2 + \|\nabla u\|_2^2$. Combining (3.15) and (3.16), we conclude

$$\begin{aligned}(F'(t))^a &\geq \frac{C_5\lambda_1}{1+\lambda_1}(\|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2),\\ (F'(t))^b &\geq \frac{C_6\lambda_1}{1+\lambda_1}(\|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2).\end{aligned}$$

Consequently,

$$(F'(t))^a + (F'(t))^b \geq \frac{\lambda_1(C_5 + C_6)}{1+\lambda_1}(\|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2) = C_7 F(t), \tag{3.17}$$

which implies

$$(F'(t))^b \left(1 + (F'(t))^{a-b}\right) \geq C_7 F(t). \tag{3.18}$$

By (3.17) and the fact that $F(t) \geq F(0) > 0$ $(F'(t) \geq 0)$, we have

$$(F'(t))^a \geq \frac{C_7}{2}F(t) \geq \frac{C_7}{2}F(0)$$

or

$$(F'(t))^b \geq \frac{C_7}{2}F(t) \geq \frac{C_7}{2}F(0),$$

which implies that

$$F'(t) \geq C_8 (F(0))^{\frac{1}{a}}$$

or

$$F'(t) \geq C_9 (F(0))^{\frac{1}{b}}.$$

Therefore, we have that $F'(t) \geq \alpha$, where $\alpha = \min\{C_8(F(0))^{\frac{1}{a}}, C_9(F(0))^{\frac{1}{b}}\}$. From (3.2), it is easy to see $a - b \leq 0$. So, combining with (3.18), we get

$$F'(t) \geq \beta(F(t))^{\frac{1}{b}}, \tag{3.19}$$

where the constant $\beta = \left(\frac{C_7}{1+\alpha^{a-b}}\right)^{\frac{1}{b}}$. By (3.19), we receive

$$\frac{F'(t)}{(F(t))^{\frac{1}{b}}} \geq \beta. \tag{3.20}$$

Integrating the inequality (3.20) from 0 to t, we see

$$(F(t))^{1-\frac{1}{b}} \leq (F(0))^{1-\frac{1}{b}} + \frac{(b-1)\beta t}{b}, \tag{3.21}$$

which implies that

$$F(t) \geq \frac{1}{[(F(0))^{1-\frac{1}{b}} + \frac{(b-1)\beta t}{b}]^{\frac{b}{1-b}}}. \tag{3.22}$$

Thus, (3.22) shows that $F(t)$ blows up at some finite time T^* such that

$$T^* \leq \frac{b(F(0))^{1-\frac{1}{b}}}{(b-1)\beta}. \tag{3.23}$$

Finally, we get the solution (u, v) blows up in $H_0^1(\Omega)$-norm in finite time. □

Remark 3.3. From (3.23), we see that the larger $F(0)$ is, the smaller the blow-up time T^* is.

4 Lower bound for blow-up time

In this section, our aim is to determine a lower bound for blow-up time of the system (1.1). The technique is the same as [2].

Theorem 4.1. *Suppose that (2.1) and (3.1) hold. Furthermore assume that $2 < p_+ < \infty$ if $n \leq 2$, $2 < p_+ \leq \frac{2n}{n-2}$ if $n \geq 3$, $u_0 \in W_0^{1,m(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega)$, $v_0 \in W_0^{1,n(\cdot)}(\Omega) \cap L^{p(\cdot)}(\Omega)$ and the solution (u, v) of the system (1.1) becomes unbounded at finite time T^* in $H_0^1(\Omega)$-norm, then a lower bounded T^* for blow-up time is given by*

$$T^* \geq \int_{F(0)}^{\infty} \frac{d\eta}{M\eta^{p_+} + N\eta^{p_-}}, \tag{4.1}$$

where M and N are suitable positive constants given later and $F(0) = \|u_0\|_{H_0^1}^2 + \|v_0\|_{H_0^1}^2$.

Proof. We define the function $F(t)$ the same as (3.9). By (3.13), it is easy to get

$$\begin{aligned} F'(t) &= 2\int_\Omega u u_t dx + 2\int_\Omega \nabla u \cdot \nabla u_t dx + 2\int_\Omega v v_t dx + 2\int_\Omega \nabla v \cdot \nabla v_t dx \\ &\leq 4\int_\Omega |uv|^{p(x)} dx. \end{aligned} \tag{4.2}$$

Let us denote the sets $\Omega_+ = \{x \in \Omega \mid |uv| \geq 1\}$ and $\Omega_- = \{x \in \Omega \mid |uv| < 1\}$. Using the Cauchy–Schwarz inequality and the Sobolev embedding inequalities , we get

$$\begin{aligned}\int_\Omega |uv|^{p(x)}dx &\leq \int_{\Omega_+} |uv|^{p_+}dx + \int_{\Omega_-} |uv|^{p_-}dx \\ &\leq \int_\Omega |uv|^{p_+}dx + \int_\Omega |uv|^{p_-}dx \\ &\leq \left(\int_\Omega |u|^{2p_+}\right)^{\frac{1}{2}} \cdot \left(\int_\Omega |v|^{2p_+}\right)^{\frac{1}{2}} + \left(\int_\Omega |u|^{2p_-}\right)^{\frac{1}{2}} \cdot \left(\int_\Omega |v|^{2p_-}\right)^{\frac{1}{2}} \\ &\leq (B_+^{p_+})^2 \|\nabla u\|_2^{p_+} \cdot \|\nabla v\|_2^{p_+} + (B_-^{p_-})^2 \|\nabla u\|_2^{p_-} \cdot \|\nabla v\|_2^{p_-}, \end{aligned} \tag{4.3}$$

where B_+, B_- are the Sobolev embedding constants for $H_0^1(\Omega) \hookrightarrow L^{p_+}(\Omega)$ and $H_0^1(\Omega) \hookrightarrow L^{p_-}(\Omega)$, respectively. From the Cauchy–Schwarz inequality, we have

$$F'(t)^2 \geq \left(\int_\Omega |\nabla u|^2 dx\right)^2 + \left(\int_\Omega |\nabla v|^2 dx\right)^2 \geq 2\int_\Omega |\nabla u|^2 dx \cdot \int_\Omega |\nabla v|^2 dx.$$

Then

$$(F'(t))^{p_+} \geq 2^{\frac{p_+}{2}} \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{p_+}{2}} \cdot \left(\int_\Omega |\nabla v|^2 dx\right)^{\frac{p_+}{2}}$$

and

$$(F'(t))^{p_-} \geq 2^{\frac{p_-}{2}} \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{p_-}{2}} \cdot \left(\int_\Omega |\nabla v|^2 dx\right)^{\frac{p_-}{2}},$$

which implies that

$$(F'(t))^{p_+} \cdot 2^{-\frac{p_+}{2}} \geq \|\nabla u\|_2^{p_+} \cdot \|\nabla v\|_2^{p_+} \tag{4.4}$$

and

$$(F'(t))^{p_-} \cdot 2^{-\frac{p_-}{2}} \geq \|\nabla u\|_2^{p_-} \cdot \|\nabla v\|_2^{p_-}. \tag{4.5}$$

Thus, the combination of (4.2)–(4.5) implies that

$$F'(t) \leq M(F(t))^{p_+} + N(F(t))^{p_-},$$

where $M = 2^{-\frac{p_+}{2}}(B_+^{P_+})^2$, $N = 2^{-\frac{p_-}{2}}(B_+^{P_-})^2$. Therefore

$$\frac{F'(t)}{M(F(t))^{p_+} + N(F(t))^{p_-}} \leq 1. \tag{4.6}$$

Integrating the inequality (4.6) from 0 to t, we get

$$\int_{F(0)}^{F(t)} \frac{d\eta}{M\eta^{p_+} + N\eta^{p_-}} \leq t.$$

If (u, v) blows up in $H_0^1(\Omega)$-norm, then we obtain a lower bound T^* given by

$$T^* \geq \int_{F(0)}^{\infty} \frac{d\eta}{M\eta^{p_+} + N\eta^{p_-}}.$$

Clearly, the integral is bound since exponents $p_+ \geq p_- > 2$. □

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